LECTURE 32

Total recap

A summary grouped in six ideas

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
November 9, 2026 · Winter 2026

How to use this

Every result here was proved in a numbered lecture; the corner note says which.

THE SHAPE OF THE MODULE

1.	**Rates** — The same payment through the lens of time.	(L1—L4)	(p3—p8)
2.	**Annuities** — Regular payments in closed form.	(L5—L9)	(p9—p13)
3.	**Projects** — is it worth doing, and what does it earn?	(L10–L11, L21–L24)	(p14–p19)
4.	**Loans** — what a repayment is made of.	(L12–L14)	(p20–p22)
5.	**Securities** — bonds, equities, property.	(L15–L20)	(p23–p27)
6.	**Curves and risk** — a rate for every term.	(L25–L31)	(p28–p35)

Today

1. Rates
2. Annuities
3. Projects
4. Loans
5. Securities
6. Curves and risk
7. The whole thing on four slides

SECTION 1

Rates

$\mathsf{A}(t_1, t_2)$ is what 1 at t_1 becomes at t_2; $\mathsf{A}(t) = \mathsf{A}(0, t)$ is a shorthand; and the **discount factor** is its reciprocal,

$$\mathsf{v}(t) = \frac{1}{\mathsf{A}(t)}.$$

CONSISTENCY: TWO ROUTES, ONE ANSWER

$\mathsf{A}(t_0, t_2) = \mathsf{A}(t_0, t_1)\mathsf{A}(t_1, t_2)$. It is forced by no-arbitrage.

It holds for compound interest $C(1+i)^t$ and continous interest $C\mathrm{e}^{t\delta}$, and fails for simple interest $C(1+rt)$. Everything after Lecture 1 is compound or continuous.

The five rates

Effective interest i, effective discount d, nominal $i^{(m)}$ and $d^{(m)}$, and the force δ. All are the same growth, quoted differently:

$$1+i \;=\; \left(1+\frac{i^{(m)}}{m}\right)^{m} \;=\; (1-d)^{-1} \;=\; \left(1-\frac{d^{(m)}}{m}\right)^{-m} \;=\; \frac{1}{\mathsf{v}} \;=\; \mathsf{e}^{\delta}.$$

KEY IDEA

Convert *to* $1+i$, then *from* it. And for $i > 0$,

$$d \;<\; d^{(m)} \;<\; \delta \;<\; i^{(m)} \;<\; i,$$

with the nominal rates converging to δ from opposite sides as $m \to \infty$.

$$d = iv = \frac{i}{1+i}, \qquad i = \frac{d}{1-d}, \qquad \frac{d^{(m)}}{m} = \frac{i^{(m)}/m}{1+i^{(m)}/m}.$$

NUMERATOR = INTEREST. DENOMINATOR = "COMPARED WITH WHAT?".

Interest is measured against what you *put in*, discount against what you *take out*.

The last identity is the first one again, on a period of length $1/m$.

The force of interest

$$\delta \;=\; \lim_{m\to\infty} i^{(m)} \;=\; \lim_{m\to\infty} d^{(m)} \;=\; \ln(1+i) \;=\; -\ln(1-d).$$

When the rate moves with time, $\delta(t) = V_t'/V_t = \frac{\mathrm{d}}{\mathrm{d}t}\ln V_t$, and integrating gives

$$\boxed{\mathsf{A}(t_1,t_2) \;=\; \mathrm{e}^{\int_{t_1}^{t_2}\delta(t)\,\mathrm{d}t}, \qquad \mathsf{v}(t) \;=\; \mathrm{e}^{-\int_0^t \delta(s)\,\mathrm{d}s}}$$

KEY IDEA

The accumulation factor is e to the *area* under the force. An instantaneous rate is integrated, never multiplied

A cashflow $\mathcal{C}$ has discrete payments c_t and a continuous stream at rate $\rho(t)$; income counts positive, outgo negative.

$$\mathrm{PV}(\mathcal{C}) = \sum_t c_t \, \mathrm{v}(t) + \int_0^\infty \mathrm{v}(t)\, \rho(t)\, \mathrm{d}t.$$

CAUTION

$\rho(t)$ is a *rate*, in euro per unit time, not an amount: $\rho(2) = 1000$ says nothing about how much changes hands at time 2. Indeed, it is zero.

Values at any date

The value at t_1 of C due at t_2 is

$$C\,\mathrm{e}^{-\int_{t_1}^{t_2}\delta(t)\,\mathrm{d}t} \;=\; C\,\frac{\mathsf{v}(t_2)}{\mathsf{v}(t_1)},$$

whether t_2 is after t_1 (discounting) or before it (accumulation). The same formula captures both, because reversing the limits of an integral reverses its sign.

THE TRANSFER PRINCIPLE

$$\mathsf{Val}_{t_1}(\mathcal{C})\,\mathsf{v}(t_1) \;=\; \mathsf{Val}_{t_2}(\mathcal{C})\,\mathsf{v}(t_2) \;=\; \mathsf{PV}(\mathcal{C}).$$

A cashflow has *one* value, quoted at a date. Value once, then slide the answer along.

SECTION 2

Annuities

The four level-annuity symbols

n payments of 1, in arrears at $1, \dots, n$ or in advance at $0, \dots, n-1$:

$$a_{\overline{n}|} = \frac{1 - \mathrm{v}^n}{i}, \quad s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}, \quad \ddot{a}_{\overline{n}|} = \frac{1 - \mathrm{v}^n}{d}, \quad \ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}.$$

KEY IDEA

Two multiplications relate all four: $\times (1+i)^n$ moves the valuation *date*; $\times (1+i)$ moves the *payments*. The first is the transfer principle again.

All are geometric series.

The edges

$$n = 0 : \text{ all four are } 0, \qquad i = 0 : \text{ all four are } n, \qquad n = \infty : \; a_{\overline{\infty}|} = \frac{1}{i}, \; \ddot{a}_{\overline{\infty}|} = \frac{1}{d}.$$

$$s_{\overline{n+1}|} = 1 + \ddot{s}_{\overline{n}|}, \qquad \ddot{a}_{\overline{n+1}|} = 1 + a_{\overline{n}|}.$$

> CAUTION
>
> At $i = 0$ the closed forms read $0/0$: A deeper look analysing convergence hows it is just the number of payments. And $s_{\overline{\infty}|}$ is not defined at all, as an accumulated value must be accumulated *to* some date.

Both identities stem from the same argument: a payment of 1 landing on the valuation date is worth exactly 1.

Paying more often, and continuously

$$a^{(m)}_{\overline{n}|} = \frac{1-\mathsf{v}^n}{i^{(m)}}, \qquad \ddot{a}^{(m)}_{\overline{n}|} = \frac{1-\mathsf{v}^n}{d^{(m)}}, \qquad \bar{a}_{\overline{n}|} = \int_0^n \mathsf{v}(t)\,\mathrm{d}t \overset{\delta \text{ const.}}{=} \frac{1-\mathsf{v}^n}{\delta}.$$

KEY IDEA

$\frac{i^{(m)}}{d^{(m)}} = (1+i)^{1/m}$, so moving the payments $\frac{1}{m}$ of a period earlier multiplies the value by $(1+i)^{1/m}$, the m-thly form of $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$.

Continuous payment is the limit: $i^{(m)}$ and $d^{(m)}$ both tend to δ, so $a^{(m)}_{\overline{n}|}$ and $\ddot{a}^{(m)}_{\overline{n}|}$ converge onto $\bar{a}_{\overline{n}|}$ from opposite ends.

One numerator, five rates

Payments	Present value	Accumulated value	Rate
Yearly, in arrears	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	i
m-thly, in arrears	$a_{\overline{n}\vert}^{(m)}$	$s_{\overline{n}\vert}^{(m)}$	$i^{(m)}$
Continuously	$\bar{a}_{\overline{n}\vert}$	$\bar{s}_{\overline{n}\vert}$	δ
m-thly, in advance	$\ddot{a}_{\overline{n}\vert}^{(m)}$	$\ddot{s}_{\overline{n}\vert}^{(m)}$	$d^{(m)}$
Yearly, in advance	$\ddot{a}_{\overline{n}\vert}$	$\ddot{s}_{\overline{n}\vert}$	d

$$\text{PV} = \frac{1-\mathrm{v}^n}{\text{rate}}, \qquad \text{AV} = \frac{(1+i)^n-1}{\text{rate}}, \qquad n=\infty: \ \text{PV} = \frac{1}{\text{rate}}.$$

Since $d < d^{(m)} < \delta < i^{(m)} < i$, the values run the other way.

KEY IDEA

$a^{(m)}_{\overline{n}|}$ is defined exactly when $nm \in \mathbb{N}$. The term must hold a whole-number-multiple of the payment intervals. The formula is unchanged; only the definition area was extended.

Deferment by k periods, for *any* of the symbols:

$$ {}_{k|}a_{\overline{n}|} \;=\; a_{\overline{k+n}|} - a_{\overline{k}|} \;=\; \mathsf{v}^{k} a_{\overline{n}|}. $$

Use the last form in practice. And choose the unit of time so that the quoted rate is *effective*: the answer never depends on the choice, but the arithmetic does.

SECTION 3

Projects

Hold the cashflow fixed and vary the rate:

$$\mathrm{NPV}(i) \;=\; \sum_t \mathsf{v}^t c_t + \int_0^T \mathsf{v}^t \rho(t)\,\mathrm{d}t \qquad (i > -1,\ \text{continuous}).$$

DEFINITION 32.1 · Yield

If $\mathrm{NPV}(i) = 0$ has exactly one root $i_0 > -1$, that root is the **yield**, or IRR, the internal rate of return.

A unique root is guaranteed when all the negative cashflows precede all the positive ones.

$$i_0 \approx i_1 + \frac{\mathrm{NPV}(i_1)}{\mathrm{NPV}(i_1) - \mathrm{NPV}(i_2)}\,(i_2 - i_1), \qquad \mathrm{NPV}(i_1)\,\mathrm{NPV}(i_2) < 0.$$

CAUTION

For projects with initial cost and subsequent income stream, NPV is convex. For convex cruves, the chord lies above the curve and interpolation *overstates* the yield.

Halving the width roughly quarters the error, so a good guess with a narrow bracket helps. Otherwise, repeat.

Note, the *quartering the error* is **not** a rule as it depend purely on the curve and any ratio can hold.

Profitability: the project pays if $(i - i_0)\ \mathrm{NPV}'(i_0) > 0$. For the usual decreasing NPV, that is just $i < i_0$.

If the unknown (e.g., i) enters only through some power of v, use substitution and solve for an appropirate power of v. A messy equation can become a polynomial in this way.

KEY IDEA

Attach probabilities, or discount harder. With $p(t) = \mathrm{e}^{-\mu t}$ the two agree:

$$\delta^* \;=\; \delta + \mu.$$

Forces add, as they have since Lecture 2; risk is one more term in the exponent.

In general, one can not always do this.

A negative yield is a legitimate root, provided $i_0 > -1$.

$$\big(A(T) \;=\; (1+i)^T\ \mathrm{NPV}(i) \implies \text{profitable}\big) \quad\Longleftrightarrow\quad (\mathrm{NPV}(i) > 0)\,.$$

Escalation of all cashflows at rate j is discounting at the reduced rate $i_0 = \frac{i-j}{1+j}$.

CAUTION

Compare projects by NPV at the investor's own rate, *not* by yield: the curves may cross at some i^*, and below it the project with the *lower* yield is the more profitable. The yield belongs to the project; the ranking belongs to the investor.

Two rates, and the payback date

If borrowing costs j_1 and lending earns $j_2 < j_1$, then NPV and yield lose their meaning. In that case, roll the balance forward year by year at whichever rate applies.

DEFINITION 32.2 · Discounted payback period

The first t_1 with $A(t_1) \geq 0$, computed at j_1 alone.

$$A(T) \;=\; (1+j_2)^{T-t_1}A(t_1) + \sum_{t \geq t_1}(1+j_2)^{T-t}c_t + \int_{t_1}^{T}(1+j_2)^{T-t}\rho(t)\,\mathrm{d}t.$$

The date is set by the borrowing rate; everything after it, by the lending rate.

THE ONE MANOEUVRE

In the special, but important, case $A(t) = -C(1+j_1)^t + R\,s_{\overline{t}|}$, we have

$$R\,\bar{s}_{\overline{t}|} \geq C(1+j_1)^t \iff \bar{a}_{\overline{t}|} \geq \frac{C}{R}, \qquad R\,s_{\overline{t}|} \geq C(1+j_1)^t \iff a_{\overline{t}|} \geq \frac{C}{R}.$$

Divide by $(1+j_1)^t$; the accumulated annuity becomes a present value and t sits in one place only.

Accumulate any discrete flows to a common date first, then restart the clock. Ignore a blip, and ignore anything falling after the crossing. When t_1 must be an integer the balance overshoots, so $A(t_1) > 0$ counts towards the profit.

SECTION 4

Loans

What is owed

With instalments X_t and balance L_t immediately after the payment at t:

$$L_t \;=\; \underbrace{\sum_{k=t+1}^{n} X_k \mathsf{v}^{k-t}}_{\text{prospective}} \;=\; \underbrace{L_0(1+i)^t - \sum_{k=1}^{t} X_k (1+i)^{t-k}}_{\text{retrospective}}.$$

KEY IDEA

The two are the same sum split at t, which is why they agree. Prospective is shorter near the end and for level instalments ($L_t = X a_{\overline{n-t}|}$); retrospective near the start, and it is the only option if the future instalments are unknown.

Valid *at a repayment date*.

What a repayment is made of

$$X_t \;=\; b_t + f_t, \qquad b_t \;=\; iL_{t-1}, \qquad f_t \;=\; X_t - iL_{t-1} \;=\; L_{t-1} - L_t.$$

Interest is paid first, on the balance outstanding; the rest repays capital. With level instalments $f_t = X\mathsf{v}^{n-t+1}$, so $f_t = f_1(1+i)^{t-1}$: the capital content grows at exactly the rate of interest.

KEY IDEA

Over a block, the capital repayments telescope: $\sum_k f_k = L_t - L_{t+r}$, and hence $\sum_k b_k = \sum_k X_k - (L_t - L_{t+r})$. *Interest paid = money handed over − reduction in the debt.*

Lecture 13

Schedules in general

Every level-annuity schedule is one table rescaled by $L_0/a_{\overline{n}|}$ — so anything about *proportions* depends on n and i alone, not on the size of the loan.

- **Varying rates**: a single factor $(1+i)$ becomes a product of the $(1+i_r)$ actually traversed. Consistency again.
- **m-thly repayments**: the interest due in one subperiod is $\frac{i^{(m)}}{m}L_t$, **not** $\frac{i}{m}L_t$.

Choose the unit of time so that the given rate is effective, and the m-thly schedule is the annual one with the clock relabelled. Lecture 14

SECTION 5

Securities

The price of a bond

Coupon rate D per annum payable m-thly, redemption R at time T, all per unit nominal:

$$P_0 \;=\; D\, a^{(m)}_{\overline{T}|} + R\mathsf{v}^T, \qquad P_0 \;=\; (1-\tau_1) D\, a^{(m)}_{\overline{T}|} + R\mathsf{v}^T$$

gross and with income tax at source; and with the tax paid k years in arrears,

$$P_0 \;=\; D\, a^{(m)}_{\overline{T}|} + R\mathsf{v}^T - \tau_1 D\, \mathsf{v}^k a_{\overline{T}|}.$$

Given i this computes a price; given a price it is an equation of value for the yield.
Everything is per unit nominal, and everything is *net*. Lecture 15–16

Capital gains tax

TEST FIRST, THEN PRICE

$$i^{(m)} \;>\; (1-\tau_1)\frac{D}{R} \quad\Longleftrightarrow\quad \text{there is a capital gain,}$$

the right-hand side being the *net running yield*. If the price is given, compare P_0 with R directly.

If there is a gain, the price appears on both sides of its own equation; gather and divide:

$$P_0 \;=\; \frac{(1-\tau_1)D\,a^{(m)}_{\overline{T}|} + (1-\tau_2)R\mathrm{v}^T}{1-\tau_2\mathrm{v}^T}.$$

If there is no gain, no τ_2 is due and the earlier formula stands.

Lecture 16

When the borrower chooses the date

With T free in $[T_1, T_2]$, price for the worst case:

Test	At redemption	Assume
$i_0^{(m)} > (1-\tau_1)\frac{D}{R}$	capital gain	latest date, $T = T_2$
$i_0^{(m)} < (1-\tau_1)\frac{D}{R}$	capital loss	earliest date, $T = T_1$
$i_0^{(m)} = (1-\tau_1)\frac{D}{R}$	neither	any date; $P_0 = R$

KEY IDEA

A gain should arrive early and a loss late, so the worst case is a *late gain* or an *early loss*. Capital gains tax changes none of this: run the test, choose the date, and only then apply τ_2.

Securities with no redemption date

Equity: $P = \dfrac{D}{j} = D\,\dfrac{1+g}{i-g}, \qquad j = \dfrac{i-g}{1+g}, \qquad$ valid for $-1 < g < i.$

Property: $P = 12X\, a^{(12)}_{\overline{n}|}\, \dfrac{(1+i)^n}{(1+i)^n-(1+g)}, \qquad$ valid for $(1+i)^n > 1+g.$

KEY IDEA

Both are growing perpetuities: growth and discounting combine into one net rate. If that rate is not positive the model returns infinity — which tells you the *assumption* is wrong, not the arithmetic.

Lecture 19

Inflation and real yields

$$\sum_k c_{t_k} \frac{Q(0)}{Q(t_k)} \mathsf{v}_{i_r}^{t_k} \;=\; 0, \qquad 1+i \;=\; (1+j)(1+i_r), \qquad i_r \;=\; \frac{i-j}{1+j}.$$

CAUTION

$i_r \neq i - j$: subtracting inflation overstates the real return. And the base date cancels, so the answer does not depend on it.

Indexing to the same index used for deflating makes the two cancel, so the nominal amounts *are* the real ones — but a time lag breaks that, protecting the payment against *base-period* inflation rather than current.

Lecture 20

SECTION 6

Curves and risk

Three languages for one curve

$$P_n \;=\; \frac{1}{(1+y_n)^n}, \qquad (1+f_{t,r})^r \;=\; \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} \;=\; \frac{P_t}{P_{t+r}},$$

$$(1+y_t)^t \;=\; \prod_{k=0}^{t-1}(1+f_k).$$

KEY IDEA

Each of the three determines the other two, and the last identity makes $1+y_t$ the *geometric mean* of the one-year growth factors. Every coupon bond is a portfolio of zero-coupon bonds, priced strand by strand — by no arbitrage.

Continuously: $F_t = -\frac{\mathrm{d}}{\mathrm{d}t}\ln P_t$, so $P_t = \mathrm{e}^{-\int_0^t F_s\,\mathrm{d}s}$. Lecture 25, 28

Summarising a curve in one number

- **Redemption yield**: the single i with $P = Da_{\overline{n}|} + Rv^n$. It belongs to the *bond* and moves with the coupon.
- **Par yield**: the coupon making the bond price at par,

$$yc_n = \frac{1 - v_n^n}{v_1 + v_2^2 + \cdots + v_n^n}.$$

  It belongs to the *market*. Its gap from y_n is the coupon bias.

Both are averages of the curve, weighted by where the money is: on a rising curve the lower-coupon bond has the higher redemption yield. Lecture 26

The shape of the curve

A yield curve may slope down, slope up, or hump. Three explanations:

- **Expectations** — a falling curve suggests rates are expected to fall.
- **Liquidity preference** — long bonds are more rate-sensitive, so risk-averse investors demand more yield.
- **Market segmentation** — banks hold short, pension funds long, and supply meets demand term by term.

> KEY IDEA
>
> Price and yield move in opposite directions, so a dip in the curve marks a term investors *want*.

Lecture 27

The three measures

$$\nu(i) \;=\; -\frac{\mathsf{NPV}'(i)}{\mathsf{NPV}(i)}, \qquad \tau(i) \;=\; (1+i)\nu(i), \qquad \mathsf{c}(i) \;=\; \frac{\mathsf{NPV}''(i)}{\mathsf{NPV}(i)}.$$

KEY IDEA

τ is the *centre of mass* of the discounted cashflows: n for a zero-coupon bond, and $\frac{D(Ia)_{\overline{n}|}+nR\mathsf{v}^n}{Da_{\overline{n}|}+R\mathsf{v}^n}$ for a coupon bond. It falls as i rises, and a smaller coupon means a larger duration.

Convexity is positive when all the cashflows are, and grows roughly with the *square* of the term while duration grows linearly. Lecture 27–29

The logarithmic derivative, four times

Quantity	Definition	Lecture
Force of interest $\delta(t)$	$\frac{\mathrm{d}}{\mathrm{d}t} \ln V_t$	2
Instantaneous forward rate F_t	$-\frac{\mathrm{d}}{\mathrm{d}t} \ln P_t$	28
Effective duration $\nu(i)$	$-\frac{\mathrm{d}}{\mathrm{d}i} \ln \mathrm{NPV}(i)$	27
Discounted mean term $\tau(i)$	$-\frac{\mathrm{d}}{\mathrm{d}\delta} \ln \mathrm{NPV}(i)$	28

KEY IDEA

Each time the reason is the same: a slope alone is not a rate, because it grows with the size of the fund. Dividing by the level removes the size and leaves the rate.

Lecture 2, 27–28

$$\frac{\mathrm{NPV}(i+\varepsilon)-\mathrm{NPV}(i)}{\mathrm{NPV}(i)} \;\approx\; -\varepsilon\,\nu(i)+\frac{\varepsilon^2}{2}\,\mathsf{c}(i).$$

REDINGTON, AT i_0

$$V_A \;=\; V_L, \qquad \nu_A \;=\; \nu_L \;\;(\text{equivalently } \tau_A \;=\; \tau_L), \qquad \mathsf{c}_A > \mathsf{c}_L\,.$$

The three terms of the expansion of the surplus $S = V_A - V_L$: the first is zero, the second changes sign with ε so must be killed, the third has a fixed sign so must be made positive.

Lecture 29

Meeting the conditions, and what they buy

For two zero-coupon bonds of terms $a < b$,

$$w_a + w_b \;=\; 1, \qquad w_a a + w_b b \;=\; \tau_L \qquad \Longrightarrow \qquad w_a \;=\; \frac{b - \tau_L}{b - a},$$

so the assets must *straddle* the liabilities.

$$\frac{\Delta V_A}{V_A} - \frac{\Delta V_L}{V_L} \;\approx\; \underbrace{-\varepsilon(\nu_A - \nu_L)}_{\text{zero under cond. 2}} + \underbrace{\frac{\varepsilon^2}{2}(\mathsf{c}_A - \mathsf{c}_L)}_{\text{positive under cond. 3}} .$$

Match values and durations and the surplus is protected against the *direction* of a move; the convexity gap then decides its size, whichever way rates go. Lecture 30–31

The caveats worth remembering

CAUTION

- Immunisation is *local*: it protects against small movements only.
- Matching values and durations but *not* convexity makes i_0 a local *maximum* of the surplus — worse in both directions.
- An unmatched fund is a bet on the direction of rates. It may profit; it may equally lose.
- Immunisation is insurance, and is paid for by surrendering the upside.

Lecture 29–31

SECTION 7

The whole thing on four slides

One: moving money

$$\mathsf{A}(t_0,t_2) \;=\; \mathsf{A}(t_0,t_1)\mathsf{A}(t_1,t_2), \qquad \mathsf{v}(t) \;=\; \frac{1}{\mathsf{A}(t)} \;=\; \mathrm{e}^{-\int_0^t \delta(s)\,\mathrm{d}s},$$

$$\mathsf{Val}_{t_1}(\mathcal{C})\,\mathsf{v}(t_1) \;=\; \mathsf{PV}(\mathcal{C}).$$

$$1+i \;=\; \left(1+\tfrac{i^{(m)}}{m}\right)^m \;=\; (1-d)^{-1} \;=\; \left(1-\tfrac{d^{(m)}}{m}\right)^{-m} \;=\; \mathsf{v}^{-1} \;=\; \mathrm{e}^{\delta},$$

$$d \;<\; d^{(m)} \;<\; \delta \;<\; i^{(m)} \;<\; i.$$

Consistency says the route does not matter. The transfer principle says the date does not either — only which date you *quote*.

Two: many payments

$$\text{PV} \;=\; \frac{1-\mathsf{v}^{n}}{\text{rate}}, \qquad \text{AV} \;=\; \frac{(1+i)^{n}-1}{\text{rate}},$$

$$n=\infty: \;\; \text{PV} \;=\; \frac{1}{\text{rate}}, \qquad \text{deferred by } k: \;\; \times \mathsf{v}^{k}.$$

$$\text{rate} \in \Big\{ \underbrace{d}_{\text{yearly, advance}} \;,\; \underbrace{d^{(m)}}_{m\text{-thly, advance}} \;,\; \underbrace{\delta}_{\text{continuous}} \;,\; \underbrace{i^{(m)}}_{m\text{-thly, arrears}} \;,\; \underbrace{i}_{\text{yearly, arrears}} \Big\}$$

One numerator, and a denominator chosen by *when* the money moves. Every symbol in the module is that sentence.

Three: is it worth it?

$$\mathrm{NPV}(i) \;=\; \sum_t \mathsf{v}^t c_t + \int_0^T \mathsf{v}^t \rho(t)\,\mathrm{d}t, \qquad \mathrm{NPV}(i_0) \;=\; 0, \qquad A(T) \;=\; (1+i)^T\,\mathrm{NPV}(i).$$

- One project: profitable iff $(i - i_0)\,\mathrm{NPV}'(i_0) > 0$.
- Two projects: rank by NPV at the investor's rate, not by yield.
- Two rates: roll the balance forward; the payback date is set by j_1.
- A bond, an equity, a property: the same NPV, with the cashflow named.

The rate is either given, and you compute a value; or the value is given — often zero — and you find the rate.

Four: what a move costs

$$\frac{\Delta \,\mathrm{NPV}}{\mathrm{NPV}} \;\approx\; -\varepsilon\,\nu + \frac{\varepsilon^2}{2}\,\mathsf{c},$$

$$\nu \;=\; -\frac{\mathrm{NPV}'}{\mathrm{NPV}}, \qquad \tau \;=\; (1+i)\nu, \qquad \mathsf{c} \;=\; \frac{\mathrm{NPV}''}{\mathrm{NPV}}.$$

KEY IDEA

A single i was always a fiction: markets quote y_t for every term. Duration says how far the money is away, convexity how spread out it is — and Redington says match the first two and keep more of the third.

That is the module. Everything else was arithmetic.

END OF LECTURE 32

Consistency, the transfer principle, one numerator over a chosen rate, and the logarithmic derivative. Four ideas, thirty-one lectures.

ilia.pirashvili@universityofgalway.ac.ie