MA3991 · 2026–27

Cashflow Models

A revision guide

The results, and how to use them.
Not the road that led to them.

Closed book. Calculation heavy. Precision matters.

How to use this guide

This is not the lecture course shortened. It is what you need to *do*, with the reasons trimmed to what helps you remember. The exam is closed book and you should assume that **no formulae are given**, so every boxed result in here is something to know. Most proofs are not examinable. What is examinable is whether you can set a problem up, calculate it, and notice when the answer cannot be right. A few short derivations are asked (going from one annuity formula to another, say), and these are marked.

Every chapter has the same shape.

> RESULT
>
> A formula or statement to know. If it is boxed like this, it is in the exam.

> HOW TO
>
> 1. Steps for attacking a type of question, in the order to do them.

> WATCH OUT
>
> The mistake that is made every year.

> SANITY CHECK
>
> A quick way to see that an answer is wrong. Use it before you move on.

> EXAMPLE · Worked example
>
> A question…
>
> SOLUTION
>
> …and its answer, with the working that earns the marks.

Each chapter opens with what you should be able to do afterwards, and closes with the whole chapter in one box and a few questions with answers. Do the questions on paper first. Numbers are given to six decimals where rounding matters; in the exam, carry *more* digits than you think you need.

> CONVENTIONS
>
> Time is measured in years from a stated origin. 5% means 0.05. Money is in euro, written €1,234.50. Where a number is written 1,000, the comma is only a thousands separator.

Contents

Notation at a glance . . . . . **v**

I Rates and values **1**

1 Interest and discount **2**
1.1 Accumulating and discounting 2
1.2 Consistency 3
1.3 Effective rates, and interest versus discount 4
1.4 Nominal rates 5

2 The force of interest **9**
2.1 The limit of the nominal rates 9
2.2 A force that varies with time 10

3 Cashflows and net present value **14**
3.1 Payments at many dates 14
3.2 Payments made continuously 15
3.3 Money in, money out 16

4 Value at any date **19**
4.1 The value of a payment at any date 19
4.2 The value of a cashflow at any date 20

II Annuities **23**

5 Level annuities **24**
5.1 Vocabulary 24
5.2 The sum behind every annuity 25
5.3 The four symbols 25

6 Annuities at work **28**
6.1 Four symbols, two multiplications 28
6.2 Two edge cases 29
6.3 One payment more 29
6.4 Payments that never stop 30
6.5 A loan repaid 31

7 Payable more often than once a period **33**
7.1 Payments made continuously 33
7.2 Payments made m times a period 34
7.3 The whole family on one page 36
7.4 Choosing the unit of time 37

8 Deferred and varying annuities **39**
8.1 Waiting before the payments begin 39
8.2 Payments that grow by one each period 40
8.3 Decreasing payments, and everything between 42
8.4 Deferring a varying annuity 43

8.5 Growth in continuous time . . . 43

III Yield and uncertainty **46**

9 The yield of a project . . . **47**
9.1 The value as a function of the rate . . . 47
9.2 The equation of value and the yield . . . 48
9.3 Finding the yield . . . 49

10 Cashflows that may not arrive . . . **53**
10.1 Allowing for uncertainty . . . 53
10.2 A project with uncertain receipts . . . 55

IV Loans **57**

11 Inside a loan repayment . . . **58**
11.1 What a repayment is doing . . . 58
11.2 The loan outstanding . . . 60
11.3 Interest and capital in closed form . . . 61
11.4 A block of instalments . . . 62
11.5 When does capital overtake interest? . . . 63

12 Loan schedules in general . . . **65**
12.1 When the rate is not constant . . . 65
12.2 The schedule . . . 66
12.3 More frequent repayments . . . 67
12.4 Fractional terms . . . 68

V Securities **70**

13 Bonds and their prices . . . **71**
13.1 Securities and fixed-interest bonds . . . 71
13.2 The price is a present value . . . 73
13.3 Income tax . . . 73
13.4 A worked example, and a method . . . 74

14 Tax on income, tax on gains . . . **77**
14.1 Income tax paid late . . . 77
14.2 Capital gains tax: will there be a gain? . . . 78
14.3 Pricing with capital gains tax . . . 79

15 Bonds redeemable at the borrower's option . . . **82**
15.1 Finding the yield when capital gains tax applies . . . 82
15.2 Securities with no fixed redemption date . . . 83
15.3 Which redemption date is the worst case? . . . 84
15.4 The worst case in practice . . . 86
15.5 A past exam question . . . 88

16 Equities and property . . . **91**
16.1 Equities . . . 91
16.2 The dividend growth model . . . 92
16.3 Bought ex dividend . . . 93
16.4 Property . . . 94

17 Inflation and real yields . . . **96**
17.1 Real and money yields . . . 96
17.2 Constant inflation . . . 97
17.3 Payments linked to inflation . . . 98

17.4 Index-linked bonds and the time lag . . . 98

VI Project appraisal **102**

18 Project appraisal . . . **103**
18.1 Escalating cashflows . . . 103
18.2 Accumulated profit . . . 105
18.3 Present value or accumulated profit? . . . 106
18.4 Profitability and the internal rate of return . . . 106

19 Comparing projects . . . **109**
19.1 Comparing two projects . . . 109
19.2 Two ways to build a system . . . 110
19.3 Two loans . . . 112

20 Borrowing dearer than lending . . . **114**
20.1 Two rates . . . 115
20.2 Three projects . . . 115
20.3 The discounted payback period . . . 117
20.4 Finding the payback date . . . 118
20.5 A level annual annuity: the integer case . . . 120

VII The term structure **123**

21 Spot rates and forward rates . . . **124**
21.1 Zero-coupon bonds and spot rates . . . 124
21.2 Forward rates . . . 126
21.3 One-year forward rates, and the spot rate as an average . . . 127

22 One number for a whole curve . . . **129**
22.1 The redemption yield . . . 129
22.2 Par yields . . . 131

23 Yield curves and interest rate risk . . . **134**
23.1 The shape of the curve . . . 134
23.2 Interest rate risk . . . 135
23.3 Effective duration . . . 136

VIII Immunisation **139**

24 The discounted mean term . . . **140**
24.1 The instantaneous forward rate . . . 140
24.2 The discounted mean term . . . 141
24.3 Basic durations . . . 142

25 Convexity and Redington's conditions . . . **145**
25.1 Convexity . . . 145
25.2 Approximate proportional change . . . 146
25.3 Immunisation . . . 147

26 Building an immunised portfolio . . . **150**
26.1 Two zero-coupon bonds . . . 150
26.2 A worked immunisation . . . 151
26.3 What the match buys . . . 153
26.4 A fund that is not immunised . . . 154

Notation at a glance

Rates and growth

$\mathsf{A}(t_1, t_2)$, $\mathsf{A}(t)$	accumulation factor from t_1 to t_2; from 0 to t	1
v, $\mathsf{v}(t)$	$\mathsf{v} = \frac{1}{1+i}$; discount factor $\frac{1}{\mathsf{A}(t)}$	1
i, d	effective rate of interest; of discount, $d = \frac{i}{1+i}$	1
$i^{(m)}$, $d^{(m)}$	nominal rates convertible m-thly	1
δ, $\delta(t)$	force of interest, $\mathsf{e}^{\delta} = 1 + i$; as a function of time	2

Values

PV, NPV(i)	present value; net present value (in minus out)	3
$\mathrm{Val}_t(\mathcal{C})$	value of the cashflow $\mathcal{C}$ at time t; $\mathrm{Val}_t\, \mathsf{v}(t) = \mathrm{PV}$	4
$A(T)$, NPV$'(i_0)$	accumulated profit $= (1+i)^T$ NPV; slope of NPV in i	18

Annuities

$a_{\overline{n}\vert}$, $s_{\overline{n}\vert}$	level, in arrears: $\frac{1-\mathsf{v}^n}{i}$, $\frac{(1+i)^n-1}{i}$	5
$\ddot{a}_{\overline{n}\vert}$, $\ddot{s}_{\overline{n}\vert}$	in advance: same numerators over d	5
$\bar{a}_{\overline{n}\vert}$, $a^{(m)}_{\overline{n}\vert}$	continuous (δ); m-thly ($i^{(m)}$); advance uses $d^{(m)}$	7
${}_{k\vert}a_{\overline{n}\vert}$	deferred k periods: $\mathsf{v}^k a_{\overline{n}\vert}$	8
$(Ia)_{\overline{n}\vert}$, $(Da)_{\overline{n}\vert}$	payments $1, \ldots, n$; $n, \ldots, 1$	8

Yield, loans, securities

i_0	yield (IRR): the root of NPV$(i) = 0$	9
$p(t)$, μ, δ^*	probability of payment $\mathsf{e}^{-\mu t}$; $\delta^* = \delta + \mu$	10
L_t, X_t, b_t, f_t	loan outstanding after payment t; instalment; interest; capital	11
D, R, τ_1, τ_2	coupon rate; redemption price; income tax; capital gains tax	13–14
g, j	dividend growth; net rate $j = \frac{i-g}{1+g}$	16
$Q(t)$, i_r	index; real yield $i_r = \frac{i-j}{1+j}$	17
i_{e}, j_1, j_2	rate adjusted for escalation; borrowing and lending rates	18, 20

Term structure, risk and immunisation

P_n, y_n	zero-coupon price; spot rate, $P_n = (1+y_n)^{-n}$	21
$f_{t,r}$, F_t	forward rate; instantaneous forward rate $-\frac{\mathrm{d}}{\mathrm{d}t}\ln P_t$	21, 24
yc_n	n-year par yield	22
$\nu(i)$	effective duration (volatility) $-$ NPV$'$ / NPV	23
$\tau(i)$	discounted mean term $(1+i)\nu$	24
$\mathsf{c}(i)$	convexity NPV$''$ / NPV	25
Redington: $V_A = V_L$, $\nu_A = \nu_L$, $\mathsf{c}_A > \mathsf{c}_L$		25–26

Last column: the chapter where the symbol is introduced.

PART I

Rates and values

CHAPTER 1

Interest and discount

LECTURE 1 · *What a rate of interest actually says*

AFTER THIS CHAPTER YOU CAN

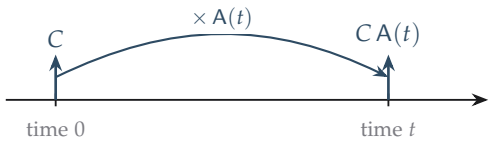
- ☐ Accumulate and discount a sum, under simple and under compound interest.
- ☐ Move between i, d, $i^{(m)}$, $d^{(m)}$ and v *in any direction*, without memorising a table.
- ☐ Say what a quoted rate means: effective or nominal, interest or discount.
- ☐ Test whether an accumulation function is consistent, and know why it must be.

1.1 Accumulating and discounting

A rate of interest is a *price*: the price of having money now rather than later. It depends on default risk, on inflation, and on term and liquidity. For this course, the rates are given. What matters is how to use them, and for that we make four standing assumptions: no expenses, charges or tax; payments are certain, in full and on time; rates are known in advance; and everything is proportional to the size of the money (**scalability**).

ACCUMULATION AND DISCOUNT FACTORS

$\mathsf{A}(t_1, t_2)$ is what 1 invested at time t_1 has become at time t_2. Write $\mathsf{A}(t) := \mathsf{A}(0, t)$. By scalability, C at t_1 becomes $C\,\mathsf{A}(t_1, t_2)$ at t_2.
The **discount factor** $\mathsf{v}(t) = 1/\mathsf{A}(t)$ is the present value of 1 due at time t: the cost of creating exactly that payment.



Discounting is the same picture read backwards: divide by $\mathsf{A}(t)$.

SIMPLE AND COMPOUND INTEREST

	$\mathsf{A}(t)$	$\mathsf{v}(t)$
Simple, rate r	$1+rt$	$\dfrac{1}{1+rt}$
Compound, rate i	$(1+i)^t$	$(1+i)^{-t}$

In the compound case $\mathsf{v} := \mathsf{v}(1) = \dfrac{1}{1+i}$, which appears constantly. Simple interest earns nothing on interest and only appears for short-dated money-market instruments ($t < 1$). **Unless told otherwise, "interest" means compound.**

EXAMPLE 1.1 · Compound, simple, and discounting

€1,000 for five years at 6%: simple interest or compound? And how much must you invest today to have €10,000 in four years, at 5%?

SOLUTION

Simple: $1000(1 + 0.06 \times 5) = €1{,}300$. Compound: $1000 \times 1.06^5 = €1{,}338.23$; the extra €38.23 is interest earned on interest.
Discounting means dividing by 1.05^4: $10000/1.05^4 = €8{,}227.02$, the **present value**. "Discounted at 5%" just means "divide by 1.05^t".

A RATE OF DISCOUNT IS NOT A RATE OF INTEREST

Short-term instruments are often quoted the other way round. A **rate of discount** d is deducted from the *final* amount, so a bill of face value F maturing at time t costs $F(1 - dt)$.
A €5,000 bill due in three months at a simple discount rate of 4% p.a. costs $5000(1 - 0.04 \times \frac{1}{4}) = €4{,}950$. You paid 4950 and received 5000: a return of $50/4950 = 1.0101\%$ for the quarter, not 1%. **Interest is measured against what you put in, discount against what you take out.**

1.2 Consistency

Invest 1 at t_0 and take it out at t_2. You may leave it alone, or withdraw at some t_1 in between and reinvest at once. In a sensible market both routes pay the same.

PRINCIPLE OF CONSISTENCY

For $t_0 \le t_1 \le t_2$,

$$\mathsf{A}(t_0, t_2) \;=\; \mathsf{A}(t_0, t_1)\,\mathsf{A}(t_1, t_2), \qquad \text{and so} \qquad \mathsf{A}(t_1, t_n) \;=\; \mathsf{A}(t_1, t_2)\mathsf{A}(t_2, t_3)\cdots\mathsf{A}(t_{n-1}, t_n).$$

Compound interest is consistent (it is the law of exponents). Simple interest is not.

Why must a market be consistent? Because otherwise there is an arbitrage: a guaranteed profit with no capital and no risk ("no free lunch").

EXAMPLE 1.2 · Consistency at work

(a) A fund returns 5% in year one and 4% in year two. What happens to €1,000? (b) A bank offers 5% simple interest to anyone, whenever they invest. Is that consistent?

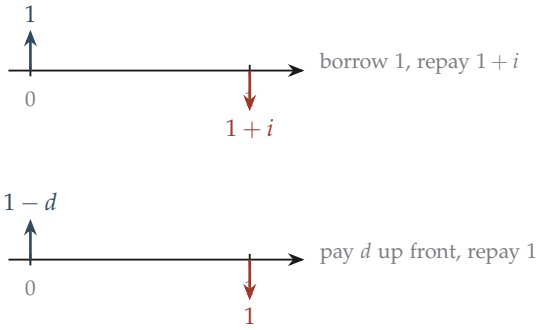
SOLUTION

(a) $\mathsf{A}(2) = 1.05 \times 1.04 = 1.092$, so €1,000 becomes €1,092: growth of 9.2%, *not* 9%. The extra 0.2% is year two's 4% earned on year one's interest.
(b) Leave 1 for two years: $1 + 0.05 \times 2 = 1.10$. Withdraw at time 1 and reinvest on the same terms: $1.05 \times 1.05 = 1.1025$. The routes disagree, so this A is not consistent: borrow on the two-year deal, lend a year at a time, keep 0.0025 per unit borrowed. For $t < 1$ the mismatch is tiny, which is why simple interest survives in real life.

1.3 Effective rates, and interest versus discount

EFFECTIVE RATE

An **effective rate** is the actual rate of change over one complete unit of time. It is a rate of *interest* if measured from the beginning to the end of the period, and a rate of *discount* if measured from the end back to the beginning.



In terms of the accumulation function, the effective rates over period n, that is over $[n-1, n]$, are

$$i_n = \frac{\mathsf{A}(n) - \mathsf{A}(n-1)}{\mathsf{A}(n-1)}, \qquad d_n = \frac{\mathsf{A}(n) - \mathsf{A}(n-1)}{\mathsf{A}(n)}.$$

BEHIND THE JARGON

Numerator = interest. Denominator = "compared with what?". That is the whole difference between i and d: the same interest, measured against the opening balance or against the closing balance.

Under *compound* interest $\mathsf{A}(n) = (1+i)^n$ and $i_n = i$ for every n: the effective rate is the same every period and equals the compound rate. Under *simple* interest it drifts: with $\mathsf{A}(t) = 1 + 0.05t$ we get $i_1 = 5\%$, $i_2 = 4.76\%$, $i_3 = 4.54\%$. The interest stays fixed while the denominator grows.

INTEREST AND DISCOUNT, ONE PERIOD

Since the present value of 1 due at the end of the period is $v = 1 - d$,

$$d = 1 - v = iv = \frac{i}{1+i}, \qquad \boxed{i = \frac{d}{1-d}}$$

Henceforth i means the *effective rate of interest per unit time, compound*.

EXAMPLE 1.3

$i = 8\%$ gives $d = 0.08/1.08 = 7.4074\%$. And $d = 5\%$ gives $i = 0.05/0.95 = 5.2632\%$. An account grows by 4% in the first year and 5% in the second. Then $A(2) = 1.04 \times 1.05 = 1.092$ and $i_1 = 4\%$, $i_2 = (1.092 - 1.04)/1.04 = 5\%$, $d_2 = (1.092 - 1.04)/1.092 = 4.7619\%$. The effective rate reads the growth of *that* period, whatever happened before.

IS $d = i/(1+i)$ BELIEVABLE?

Always do sanity checks, as you would check units in physics. For $i > 0$:

- $d < i$: money received *early* is worth more, so a smaller number does the same job.
- $d < 1$ however large i is: you cannot deduct more than the whole value in advance.
- $d \approx i - i^2$ for small i only: $i = 2\%$ gives $d = 1.9608\%$; but $i = 50\%$ gives $d = 33.3\%$, not 25%.

1.4 Nominal rates

NOMINAL RATE

The **nominal rate of interest payable m times per period**, written $i^{(m)}$, means a rate of $i^{(m)}/m$ applied in each m-th of a period. The nominal rate of discount $d^{(m)}$ is defined the same way: $d^{(m)}/m$ applied in each m-th of a period. Also said to be **convertible** or **compounded** m-thly.

A NOMINAL RATE IS A QUOTATION, NOT A RATE ANYONE EARNS

A nominal 12% p.a. convertible monthly means 1% per month, nothing more. After a year, $100(1.01)^{12} = €112.68$, so what you actually earned is the *effective* 12.68%. The 12% is just twelve copies of the monthly rate. Because interest compounds within the year, $i^{(m)} < i$ for $m > 1$.

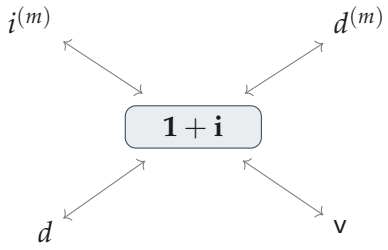
NOMINAL RATES THROUGH THE EFFECTIVE RATE

$$1+i \;=\; \left(1+\frac{i^{(m)}}{m}\right)^{m}, \qquad 1-d \;=\; \left(1-\frac{d^{(m)}}{m}\right)^{m}, \qquad \left(1+\frac{i^{(m)}}{m}\right)\left(1-\frac{d^{(m)}}{m}\right) \;=\; 1.$$

Equivalently $i^{(m)} = m\left[(1+i)^{1/m}-1\right]$ and $d^{(m)} = m\left[1-(1-d)^{1/m}\right]$, and the last identity gives the direct link $\dfrac{d^{(m)}}{m} = \dfrac{i^{(m)}}{m+i^{(m)}}$. Put $m=1$ and you recover $d = i/(1+i)$: there is only ever one relationship.

The method: one bridge, $1+i$

You never need a table of conversions. Convert *to* $1+i$, then convert *from* $1+i$.



$$1+i = \left(1+\frac{i^{(m)}}{m}\right)^{m}$$
$$1+i = \left(1-\frac{d^{(m)}}{m}\right)^{-m}$$
$$1+i = (1-d)^{-1}$$
$$1+i = \mathsf{v}^{-1}$$

CONVERTING BETWEEN QUOTATIONS

1. Write down the rate that is actually *applied*: $i^{(m)}/m$ per m-th of a period (never the quoted number itself).
2. **Match periods.** Ask how many of the small periods fit into the one you want, and compound that many times. Three months make a quarter, so three monthly steps make one quarterly step. This is shorter than going through i, and avoids rounding.
3. Check the ordering at the end (see below).

EXAMPLE 1.4 · Four conversions

(a) A nominal 3% p.a. convertible monthly: find the nominal rate convertible quarterly. (b) A nominal 8% p.a. convertible quarterly: find the effective annual rate, and the equivalent nominal rate convertible monthly. (c) The nominal discount rate convertible every other month is 1.2% p.a.: find the effective annual rate of discount. (d) The nominal rate of interest convertible monthly is 3% p.a.: find the effective annual rate of discount.

SOLUTION

(a) Three monthly steps make one quarterly step:

$$1+\frac{i^{(4)}}{4} \;=\; \left(1+\frac{0.03}{12}\right)^{3} \;=\; 1.0025^{3}, \qquad i^{(4)} \;=\; 4\left(1.0025^{3}-1\right) \;=\; 0.030075.$$

Through i instead: $1+i = 1.0025^{12} = 1.030416$ and $i^{(4)} = 4[1.030416^{1/4} - 1] = 0.030075$. Same answer; the first route never computes i.
(b) The quarterly rate is 2%, so $1+i = 1.02^4 = 1.082432$ and $i = 8.2432\%$. One month is a third of a quarter, so $1 + \frac{i^{(12)}}{12} = 1.02^{1/3} = 1.006623$ and $i^{(12)} = 12 \times 0.006623 = 7.9472\%$. Sanity check: $i^{(12)} = 7.9472\% < i^{(4)} = 8\% < i = 8.2432\%$.
(c) Here $m = 6$: $d = 1 - (1 - 0.012/6)^6 = 1 - 0.998^6 = 0.01194$.
(d) From (a), $i = 0.030416$, so $d = i/(1+i) = 0.030416/1.030416 = 0.029518$.

RATE NAMES IN THE WILD

A card charging a nominal 18% convertible monthly is charging 1.5% per month, so you actually pay $1.015^{12} - 1 = 19.56\%$. **Nominal** is the number *before* compounding, which lenders like to advertise. **AER**, the effective annual rate, is the number *after*, which savings accounts like to advertise.

WHERE THE QUOTATIONS SIT

For $i > 0$ and $n < m$:

$$d \;<\; d^{(n)} \;<\; d^{(m)} \;<\; i^{(m)} \;<\; i^{(n)} \;<\; i,$$

with $d^{(m)}$ increasing and $i^{(m)}$ decreasing in m. They all describe the same balance change, so nominal rates with different m cannot be compared directly. At $i = 8\%$: $d = 7.4074\% < d^{(2)} = 7.5499\% < i^{(2)} = 7.8461\% < i = 8\%$. They squeeze onto a limit in the middle, which is the subject of the next chapter.

THE CHAPTER IN ONE BOX

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = \frac{1}{1-d} = \left(1 - \frac{d^{(m)}}{m}\right)^{-m} = \frac{1}{v} = 1 + i.$$

- $A(t_1, t_2)$: what 1 at t_1 becomes at t_2; $v(t) = 1/A(t)$. Compound: $A(t) = (1+i)^t$.
- Consistency: $A(t_0, t_2) = A(t_0, t_1)A(t_1, t_2)$. Compound yes, simple no.
- $i_n = \dfrac{A(n) - A(n-1)}{A(n-1)}$; d_n is the same but over $A(n)$. One period, two ends: $d = iv = \dfrac{i}{1+i} < i$.
- **Effective**: what one whole period does. **Nominal**: $i^{(m)}$ means $i^{(m)}/m$, applied m times. Convert by matching periods.

TRY IT YOURSELF

1. An account pays a nominal 4.8% p.a. convertible monthly. Find (a) the effective annual rate of interest; (b) the accumulated value of €2,500 after 30 months; (c) the equivalent nominal rate convertible quarterly.
2. (a) A bill for €20,000 due in 91 days is bought at a simple discount rate of 3.5% p.a. (365-day year). Find the price paid and the equivalent effective annual rate of interest.

(b) A shop offers "2% off if you pay today instead of in three months". What effective annual rate of interest is it charging customers who wait?

3. A bank offers 6% simple interest: $\mathsf{A}(t) = 1 + 0.06t$ for a deposit at time 0, and the same terms to anyone depositing later. (a) Find i_1, i_2, i_3. (b) Show the market is not consistent, and find the risk-free profit that 1 of borrowed capital yields over two years. (c) Find the effective compound rate that accumulates 1 to the same amount over two years.

ANSWERS

1. Rate applied: 0.004 per month. (a) $1.004^{12} - 1 = 4.9070\%$. (b) $2500 \times 1.004^{30} = €2{,}818.07$. (c) $4(1.004^3 - 1) = 4.8193\%$.
2. (a) Price $20000(1 - 0.035 \times \frac{91}{365}) = €19{,}825.48$; growth over the 91 days is $20000/19825.48 = 1.008803$, so $1 + i = 1.008803^{365/91}$ and $i = 3.5779\%$. (b) $\mathsf{v}(1/4) = 0.98$, so $1 + i = (1/0.98)^4$ and $i = 8.4166\%$.
3. (a) 6%, $0.06/1.06 = 5.6604\%$, $0.06/1.12 = 5.3571\%$. (b) Leaving it gives 1.12; rolling over gives $1.06^2 = 1.1236$; the gap is 0.0036 per unit borrowed. (c) $1.12^{1/2} - 1 = 5.8301\%$.

CHAPTER 2

The force of interest

LECTURE 2 · *A rate for an instant, and what happens when it moves*

AFTER THIS CHAPTER YOU CAN

- ☐ Move between δ and i, d, $i^{(m)}$, $d^{(m)}$.
- ☐ Accumulate and discount under a constant force without rounding along the way.
- ☐ Find $\delta(t)$ from a fund value V_t.
- ☐ Compute $\mathsf{A}(t_1, t_2)$ and $\mathsf{v}(t)$ from a varying $\delta(t)$, including a force that changes formula part-way.
- ☐ Find the constant rate equivalent to a varying force.

2.1 The limit of the nominal rates

Pay interest (compound) ever more often. At $i = 8\%$ the nominal rate falls, but does not fall to zero. It settles, and fast:

m	1	2	4	12	52	365
$i^{(m)}$	0.080000	0.078461	0.077706	0.077208	0.077018	0.076969

The limit is $0.076961\ldots$ It is the most canonical number of the lot.

FORCE OF INTEREST

The **force of interest** corresponding to an effective rate i is the limiting nominal rate $\delta = \lim_{m\to\infty} i^{(m)}$. It is the nominal rate of interest per unit time convertible **continuously**.

ONE NUMBER FOR THE WHOLE COLLECTION

$$\delta \;=\; \ln(1+i) \;=\; -\ln(1-d) \;=\; -\ln \mathsf{v} \;=\; \lim_{m\to\infty} i^{(m)} \;=\; \lim_{m\to\infty} d^{(m)}.$$

Exponentiating, $\mathrm{e}^{\delta} = 1 + i$, and therefore for a *constant* force

$$\mathsf{A}(t) \;=\; (1+i)^t \;=\; \mathrm{e}^{\delta t}, \qquad\qquad \mathsf{v}(t) \;=\; \mathrm{e}^{-\delta t}.$$

The bridge of Chapter 1 gets a fifth leg: $1 + i = \mathrm{e}^{\delta}$.

WHERE EVERYTHING SITS

For $i > 0$:

$$d \;<\; d^{(m)} \;<\; \delta \;<\; i^{(m)} \;<\; i,$$

with $d^{(m)}$ increasing and $i^{(m)}$ decreasing in m, squeezing onto δ. Paid infinitely often, interest and discount stop being different things.

EXAMPLE 2.1 · Everything from the force

(a) A bank credits interest continuously at a force of 5% p.a. Find i, d, and the accumulated value of €1,000 after three years. (b) The nominal rate of discount convertible monthly is 6% p.a. Find d, i and δ.

SOLUTION

(a) $1 + i = \mathrm{e}^{0.05} = 1.051271$, so $i = 5.1271\%$. Then $d = 1 - \mathrm{v} = 1 - 1.051271^{-1} = 1 - 0.951229 = 4.8771\%$. And $1000\,\mathrm{A}(3) = 1000\,\mathrm{e}^{0.15} = €1{,}161.83$. With a constant force $(1+i)^t$ and $\mathrm{e}^{\delta t}$ are the same computation, but the second needs no intermediate rounding.
(b) Twelve monthly discounts of 0.5% give $\mathrm{v} = 0.995^{12} = 0.941623$. So $d = 1 - 0.941623 = 5.8377\%$, $1 + i = 1/0.941623$ gives $i = 6.1996\%$, and $\delta = -\ln \mathrm{v} = -12 \ln 0.995 = 0.060151$. Sanity check:

$$\underbrace{5.8377\%}_{d} \;<\; \underbrace{6\%}_{d^{(12)}} \;<\; \underbrace{6.0151\%}_{\delta} \;<\; \underbrace{6.0301\%}_{i^{(12)}} \;<\; \underbrace{6.1996\%}_{i}.$$

WHY ANYONE BOTHERS WITH δ

- Forces **add** where rates **multiply**: $\mathrm{e}^{\delta s}\mathrm{e}^{\delta(t-s)} = \mathrm{e}^{\delta t}$.
- **No period is attached.** i needs "per annum" and $i^{(12)}$ needs "convertible monthly"; δ is instantaneous, so it suits payments at arbitrary times.
- It **differentiates and integrates cleanly**, which is what lets the rate itself move. That is the next section.

2.2 A force that varies with time

So far the rate has been one number for all time. Nothing in $\mathrm{A}(t)$ required that.

FORCE OF INTEREST AS A FUNCTION OF TIME

With V_t the value of a fund at time t,

$$\delta(t) \;=\; \frac{V_t'}{V_t} \;=\; \frac{\mathrm{d}}{\mathrm{d}t} \ln V_t.$$

It is the instantaneous change in the fund as an annualised proportion of its *current* value.

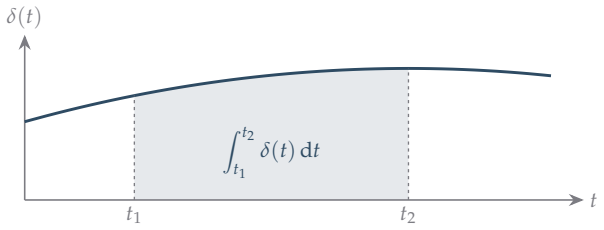
The slope alone is not a rate: €1,000,000 growing by €50,000 a year and €1,000 growing by €50 have very different slopes and identical performance. Dividing by the height removes

the size of the fund and leaves the rate.

ACCUMULATION AND DISCOUNT FROM A VARYING FORCE

$$\boxed{\mathsf{A}(t_1, t_2) \;=\; \mathsf{e}^{\int_{t_1}^{t_2} \delta(s)\,\mathrm{d}s}} \qquad\qquad \mathsf{v}(t) \;=\; \mathsf{e}^{-\int_0^t \delta(s)\,\mathrm{d}s}.$$

If δ is a constant this is $\mathsf{e}^{\delta t}$ and $\mathsf{e}^{-\delta t}$ again. So everything in Section 2.1 is a special case of this.



The accumulation factor is e raised to the *shaded area*. Two different-looking forces enclosing the same area accumulate money identically. And since integrals over adjacent intervals add, any model built from a force is automatically consistent: $\mathsf{A}(t_1, t_2)\mathsf{A}(t_2, t_3) = \mathsf{A}(t_1, t_3)$.

ACCUMULATING OR DISCOUNTING UNDER A VARYING FORCE

1. Integrate δ over the interval *between the two dates*. Nothing outside it matters. If δ changes formula, split the integral there.
2. Exponentiate: $+$ to accumulate, $-$ to discount.
3. Multiply by the amount.
4. For the *equivalent constant rate* over $[0, T]$, divide the integral by T to get the average force $\bar{\delta}$; then $i = \mathsf{e}^{\bar{\delta}} - 1$.

EXAMPLE 2.2 · Finding the force from a fund

A fund has value $V_t = 1000(1 + 0.02t)^2$. Find $\delta(t)$.

SOLUTION

$V_t' = 1000 \cdot 2(1 + 0.02t)(0.02) = 40(1 + 0.02t)$, so

$$\delta(t) \;=\; \frac{V_t'}{V_t} \;=\; \frac{40(1 + 0.02t)}{1000(1 + 0.02t)^2} \;=\; \frac{0.04}{1 + 0.02t}.$$

Here $\delta(0) = 4\%$ and $\delta(5) = 0.04/1.1 = 3.6364\%$: the fund grows, but its growth *rate* falls.

EXAMPLE 2.3 · A force that drifts upwards

The force of interest is $\delta(t) = 0.04 + 0.01t$ p.a. Find the accumulated value at time 5 of €1,000 invested at time 0, and the constant effective annual rate that does the same.

SOLUTION

$\int_0^5 (0.04 + 0.01t)\,\mathrm{d}t = \left[0.04t + 0.005t^2\right]_0^5 = 0.2 + 0.125 = 0.325$. So $\mathsf{A}(5) = \mathrm{e}^{0.325} = 1.384031$ and the fund is worth €1,384.03.
For the constant rate, $(1+i)^5 = \mathrm{e}^{0.325}$ gives $i = \mathrm{e}^{0.065} - 1 = 6.7159\%$: the average force $0.325/5 = 0.065$, exponentiated.

EXAMPLE 2.4 · A force that changes once

The force is 5% p.a. for three years and 4% thereafter. Find the present value of €10,000 due at time 5.

SOLUTION

Split the integral where the integrand changes: $\int_0^5 \delta(s)\,\mathrm{d}s = \int_0^3 0.05\,\mathrm{d}s + \int_3^5 0.04\,\mathrm{d}s = 0.15 + 0.08 = 0.23$. So $\mathsf{v}(5) = \mathrm{e}^{-0.23} = 0.794534$ and the present value is €7,945.34. (Splitting is consistency written in exponents: $\mathrm{e}^{0.15}\mathrm{e}^{0.08} = \mathrm{e}^{0.23}$.)

AN INSTANTANEOUS RATE IS INTEGRATED, NOT MULTIPLIED

The force is 6% for the first half-year and 4% for the second. Then $\mathsf{A}(1) = \mathrm{e}^{0.03}\mathrm{e}^{0.02} = \mathrm{e}^{0.05} = 1.051271$, a year's growth of 5.1271%: not 6, not 4, not 5. In general

$$\mathsf{A}(t, t+h) \;=\; \mathrm{e}^{\int_t^{t+h} \delta(s)\,\mathrm{d}s} \qquad (\approx\; 1 + \delta(t)h \;\text{ for small } h)\,,$$

and the familiar $1 + rh$ is only its first term.

THE CHAPTER IN ONE BOX

$$\delta \;=\; \ln(1+i) \;=\; -\ln(1-d) \;=\; \lim_{m\to\infty} i^{(m)} \;=\; \lim_{m\to\infty} d^{(m)}, \qquad \mathrm{e}^{\delta} = 1 + i.$$

$$\mathsf{A}(t_1, t_2) \;=\; \mathrm{e}^{\int_{t_1}^{t_2} \delta(t)\,\mathrm{d}t}, \qquad \mathsf{v}(t) \;=\; \mathrm{e}^{-\int_0^t \delta(s)\,\mathrm{d}s}, \qquad \delta(t) \;=\; \frac{V_t'}{V_t}.$$

Rates multiply and forces add, because consistency is the law of exponents. Only the area under δ matters. Order: $d < d^{(m)} < \delta < i^{(m)} < i$.

TRY IT YOURSELF

1. The force of interest is $\delta(t) = 0.03 + 0.005t$ p.a. Find $\mathsf{A}(4)$ and the constant effective annual rate that gives the same accumulation over those four years.
2. The force is $\delta(t) = 0.06 - 0.002t$ p.a. Find the present value of €500 due at time 2 together with €800 due at time 6.
3. From $\mathsf{A}(t_1, t_2) = \mathrm{e}^{\int_{t_1}^{t_2} \delta}$, show that $\mathsf{A}(t_1, t_2)\mathsf{A}(t_2, t_3) = \mathsf{A}(t_1, t_3)$.

ANSWERS

1. $\int_0^4 = 0.12 + 0.04 = 0.16$, so $\mathsf{A}(4) = \mathsf{e}^{0.16} = 1.173511$. Average force 0.04, so $i = \mathsf{e}^{0.04} - 1 = 4.0811\%$.
2. $\int_0^t = 0.06t - 0.001t^2$, so $\mathsf{v}(2) = \mathsf{e}^{-0.116} = 0.890475$ and $\mathsf{v}(6) = \mathsf{e}^{-0.324} = 0.723250$. Present value $445.24 + 578.60 = €1{,}023.84$.
3. Integrals over adjacent intervals add, and $\mathsf{e}^x \mathsf{e}^y = \mathsf{e}^{x+y}$.

CHAPTER 3

Cashflows and net present value

LECTURE 3 · *Many payments, one number*

AFTER THIS CHAPTER YOU CAN

- ☐ Find the present value of lumps at several dates.
- ☐ Find the present value of a continuous payment stream, including one that grows.
- ☐ Build a net cashflow with the right signs and compute its NPV.
- ☐ Build $\mathsf{v}(t)$ once for a force that changes formula, then use it.

3.1 Payments at many dates

You may not add euro at *different* times, but you may add them at *the same* time, e.g. today. Each payment is dragged back to time 0 by its own discount factor.

PRESENT VALUE OF A DISCRETE CASHFLOW

Sums $c_{t_1}, \dots, c_{t_n}$ due at times $t_1 < \cdots < t_n$ have present value

$$\mathrm{PV} = \sum_{j=1}^{n} c_{t_j}\, \mathsf{v}(t_j).$$

The same formula works if the payments never stop, provided the series converges.

EXAMPLE 3.1 · A table is all it takes

The force is a constant 5%. Find the present value of €1,000 at time 1, €2,000 at time 3 and €1,500 at time 5.

SOLUTION

$\mathsf{v}(t) = \mathrm{e}^{-0.05t}$, so

t_j	c_{t_j}	$\mathsf{v}(t_j)$	$c_{t_j}\mathsf{v}(t_j)$
1	1,000	0.951229	951.23
3	2,000	0.860708	1,721.42
5	1,500	0.778801	1,168.20
		Present value	€3,840.85

€4,500 of promised money is worth €3,840.85 today.

EXAMPLE 3.2 · A force that changes formula: build $v(t)$ once

The force is 0.04 for $t < 10$ and 0.03 for $t \geq 10$. Find $v(t)$ for every $t \geq 0$.

SOLUTION

For $t < 10$, $v(t) = e^{-0.04t}$. For $t \geq 10$, split the integral: $\int_0^{10} 0.04\,ds + \int_{10}^{t} 0.03\,ds = 0.4 + 0.03(t-10) = 0.1 + 0.03t$. So

$$v(t) = \begin{cases} e^{-0.04t} & t < 10, \\ e^{-(0.1+0.03t)} & t \geq 10. \end{cases}$$

Check: both pieces give $e^{-0.4}$ at $t = 10$. Interest and discount cannot jump, so $v(t)$ must be continuous.

3.2 Payments made continuously

When the lumps are small and frequent it is easier to model them as a flow, at a **rate of payment** $\rho(t)$ euro per unit time.

A RATE IS NOT AN AMOUNT

$\rho(2) = 1000$ does *not* mean €1,000 changed hands at time 2; at an instant, no money changes hands. A car doing 100 km/h at time 2 has not travelled 100 km at time 2. Money paid is the *integral* of the rate: $\int_\alpha^\beta \rho(t)\,dt$.

Between t and $t + dt$ about $\rho(t)\,dt$ is paid, worth $v(t)\rho(t)\,dt$ today. Add up the strips:

PRESENT VALUE OF A CONTINUOUS CASHFLOW

$$\text{PV} = \int_0^T v(t)\,\rho(t)\,dt, \qquad v(t) = e^{-\int_0^t \delta(s)\,ds}.$$

The discount factor sits *inside* the integral: each strip must be discounted from its own date. For a constant force, the integral to know:

$$\int_\alpha^\beta e^{-kt}\,dt = \frac{e^{-k\alpha} - e^{-k\beta}}{k}, \qquad \int_0^T e^{-\delta t}\,dt = \frac{1 - e^{-\delta T}}{\delta}, \qquad \int_0^\infty e^{-\delta t}\,dt = \frac{1}{\delta}.$$

PRESENT VALUE OF A STREAM

1. Write down $v(t)$ (Chapter 2).
2. Multiply by $\rho(t)$ and *simplify before integrating*. Exponentials combine: a stream growing at force g discounted at force δ behaves exactly like a level stream discounted at force $\delta - g$.
3. Integrate, splitting where $v(t)$ or ρ changes formula.

EXAMPLE 3.3 · Level, growing and never-ending streams

The force is a constant 5%. Find the present value of (a) €1,000 p.a. continuously for five years, (b) a stream at rate $1000\,\mathrm{e}^{0.03t}$ for ten years, (c) €1 p.a. forever.

SOLUTION

(a) $1000 \cdot \dfrac{1-\mathrm{e}^{-0.25}}{0.05} = €4{,}423.98.$

(b) Combine first: $\displaystyle\int_0^{10} \mathrm{e}^{-0.05t}\,1000\,\mathrm{e}^{0.03t}\,\mathrm{d}t = 1000\int_0^{10}\mathrm{e}^{-0.02t}\,\mathrm{d}t = 1000\cdot\frac{1-\mathrm{e}^{-0.2}}{0.02} =$ €9,063.46.

(c) $\int_0^\infty \mathrm{e}^{-0.05t}\,\mathrm{d}t = 1/0.05 =$ €20. An income of €1 a year for all eternity is worth €20 today. Discounting makes the far future almost worthless: everything after year 60 is together worth under €1.

IS THE STREAM WORTH THE RIGHT AMOUNT?

- A present value is less than the undiscounted total. (In (a), €4,423.98 against €5,000.)
- The same €5,000 paid *discretely* at the ends of years $1,\dots,5$ is worth $1000\sum_{k=1}^{5}\mathrm{e}^{-0.05k} =$ €4,314.31, which is €109.68 less: continuous money arrives earlier, on average halfway through each year instead of at its end.

3.3 Money in, money out

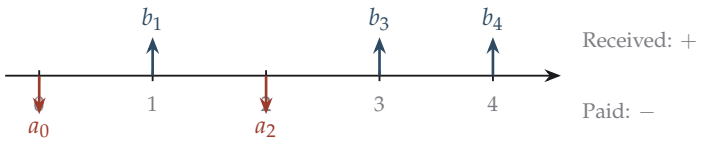
CASHFLOW

A **cashflow** $\mathcal{C}$ is discrete payments c_t at times t (only countably many non-zero) together with a continuous stream at rate $\rho(t)$. Its present value is

$$\mathrm{PV}(\mathcal{C}) = \sum_t c_t\,\mathsf{v}(t) + \int_0^T \rho(t)\mathsf{v}(t)\,\mathrm{d}t.$$

Lumps are summed against $\mathsf{v}(t)$; flows are integrated against it.

Payments go both ways. Let b_t be received and a_t paid at time t; the **net** discrete cashflow is $c_t = b_t - a_t$, and likewise $\rho = \rho_2 - \rho_1$ for the rates of receiving and paying.



NET PRESENT VALUE

NPV$(\mathcal{C})$ is PV applied to the net payments:

$$\mathrm{NPV}(\mathcal{C}) \;=\; \sum_t c_t\, \mathsf{v}(t) + \int \mathsf{v}(t)\,\rho(t)\,\mathrm{d}t, \qquad c_t = \text{in} - \text{out}.$$

Netting works because present values add: two payments on one date may be replaced by their difference. A positive NPV means the project is worth doing at this force of interest.

EXAMPLE 3.4 · Lump out, stream in

An investor pays €5,000 now and receives €2,000 p.a. continuously for four years, at a constant force of 5%. Is it worth doing?

SOLUTION

NPV $= -5000 + \int_0^4 2000\,\mathrm{e}^{-0.05t}\,\mathrm{d}t = -5000 + 7250.77 = €2{,}250.77$. Positive, so yes.

EXAMPLE 3.5 · A force that changes, a flow that does not

The force is 4% for ten years, then 3% (so $\mathsf{v}(t)$ is as above). An investment of €10 buys an income of €1 p.a., paid continuously for fifteen years. Is it worth making?

SOLUTION

The flow has $\rho \equiv 1$, and $\mathsf{v}(t)$ has two formulas, so split at $t = 10$:

$$\int_0^{10} \mathrm{e}^{-0.04t}\,\mathrm{d}t + \mathrm{e}^{-0.1}\int_{10}^{15} \mathrm{e}^{-0.03t}\,\mathrm{d}t \;=\; \underbrace{\frac{1-\mathrm{e}^{-0.4}}{0.04}}_{8.2420} + \underbrace{\frac{\mathrm{e}^{-0.1}}{0.03}\left(\mathrm{e}^{-0.3}-\mathrm{e}^{-0.45}\right)}_{3.1123} \;=\; 11.35.$$

So NPV $= -10 + 11.35 = €1.35 > 0$. Worth making.
Sanity check: undiscounted the income is €15, and discounting cost about a quarter. Had the force stayed at 0.04 throughout we would get $(1-\mathrm{e}^{-0.6})/0.04 = €11.28$; the fall to 0.03 raises the value, as it must.

THE CHAPTER IN ONE BOX

$$\mathrm{NPV}(\mathcal{C}) \;=\; \sum_t c_t\, \mathsf{v}(t) \;+\; \int_0^T \mathsf{v}(t)\,\rho(t)\,\mathrm{d}t, \qquad c_t = \text{in} - \text{out}.$$

$$\int_\alpha^\beta \mathrm{e}^{-kt}\,\mathrm{d}t = \frac{\mathrm{e}^{-k\alpha} - \mathrm{e}^{-k\beta}}{k}, \qquad \int_0^\infty \mathrm{e}^{-\delta t}\,\mathrm{d}t = \frac{1}{\delta}.$$

Lumps are summed against $\mathsf{v}(t)$, flows integrated against it. Simplify before integrating. A rate ρ is not an amount. Move everything to a common date (e.g. today) and sum up.

TRY IT YOURSELF

1. The force is $\delta(t) = 0.05 - 0.002t$. Find the present value of €1,000 due at time 3 together with €2,000 due at time 8.
2. An investor is paid continuously for ten years at the rate $\rho(t) = 800\,\mathrm{e}^{0.02t}$ p.a., the force being a constant 5%. Find the present value.
3. An investor pays €10,000 now and receives €3,000 at each of times 2, 4, 6. With $\delta = 0.06$ constant, find the NPV and say whether the investment is worth making.

ANSWERS

1. $\mathsf{v}(t) = \mathrm{e}^{-(0.05t - 0.001t^2)}$; $\mathsf{v}(3) = \mathrm{e}^{-0.141} = 0.868489$, $\mathsf{v}(8) = \mathrm{e}^{-0.336} = 0.714623$. PV $= 868.49 + 1429.25 = €2{,}297.74$.
2. Combine: $800\int_0^{10} \mathrm{e}^{-0.03t}\,\mathrm{d}t = 800(1 - \mathrm{e}^{-0.3})/0.03 = €6{,}911.51$.
3. $3000(\mathrm{e}^{-0.12} + \mathrm{e}^{-0.24} + \mathrm{e}^{-0.36}) = 3000 \times 2.371225 = 7113.67$, so NPV $= -€2{,}886.33 < 0$: not worth making.

CHAPTER 4

Value at any date

LECTURE 4 · *One value, quoted at a date*

AFTER THIS CHAPTER YOU CAN

- ☐ Value one payment at any date, before or after it is due.
- ☐ Value a whole cashflow at any date, and slide a value from one date to another.
- ☐ Turn calendar dates into times, choosing a convenient origin.

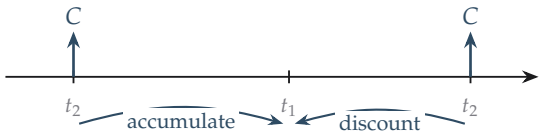
4.1 The value of a payment at any date

Present value answers "what is this worth now?". Often you want something else: a pension pot at retirement, payments in five years' time, two offers on different dates. So we need the value at a general time t_1, with the payment due before or after it.

VALUE AT TIME t_1 OF C DUE AT TIME t_2

$$C\,\mathrm{e}^{-\int_{t_1}^{t_2}\delta(t)\,\mathrm{d}t} = C\,\frac{\mathrm{v}(t_2)}{\mathrm{v}(t_1)}$$

If $t_1 < t_2$ this *discounts*; if $t_1 \geq t_2$ it *accumulates*. One formula covers both, because swapping the limits flips the sign of the exponent. Only the force *between* t_1 and t_2 matters.



EXAMPLE 4.1 · A payment moved

(a) With $\delta = 0.06$: the value at time 2 of 100 due at time 5, and the value at time 5 of 100 due at time 2. (b) With $\delta(t) = 0.04 + 0.002t$: the value at time 3 of €2,000 due at time 6.

SOLUTION

(a) $100\,\mathrm{e}^{0.06(2-5)} = €83.53$, and $100\,\mathrm{e}^{0.06(5-2)} = €119.72$.

(b) Integrate only from 3 to 6: $\left[0.04t + 0.001t^2\right]_3^6 = 0.276 - 0.129 = 0.147$. The value is

$2000\,\mathrm{e}^{-0.147} = €1{,}726.59.$

4.2 The value of a cashflow at any date

Values add, so apply the factor to every payment.

VALUE OF A CASHFLOW AT TIME t_1

$$\mathrm{Val}_{t_1}(\mathcal{C}) \;=\; \sum_t c_t \,\frac{\mathsf{v}(t)}{\mathsf{v}(t_1)} \;+\; \int_{-\infty}^{\infty} \rho(t)\,\frac{\mathsf{v}(t)}{\mathsf{v}(t_1)}\,\mathrm{d}t.$$

Payments can lie on *both* sides of t_1; the formula handles it. For $t < 0$, $\mathsf{v}(t) > 1$, so past payments are accumulated automatically.

TWO WORDS FOR TWO SITUATIONS

If all payments are due at or *after* t_1, their value at t_1 is their **discounted value**. If all are at or *before* t_1, it is their **accumulation**. In general $\mathrm{Val}_{t_1}(\mathcal{C})$ is the sum of the two. With $t_1 = 0$ and everything in the future, it is the **present value**.

THE TRANSFER PRINCIPLE

$$\mathrm{Val}_{t_1}(\mathcal{C})\,\mathsf{v}(t_1) \;=\; \mathrm{Val}_{t_2}(\mathcal{C})\,\mathsf{v}(t_2) \;=\; \mathrm{PV}(\mathcal{C}),$$

$$\mathrm{Val}_{t_1}(\mathcal{C}) \;=\; \mathrm{Val}_{t_2}(\mathcal{C}) \times \frac{\mathsf{v}(t_2)}{\mathsf{v}(t_1)}, \qquad \mathrm{Val}_t(\mathcal{C}) \;=\; \mathrm{PV}(\mathcal{C})\,\mathsf{A}(t).$$

A cashflow has *one* value; the date only decides how it is written down. Value once, at whichever date suits you, then slide the answer along. You never need to go back to the individual payments.

VALUING A CASHFLOW AT A DATE

1. Choose the origin (time 0) at a convenient date, often the valuation date or the first payment, and measure time in years. Use exact fractions (14 months $= \frac{14}{12}$) unless a day-count convention is specified.
2. Value every payment at that origin: lumps by $c_t\mathsf{v}(t)$, flows by integrating.
3. Slide to the date asked for: multiply by $\mathsf{A}(t)$ to move later, divide to move earlier.
4. Check one payment directly: $C\,\mathrm{e}^{-\int_{t_1}^{t_2}\delta}$.

EXAMPLE 4.2 · A lump and a stream, at two dates

An investor receives €1,000 at time 2 and a continuous stream of €300 p.a. between times 0 and 4. The force is a constant 4%. Find the value at time 0 and at time 4.

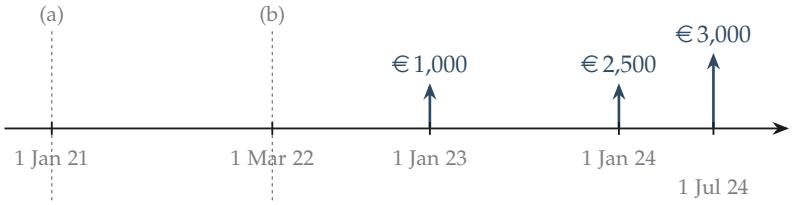
SOLUTION

$$\text{PV} = 1000\,\mathrm{e}^{-0.08} + \int_0^4 300\,\mathrm{e}^{-0.04t}\,\mathrm{d}t = 923.12 + 300 \cdot \frac{1-\mathrm{e}^{-0.16}}{0.04} = €2{,}032.04.$$

Every payment is at or before time 4, so this is an accumulation: $\text{Val}_4 = 2032.04\,\mathrm{e}^{0.16} =$ €2,384.62. No need to revisit the lump and the integral.

EXAMPLE 4.3 · Alfred's payments

Alfred is owed €1,000 on 1 Jan 2023, €2,500 on 1 Jan 2024 and €3,000 on 1 Jul 2024, at a constant force 0.06. Find the value on (a) 1 Jan 2021, (b) 1 Mar 2022.



SOLUTION

(a) Take 1 Jan 2021 as time 0; the payments are at $t = 2,\ 3,\ 3.5$:

$$1000\,\mathrm{e}^{-0.12} + 2500\,\mathrm{e}^{-0.18} + 3000\,\mathrm{e}^{-0.21} = 886.92 + 2088.18 + 2431.75 = €5{,}406.85.$$

(b) 1 Mar 2022 is fourteen months in: $t_1 = \frac{14}{12} = \frac{7}{6}$. Slide:

$$\text{Val}_{7/6} = 5406.85\,\mathrm{e}^{0.06\times\frac{7}{6}} = 5406.85\,\mathrm{e}^{0.07} = €5{,}798.89.$$

Valuing each payment directly at t_1 by $C\,\mathrm{e}^{-0.06(t-t_1)}$ gives $951.23 + 2239.59 + 2608.07 =$ €5,798.89, identical.

THE CHAPTER IN ONE BOX

$$\text{Val}_{t_1}(\mathcal{C}) = \sum_t c_t \frac{\mathrm{v}(t)}{\mathrm{v}(t_1)} + \int \rho(t)\frac{\mathrm{v}(t)}{\mathrm{v}(t_1)}\,\mathrm{d}t, \qquad \text{Val}_{t_1}(\mathcal{C})\,\mathrm{v}(t_1) = \text{PV}(\mathcal{C}).$$

One value, quoted at a date. Value once, at whichever date suits you, then slide the answer along. Payments before the date accumulate and payments after it discount, but a single formula captures both.

TRY IT YOURSELF

1. Payments of €500 and €700 are due at times 1 and 4, and $\delta = 0.05$ is constant. Find the value at time 3, and check it against $\text{PV}/\mathrm{v}(3)$.

2. With $\delta = 0.05$, find the value at time 0 of €800 paid at time -2 together with €500 due at time 3. Which is accumulated?
3. Show directly from the definition that $\mathrm{Val}_{t_1}(\mathcal{C})\, \mathsf{v}(t_1)$ does not depend on t_1.

ANSWERS

1. $500\,\mathsf{e}^{0.1} + 700\,\mathsf{e}^{-0.05} = 552.59 + 665.86 = €1{,}218.45$. Check: PV $= 1048.73$ and $1048.73\,\mathsf{e}^{0.15} = 1218.45$.
2. $800\,\mathsf{e}^{0.1} + 500\,\mathsf{e}^{-0.15} = 884.14 + 430.35 = €1{,}314.49$. The €800 is accumulated (it is in the past); the €500 is discounted.
3. Multiply the definition by $\mathsf{v}(t_1)$: every denominator cancels and no t_1 is left.

PART II

Annuities

CHAPTER 5

Level annuities

LECTURE 5 · *Constant payments at regular intervals*

AFTER THIS CHAPTER YOU CAN

- ☐ Classify an annuity: level, certain, in arrears or in advance, immediate or deferred.
- ☐ Count the payments correctly.
- ☐ Write down $a_{\overline{n}|}$, $s_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$ from memory, and say where each is valued.
- ☐ Value an annuity at a date other than its symbol's date.
- ☐ Derive one annuity formula from another.

5.1 Vocabulary

An **annuity** is a regular series of payments: premiums, dividends, mortgage repayments, salaries, rents, pensions. An annuity is called

- **certain** if payable for a definite period; **level** if every payment is the same; **increasing** (**decreasing**) if payments rise (fall) by the same amount;
- **in arrears** if paid at the end of each period; **in advance** (or **annuity due**) if paid at the beginning, so the first payment is at $t = 0$;
- **immediate** if the first payment is at $t = 1$; **deferred** if it is at $t \geq 2$.

This chapter is all *level annuities certain*, in arrears or in advance.

> "IMMEDIATE" DOES NOT MEAN "NOW"
>
> An annuity-immediate in arrears pays nothing at time 0; its first payment is at time 1. The word only promises that payments start in the first period. The annuity that pays *now* is the annuity *due*.

5.2 The sum behind every annuity

A payment of 1 at time t is worth v^t today, so valuing an annuity means summing a **geometric progression**.

GEOMETRIC SUMS

For $r \neq 1$,

$$\sum_{k=0}^{n} ar^k = \frac{a\left(1-r^{n+1}\right)}{1-r}, \qquad \sum_{k=1}^{n} ar^k = \frac{ar\left(1-r^{n}\right)}{1-r}.$$

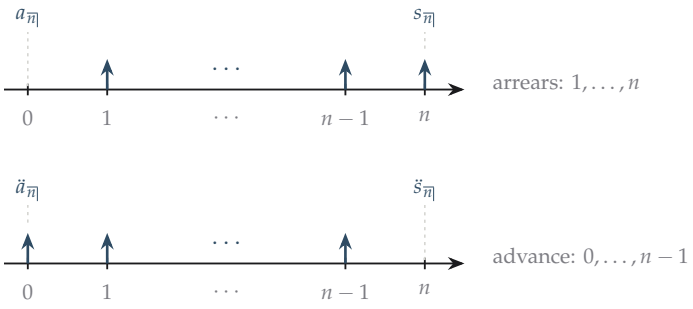
To remember it: $S - rS$ cancels everything except the first term and the term *after* the last.

ALWAYS ASK: HOW MANY PAYMENTS?

The first sum has $n+1$ terms, not n: the exponents run $0, 1, \ldots, n$. Miscounting terms is the commonest error in annuity questions. Payments at times $0, 1, \ldots, n$ are $n+1$ payments.

5.3 The four symbols

Take n payments of 1 at effective rate $i \neq 0$ per period. The angle carries the number of *payments*, not the number of years to anything. Dots ("umlaut") on top mean *in advance*.



THE FOUR LEVEL ANNUITIES

symbol	formula	payments at	valued at
$a_{\overline{n}\vert}$	$\dfrac{1-v^n}{i}$	$1,\dots,n$	0
$s_{\overline{n}\vert}$	$\dfrac{(1+i)^n-1}{i}$	$1,\dots,n$	n
$\ddot{a}_{\overline{n}\vert}$	$\dfrac{1-v^n}{d}$	$0,\dots,n-1$	0
$\ddot{s}_{\overline{n}\vert}$	$\dfrac{(1+i)^n-1}{d}$	$0,\dots,n-1$	n

The dots say when the *payments* fall, not when the value is taken. The transfer principle links the values:

$$a_{\overline{n}\vert} = v^n s_{\overline{n}\vert}, \qquad \ddot{a}_{\overline{n}\vert} = (1+i)\, a_{\overline{n}\vert}, \qquad \ddot{s}_{\overline{n}\vert} = (1+i)\, s_{\overline{n}\vert}.$$

The factor $(1+i)$: every annuity-due payment falls one period earlier, so earns one more period of interest. The derivation to expect is one line: the closed forms differ only in the denominator, so $\ddot{a}_{\overline{n}\vert}/a_{\overline{n}\vert} = i/d = 1+i$.

VALUING AN ANNUITY

1. **How many payments?** That is n.
2. Arrears or advance? Is the first payment at the start of the first period or the end?
3. Payment $\times$ the matching symbol. That is the value at the symbol's date.
4. If the date you are asked for is not the symbol's date, slide: multiply by $(1+i)^k$ for k periods later, by v^k for k periods earlier.
5. Sanity check: the value is below the undiscounted total, and an annuity-due is worth more than the same annuity in arrears.

EXAMPLE 5.1 · Arrears and advance

(a) An annuity pays €400 at the end of each year for twelve years, at $i = 5\%$. Find its present value. (b) A tenant pays €900 at the start of each year for six years, at $i = 5\%$. Find the present value of the rent.

SOLUTION

(a) Twelve payments in arrears: $a_{\overline{12}\vert}$. $v^{12} = 1.05^{-12} = 0.556837$, so $a_{\overline{12}\vert} = (1 - 0.556837)/0.05 = 8.863252$ and the value is $400 \times 8.863252 = €3{,}545.30$. (Undiscounted: €4,800.)

(b) Six payments in advance: $\ddot{a}_{\overline{6}\vert}$. $v^6 = 0.746215$ and $d = 0.05/1.05 = 0.047619$, so $\ddot{a}_{\overline{6}\vert} = (1 - 0.746215)/0.047619 = 5.329477$ and the value is $900 \times 5.329477 = €4{,}796.53$. In arrears the same rent would be worth $900\, a_{\overline{6}\vert} = €4{,}568.12$, lower by exactly the factor 1.05.

EXAMPLE 5.2 · All four on one set of numbers

At $i = 6\%$ and $n = 10$ (so $v^{10} = 0.558395$, $d = 0.056604$):

	payments at	valued at	value
$a_{\overline{10}\vert}$	$1, \ldots, 10$	0	7.360087
$\ddot{a}_{\overline{10}\vert}$	$0, \ldots, 9$	0	7.801692
$s_{\overline{10}\vert}$	$1, \ldots, 10$	10	13.180795
$\ddot{s}_{\overline{10}\vert}$	$0, \ldots, 9$	10	13.971643

Each "due" value is 1.06 times the one above it, and each value at time 10 is 1.06^{10} times the one at time 0.

CHECKING AN ANNUITY ANSWER

A quick test at $i = 6\%$: $s_{\overline{3}\vert} = (1.06^3 - 1)/0.06 = 3.1836$ is just $1.06^2 + 1.06 + 1$, and $a_{\overline{3}\vert} = 2.673012$ is $v + v^2 + v^3$. If you forget a formula, write the sum and add it up.

THE CHAPTER IN ONE BOX

$$a_{\overline{n}\vert} = \frac{1 - v^n}{i}, \qquad s_{\overline{n}\vert} = \frac{(1+i)^n - 1}{i}, \qquad \ddot{a}_{\overline{n}\vert} = \frac{1 - v^n}{d}, \qquad \ddot{s}_{\overline{n}\vert} = \frac{(1+i)^n - 1}{d}.$$

Count the payments. Arrears: times $1, \ldots, n$. Advance: times $0, \ldots, n-1$. a is valued at time 0, s at time n; due $= (1+i)\times$ immediate; slide to any other date by $(1+i)^k$.

TRY IT YOURSELF

1. An annuity pays €2,000 at the end of each year for fifteen years, at $i = 4\%$. Find its present value and its accumulated value at the end, and check them against each other.
2. Payments of 1 are made at times $0, 1, \ldots, n$. How many payments are there? Write the value at time 0 as a single "due" symbol, and again as 1 plus an in-arrears symbol, and check they agree at $i = 6\%$, $n = 10$.
3. Prove $\ddot{a}_{\overline{n}\vert} = (1+i)\, a_{\overline{n}\vert}$ from the closed forms, and say what the factor $(1+i)$ is doing.

ANSWERS

1. $a_{\overline{15}\vert} = 11.118387$, $s_{\overline{15}\vert} = 20.023588$. Present value €22,236.77, accumulated value €40,047.18; check $22236.77 \times 1.04^{15} = 40047.18$.
2. $n+1$ payments. $\ddot{a}_{\overline{n+1}\vert} = 1 + a_{\overline{n}\vert}$; at 6%: 8.360087 both ways.
3. $\dfrac{\ddot{a}_{\overline{n}\vert}}{a_{\overline{n}\vert}} = \dfrac{(1 - v^n)/d}{(1 - v^n)/i} = \dfrac{i}{d} = 1 + i$, since $d = \frac{i}{1+i}$. The factor is one period's accumulation: the two annuities have the same payments and valuation date, but every payment of the annuity-due falls one period earlier.

CHAPTER 6

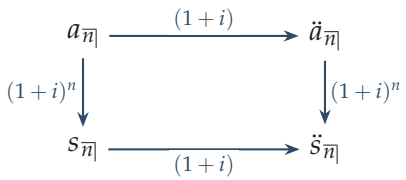
Annuities at work

LECTURE 6 · *Edge cases, perpetuities and loan repayments*

AFTER THIS CHAPTER YOU CAN

- ☐ Get from any of the four symbols to any other, by moving payments or moving the valuation date.
- ☐ Say what happens at $n = 0$, at $i = 0$ and as $n \to \infty$, and why.
- ☐ Use $s_{\overline{n+1}|} = 1 + \ddot{s}_{\overline{n}|}$ and $\ddot{a}_{\overline{n+1}|} = 1 + a_{\overline{n}|}$.
- ☐ Value a perpetuity, in arrears and in advance.
- ☐ Find the instalment that repays a loan.

6.1 Four symbols, two multiplications



Horizontal: the payments move one period earlier. *Vertical:* the valuation date moves n periods later. Going across and then down gives the same number as going down and then across.

EXAMPLE 6.1 · Moving around the square

At $i = 6\%$, $a_{\overline{10}|} = 7.360087$. Find $\ddot{a}_{\overline{10}|}$, $s_{\overline{10}|}$, $\ddot{s}_{\overline{10}|}$.

SOLUTION

$\ddot{a}_{\overline{10}|} = 1.06 \times 7.360087 = 7.801692$. $s_{\overline{10}|} = 1.06^{10} \times 7.360087 = 13.180795$. $\ddot{s}_{\overline{10}|} = 1.06\, s_{\overline{10}|} = 1.06^{10}\, \ddot{a}_{\overline{10}|} = 13.971643$.

6.2 Two edge cases

NO PAYMENTS, AND NO INTEREST

- $n = 0$: no payments, so $a_{\overline{0}|} = s_{\overline{0}|} = \ddot{a}_{\overline{0}|} = \ddot{s}_{\overline{0}|} = 0$. The closed forms agree, since $v^0 = 1$.
- $i = 0$: all four symbols equal n. Every payment of 1 is worth 1 at every date, and it no longer matters when they fall. This is the only case in which the four symbols coincide: interest is precisely what tells them apart.

AT $i = 0$ THE CLOSED FORM IS $0/0$: USE A LIMIT

The value n is a *limit*, not a substitution. For small i, $(1+i)^{-n} = 1 - ni + \frac{n(n+1)}{2}i^2 - \cdots$, so

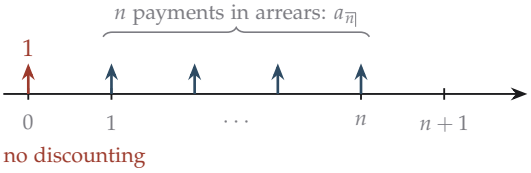
$$a_{\overline{n}|} = \frac{1 - \left(1 - ni + \frac{n(n+1)}{2}i^2 - \cdots\right)}{i} = n - \frac{n(n+1)}{2}\,i + \cdots \xrightarrow[i\to 0]{} n.$$

Never evaluate a function at a point where the denominator is zero, even if the numerator is zero too. Always use limits.

SENSIBLE AT BOTH ENDS

Ten payments of 1, valued at time 0: as $i \to 0$, $a_{\overline{10}|}$ rises to 10 and nothing is discounted. As i grows, $v \to 0$ and later payments become meaningless. The value never leaves $(0, n]$ and falls as i rises.

6.3 One payment more



ONE PAYMENT MORE

$$\ddot{a}_{\overline{n+1}|} = 1 + a_{\overline{n}|}, \qquad s_{\overline{n+1}|} = 1 + \ddot{s}_{\overline{n}|}.$$

$n+1$ payments in advance, valued at time 0: the first falls *on* the valuation date and contributes exactly 1; the other n are an annuity in arrears. Mirror image: $n+1$ payments in arrears, valued at time $n+1$: the last falls on the valuation date, earns nothing and contributes 1; the other n are an annuity-due, valued n periods later. **A payment that lands on the valuation date is worth exactly 1.**

PROVING IT FROM THE CLOSED FORMS

Either identity follows by algebra: write both symbols over the denominator i, using $\ddot{s}_{\overline{n}|} = \dfrac{(1+i)^n - 1}{d}$ and $\dfrac{1}{d} = \dfrac{1+i}{i}$. Then $s_{\overline{n+1}|} - \ddot{s}_{\overline{n}|} = \dfrac{(1+i)^{n+1} - 1 - (1+i)^{n+1} + (1+i)}{i} = \dfrac{i}{i} = 1$. This is the level of derivation the exam can ask: go from one annuity formula to another.

6.4 Payments that never stop

GEOMETRIC SERIES, INFINITELY MANY TERMS

$\displaystyle\sum_{k=0}^{\infty} ar^k := \lim_{n\to\infty} \sum_{k=0}^{n} ar^k$. For $a \neq 0$ it exists *if and only if* $|r| < 1$, and then $r^{n+1} \to 0$, giving $\displaystyle\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$. If $|r| \geq 1$ the terms do not shrink and the total is infinite (or oscillates, e.g. $r = -1$).

An annuity whose payments never stop is a **perpetuity**. If $i > 0$ then $\mathrm{v} < 1$, so $\mathrm{v}^n \to 0$.

PERPETUITIES

$$a_{\overline{\infty}|} = \lim_{n\to\infty} \frac{1-\mathrm{v}^n}{i} = \frac{1}{i}, \qquad \ddot{a}_{\overline{\infty}|} = \frac{1}{d} = 1 + \frac{1}{i}.$$

There is no $s_{\overline{\infty}|}$ or $\ddot{s}_{\overline{\infty}|}$. An accumulated value has to be accumulated *to* some date, and a perpetuity has none.

WHY $1/i$ IS OBVIOUS IN HINDSIGHT

Put $1/i$ in the bank at rate i. Each year it earns $i \times \frac{1}{i} = 1$. Withdraw exactly that 1 and the capital is untouched, ready to do the same next year, for ever. An infinite stream of payments has a finite value because eventually each one is almost worthless.

EXAMPLE 6.2 · Arrears and advance

Find the value of a perpetuity paying €500 a year at $i = 5\%$, in (a) arrears, (b) advance.

SOLUTION

(a) $500/0.05 = €10{,}000$. (b) $d = 0.05/1.05 = 0.047619$, so $1/d = 21$ and the value is $500 \times 21 = €10{,}500$. Equivalently $500(1 + \frac{1}{i})$. The difference is exactly €500: one extra payment, made today, needing no discounting.

6.5 A loan repaid

A loan is an annuity seen from the other side: the lender hands over the loan at time 0 and receives the repayments, so *at time* 0 *the two must have the same value.*

INSTALMENT THAT REPAYS A LOAN

1. Draw the timeline: loan L at time 0; n equal instalments.
2. Equate values at time 0: $L = X\,a_{\overline{n}|}$ (arrears) or $L = Y\,\ddot{a}_{\overline{n}|}$ (advance).
3. Solve: $X = L/a_{\overline{n}|}$.
4. **Shortcut for advance.** Both annuities repay the same loan, so $Xa_{\overline{n}|} = Y\ddot{a}_{\overline{n}|} = Y(1+i)a_{\overline{n}|}$, giving $X = (1+i)Y$, i.e. $Y = \mathrm{v}X$. Paying in advance hands over every instalment a year earlier, so each can be smaller by one year's discount factor.

EXAMPLE 6.3 · A €4,800 loan over twenty years

A loan of €4,800 is repaid in 20 equal annual instalments at $i = 10\%$. Find the instalment if paid in (a) arrears, (b) advance.

SOLUTION

(a) $\mathrm{v} = 1/1.1 = 0.909091$ and $\mathrm{v}^{20} = 0.148644$, so

$$4800 \;=\; X\,a_{\overline{20}|} \;=\; X\,\frac{1-0.148644}{0.1} \;=\; X \times 8.513564 \quad\Longrightarrow\quad X \;=\; €563.81.$$

(b) $Y = \mathrm{v}X = 563.81/1.1 = €512.55$. Directly: $d = 0.090909$, $\ddot{a}_{\overline{20}|} = (1 - 0.148644)/0.090909 = 9.364920$ and $4800/9.364920 = €512.55$, the same.

DO NOT ROUND v

Keep $\mathrm{v} = 0.909091$, not 0.9. Raised to the twentieth power the error is huge: $0.9^{20} = 0.1216$ against 0.1486. That is an 18% error in v^{20}, and about 3% in the instalment. Carry six digits, or work with $(1+i)^{-n}$ in one go on the calculator.

WHAT THE BORROWER ACTUALLY PAYS

	instalment	$\times\,20$	interest paid
In arrears	563.81	11,276.20	6,476.20
In advance	512.55	10,251.00	5,451.00

Over twenty years at 10% the borrower repays more than twice what was borrowed. The "in advance" schedule looks cheaper in total, but nothing has been gained: the money is simply handed over sooner, and *at time* 0 *both schedules are worth exactly* €4,800.

THE CHAPTER IN ONE BOX

$$n = 0:\ \text{all four are } 0; \qquad i = 0:\ \text{all four are } n; \qquad a_{\overline{\infty}|} = \tfrac{1}{i},\ \ \ddot{a}_{\overline{\infty}|} = \tfrac{1}{d} = 1 + \tfrac{1}{i}.$$

$$\ddot{a}_{\overline{n+1}|} = 1 + a_{\overline{n}|}, \qquad s_{\overline{n+1}|} = 1 + \ddot{s}_{\overline{n}|}.$$

A payment landing on the valuation date is worth 1. At $i = 0$ the formulas read $0/0$ and must be replaced by limits. No $s_{\overline{\infty}|}$. To price a loan, equate the advance with the repayments at one date; in advance, $Y = vX$.

TRY IT YOURSELF

1. A loan of €20,000 is repaid by ten equal annual instalments at $i = 7\%$. Find the instalment if the payments are in arrears, and again if they are in advance.
2. Prove $s_{\overline{n+1}|} = 1 + \ddot{s}_{\overline{n}|}$ from the closed forms, and say in one sentence what the 1 is.
3. A perpetuity pays €750 a year at $i = 4\%$. Find its value in arrears and in advance, and explain why the two differ by exactly one payment.

ANSWERS

1. $v^{10} = 0.508349$, $a_{\overline{10}|} = 7.023582$, so $X = €2{,}847.55$. In advance $Y = X/1.07 = €2{,}661.26$; check $\ddot{a}_{\overline{10}|} = 1.07 a_{\overline{10}|} = 7.515232$ and $20000/7.515232 = 2661.26$.
2. $s_{\overline{n+1}|} - \ddot{s}_{\overline{n}|} = \dfrac{(1+i)^{n+1} - 1}{i} - \dfrac{(1+i)^{n+1} - (1+i)}{i} = \dfrac{i}{i} = 1$, using $\ddot{s}_{\overline{n}|} = \frac{(1+i)}{i}\big((1+i)^n - 1\big)$. The 1 is the last payment: it lands on the valuation date $n+1$ and earns no interest.
3. Arrears $750/0.04 = €18{,}750$. Advance $d = 0.04/1.04$, $1/d = 26$, so $750 \times 26 = €19{,}500$. Difference €750: the two streams are the same apart from one payment made today, which needs no discounting. In symbols, $\frac{1}{d} = 1 + \frac{1}{i}$.

CHAPTER 7

Payable more often than once a period

LECTURES 7–8 · *Continuously, m times a period, and the whole family*

AFTER THIS CHAPTER YOU CAN

- ☐ Write down $\bar{a}_{\overline{n}|}$, $\bar{s}_{\overline{n}|}$, $a^{(m)}_{\overline{n}|}$, $s^{(m)}_{\overline{n}|}$, $\ddot{a}^{(m)}_{\overline{n}|}$, $\ddot{s}^{(m)}_{\overline{n}|}$ and say what changes between them.
- ☐ Value a salary, rent or mortgage paid monthly or quarterly.
- ☐ Put the five annuities in order, and see why.
- ☐ Choose the unit of time that makes the question easy.
- ☐ Value perpetuities at any frequency.

The one idea. Every symbol in this chapter has the same two numerators as in Chapter 5, and *only the rate underneath changes*:

$$\text{PV} = \frac{1-\mathrm{v}^n}{\text{rate}}, \qquad \text{AV} = \frac{(1+i)^n-1}{\text{rate}}.$$

The rate is decided by *how often* and *when* the money moves.

7.1 Payments made continuously

A level stream at rate 1 per unit time over $[0, n]$ pays the same total n as n annual payments of 1; the difference is *when*. Water from a tap, as opposed to bottles arriving at year end. This is Chapter 3 with $\rho \equiv 1$.

CONTINUOUSLY PAYABLE ANNUITY

$\bar{a}_{\overline{n}|} := \int_0^n \mathrm{v}(t)\,\mathrm{d}t$. This is a definition, not a formula: it holds whatever $\mathrm{v}(t)$ is. $\bar{s}_{\overline{n}|}$ is the same flow valued at time n.

CONTINUOUS ANNUITY, CONSTANT FORCE

$$\bar{a}_{\overline{n}|} = \frac{1-\mathrm{v}^n}{\delta}, \qquad \bar{s}_{\overline{n}|} = \frac{(1+i)^n-1}{\delta}, \qquad \bar{a}_{\overline{n}|} = \mathrm{v}^n \bar{s}_{\overline{n}|}.$$

The denominator is δ, *not* i. (The first is the integral $\int_0^n \mathrm{e}^{-\delta t}\,\mathrm{d}t$. For $\bar{s}_{\overline{n}|}$, put $u = n - t$; or simply use $\bar{s}_{\overline{n}|} = (1+i)^n \bar{a}_{\overline{n}|}$, as it is the same cashflow quoted at time n.)

EXAMPLE 7.1 · A flow for eight years, and a salary

(a) €2,000 p.a. continuously for eight years at a force of 4%: present value? (b) A salary of €30,000 p.a., treated as continuous, for five years at $i = 4\%$: present value and accumulated value at the end.

SOLUTION

(a) $\bar{a}_{\overline{8}|} = \dfrac{1-\mathrm{e}^{-0.32}}{0.04} = \dfrac{1-0.726149}{0.04} = 6.846274$, so the value is $2000 \times 6.846274 =$ €13,692.55 (undiscounted: €16,000).
(b) $\delta = \ln 1.04 = 0.039221$ and $\mathrm{v}^5 = 0.821927$:

$$\bar{a}_{\overline{5}|} = \frac{1-0.821927}{0.039221} = 4.540277, \qquad \bar{s}_{\overline{5}|} = \frac{1.04^5-1}{0.039221} = 5.523941,$$

so $30000\,\bar{a}_{\overline{5}|} =$ €136,208.30 and $30000\,\bar{s}_{\overline{5}|} =$ €165,718.23.

WHAT CONTINUITY IS WORTH

Since the symbols differ only in the denominator,

$$\bar{a}_{\overline{n}|} = \frac{i}{\delta}\,a_{\overline{n}|}, \qquad \bar{s}_{\overline{n}|} = \frac{i}{\delta}\,s_{\overline{n}|}.$$

As $\delta < i$, $i/\delta > 1$: continuous payment is worth *more* than the same total paid annually in arrears, because it arrives sooner. The ratio does *not* depend on n. At $i = 6\%$, $\delta = 0.058269$ and $i/\delta = 1.029709$: about 3% more.

For the salary of (b), paid annually in arrears instead, $a_{\overline{5}|} = 4.451822$ and the present value would be €133,554.67, which is €2,653.63 less.

7.2 Payments made m times a period

Payments of $\frac{1}{m}$ are made m times per unit time, so the total is again 1 per unit time. There are nm payments, at times $\frac{k}{m}$.



SHORTHAND

$q := (1+i)^{1/m}$ (accumulate one m-th of a period) and $w := \mathrm{v}^{1/m} = q^{-1}$ (discount one m-th). Then

$$i^{(m)} = m(q-1), \qquad d^{(m)} = m(1-w), \qquad \frac{i^{(m)}}{d^{(m)}} = \frac{m(q-1)}{m(1-q^{-1})} = q = (1+i)^{1/m}.$$

THE FOUR m-THLY SYMBOLS

Arrears (payments at $\frac{1}{m}, \dots, n$) goes with $i^{(m)}$; advance (payments at $0, \dots, n - \frac{1}{m}$) goes with $d^{(m)}$:

$$a^{(m)}_{\overline{n}|} = \frac{1 - v^n}{i^{(m)}} = \frac{i}{i^{(m)}} a_{\overline{n}|}, \qquad s^{(m)}_{\overline{n}|} = \frac{(1+i)^n - 1}{i^{(m)}} = \frac{i}{i^{(m)}} s_{\overline{n}|},$$

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1 - v^n}{d^{(m)}} = \frac{i}{d^{(m)}} a_{\overline{n}|}, \qquad \ddot{s}^{(m)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(m)}} = \frac{i}{d^{(m)}} s_{\overline{n}|},$$

and, from $i^{(m)}/d^{(m)} = (1+i)^{1/m}$,

$$\ddot{a}^{(m)}_{\overline{n}|} = (1+i)^{1/m}\, a^{(m)}_{\overline{n}|}, \qquad \ddot{s}^{(m)}_{\overline{n}|} = (1+i)^{1/m}\, s^{(m)}_{\overline{n}|}.$$

WHERE IT COMES FROM, AND WHAT IT SAYS

The derivation is a geometric series in q (or w) with nm terms, and the only work is recognising $m(q-1)$ and $m(1-w)$ at the end; it is not examinable. *Interest goes with arrears, discount with advance, at every frequency.* In advance, each payment falls $\frac{1}{m}$ of a period earlier and so is worth one more factor of $(1+i)^{1/m}$. At $m = 1$ this is $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$ again.

VALUING AN m-THLY ANNUITY

1. Find the *total* paid per unit time (€3,000 a month is €36,000 a year) and the number of payments per unit, m.
2. Fix the unit of time (see below) and the effective rate i per unit.
3. Get $i^{(m)}$ (arrears) or $d^{(m)}$ (advance) from the bridge: $i^{(m)} = m[(1+i)^{1/m} - 1]$, $d^{(m)} = m[1 - (1+i)^{-1/m}]$.
4. Value = total $\times \dfrac{1 - v^n}{\text{that rate}}$.

EXAMPLE 7.2 · A quarterly annuity

An annuity pays €1,000 at the end of each quarter for 20 years, at $i = 6\%$ effective. Find its present value.

SOLUTION

Total €4,000 p.a., $m = 4$. $i^{(4)} = 4(1.06^{1/4} - 1) = 0.058695$ and $v^{20} = 0.311805$, so $a^{(4)}_{\overline{20}|} = \dfrac{1 - 0.311805}{0.058695} = 11.724862$ and the value is $4000 \times 11.724862 = €46{,}899.45$. Paid once a year in arrears the same total would be worth $4000\, a_{\overline{20}|} = €45{,}879.68$, about €1,020 less: three quarters of each year's money arrive later.

7.3 The whole family on one page

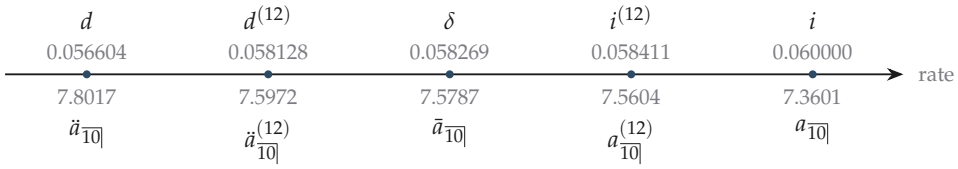
FIVE ANNUITIES, FIVE RATES

Payments	PV	AV	Rate
Yearly, in arrears	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	i
m-thly, in arrears	$a^{(m)}_{\overline{n}\vert}$	$s^{(m)}_{\overline{n}\vert}$	$i^{(m)}$
Continuously	$\bar{a}_{\overline{n}\vert}$	$\bar{s}_{\overline{n}\vert}$	δ
m-thly, in advance	$\ddot{a}^{(m)}_{\overline{n}\vert}$	$\ddot{s}^{(m)}_{\overline{n}\vert}$	$d^{(m)}$
Yearly, in advance	$\ddot{a}_{\overline{n}\vert}$	$\ddot{s}_{\overline{n}\vert}$	d

Dividing by a *smaller* rate gives a *bigger* value, so with $d < d^{(m)} < \delta < i^{(m)} < i$ the annuities run *in the opposite order*:

$$\ddot{a}_{\overline{n}\vert} > \ddot{a}^{(m)}_{\overline{n}\vert} > \bar{a}_{\overline{n}\vert} > a^{(m)}_{\overline{n}\vert} > a_{\overline{n}\vert}.$$

Continuous sits in the middle ("on average the money arrives mid-year"), and $a^{(m)}_{\overline{n}\vert}$, $\ddot{a}^{(m)}_{\overline{n}\vert}$ squeeze onto $\bar{a}_{\overline{n}\vert}$ as $m \to \infty$.



THE CHECK AT $i = 6\%$, $n = 10$

All five pay a total of 10 and differ only in timing. The figure shows their rates and values: whatever you compute for a ten-year annuity of 1 a year must lie between 7.3601 (arrears) and 7.8017 (advance), and in the right slot for its type.

PERPETUITIES AT EVERY FREQUENCY

n appears only in $v^n \to 0$, so each present value survives with numerator 1: value $= 1/\text{rate}$. At $i = 6\%$, $m = 4$:

Payments	Perpetuity value	at $i = 6\%$, $m = 4$
Yearly, in arrears	$1/i$	16.6667
m-thly, in arrears	$1/i^{(m)}$	17.0371
Continuously	$1/\delta$	17.1618
m-thly, in advance	$1/d^{(m)}$	17.2871
Yearly, in advance	$1/d$	17.6667

Still **no** $s_{\overline{\infty}\vert}$ at any frequency.

EXAMPLE 7.3 · A charitable trust

A trust pays €300 at the end of each quarter for ever. At $i = 6\%$ effective, what capital is needed?

SOLUTION

Total €1,200 p.a., quarterly in arrears: $1200/i^{(4)} = 1200/0.058695 = €20{,}444.54$. Paid once a year in arrears the capital would be $1200/0.06 = €20{,}000$; paying the same total earlier costs about €445 more capital, for ever.

7.4 Choosing the unit of time

A QUESTION THAT LOOKS HARDER THAN IT IS

1. Formulas need an *effective* rate per unit time. If the question gives a nominal rate, **let the given rate be effective**: "12% convertible half-yearly" means 6% per half-year, so take the half-year as the unit.
2. Re-express the term, the payment frequency m and the payment rate in that unit. Keep m and n whole numbers if you can.
3. Which unit you choose does not matter, as long as you are consistent: the same payments valued at the same date. Choose the convenient one.

EXAMPLE 7.4 · Quarterly payments, rate convertible half-yearly

An investor buys a level annuity of €120 p.a. payable quarterly in arrears for five years. Find the price on the basis of 12% p.a. convertible half-yearly.

SOLUTION

In half-years: $i = 6\%$ effective, $n = 10$, $m = 2$ payments per unit, €60 per unit. $i^{(2)} = 2(1.06^{1/2} - 1) = 0.059126$, so

$$60\, a^{(2)}_{\overline{10}|} \;=\; 60\,\frac{1 - 0.558395}{0.059126} \;=\; €448.13.$$

In years instead: $i = 1.06^2 - 1 = 0.1236$, $m = 4$, $n = 5$, rate of payment €120: $i^{(4)} = 4(1.1236^{1/4} - 1) = 0.118252$ and $120\, a^{(4)}_{\overline{5}|} = €448.13$. Identical to the cent, as it must be.

THE CHAPTER IN ONE BOX

$$\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta}, \quad a^{(m)}_{\overline{n}|} = \frac{1 - v^n}{i^{(m)}}, \quad \ddot{a}^{(m)}_{\overline{n}|} = \frac{1 - v^n}{d^{(m)}}, \quad \text{AV: replace } 1 - v^n \text{ by } (1+i)^n - 1.$$

$$\bar{a}_{\overline{n}|} = \tfrac{i}{\delta} a_{\overline{n}|}, \qquad a^{(m)}_{\overline{n}|} = \tfrac{i}{i^{(m)}} a_{\overline{n}|}, \qquad \tfrac{i^{(m)}}{d^{(m)}} = (1+i)^{1/m}, \qquad \bar{a}_{\overline{\infty}|} = \tfrac{1}{\delta}.$$

Same numerators, only the rate changes. $d < d^{(m)} < \delta < i^{(m)} < i$, so the annuities run the other way. Total per unit time $\times$ symbol. Make the given rate effective and choose

the unit to suit. No accumulated perpetuity.

TRY IT YOURSELF

1. At $i = 8\%$ and $n = 12$, compute $\bar{a}_{\overline{12}|}$ and $\bar{s}_{\overline{12}|}$, and verify $\bar{a}_{\overline{12}|} = \frac{i}{\delta} a_{\overline{12}|}$.
2. An investor receives €1,200 p.a., payable continuously, for twenty years at $i = 7\%$. Find the present value, and compare it with the same total paid annually in arrears.
3. An annuity pays €3,000 at the end of each month for fifteen years at $i = 7\%$. Find its present value, and the value if payments fell at the start of each month. Check the ratio of the two.
4. A mortgage of €200,000 is repaid by level monthly instalments in arrears over twenty-five years at $i = 5\%$. Find the monthly instalment.
5. A perpetuity pays €500 at the end of each quarter, at $i = 8\%$. Find the capital required, and again if the payments fell at the start of each quarter.
6. Show from the definitions that $d^{(m)} < i^{(m)}$ whenever $i > 0$, and say what that means for the two present values.

ANSWERS

1. $\delta = \ln 1.08 = 0.076961$, $v^{12} = 0.397114$: $\bar{a}_{\overline{12}|} = 7.833655$, $\bar{s}_{\overline{12}|} = 19.726476$. $a_{\overline{12}|} = 7.536078$ and $\frac{i}{\delta} = 1.039487$, product 7.833655. ✓
2. $\delta = 0.067659$, $v^{20} = 0.258419$: $\bar{a}_{\overline{20}|} = 10.960624$, value €13,152.75. Annual: $a_{\overline{20}|} = 10.594014$, €12,712.82, lower by €439.93; the ratio is $i/\delta = 1.034601$ for any term.
3. Total €36,000, $v^{15} = 0.362446$, $i^{(12)} = 0.067850$, $d^{(12)} = 0.067468$. Arrears $a^{(12)}_{\overline{15}|} = 9.396557$, value €338,276.05; advance $\ddot{a}^{(12)}_{\overline{15}|} = 9.449686$, value €340,188.71. Ratio $1.005654 = 1.07^{1/12}$, one month's accumulation.
4. X = total per annum: $i^{(12)} = 0.048889$, $v^{25} = 0.295303$, $a^{(12)}_{\overline{25}|} = 14.414086$, so $X =$ €13,875.32 and the instalment is $X/12 =$ €1,156.28.
5. Total €2,000. Arrears $2000/i^{(4)} = 2000/0.077706 =$ €25,737.98; advance $2000/d^{(4)} = 2000/0.076225 =$ €26,237.98. Difference exactly €500, one payment made today.
6. $d^{(m)} = m(1 - q^{-1}) = m\dfrac{q-1}{q} = \dfrac{i^{(m)}}{q}$ with $q > 1$, so $d^{(m)} < i^{(m)}$. Same numerator $1 - v^n > 0$, smaller denominator, so $a^{(m)}_{\overline{n}|} < \ddot{a}^{(m)}_{\overline{n}|}$: the advance annuity pays the same money earlier.

CHAPTER 8

Deferred and varying annuities

LECTURE 9 · *Payments that wait, grow and shrink*

AFTER THIS CHAPTER YOU CAN

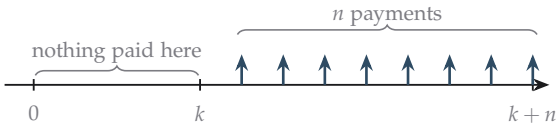
- ☐ Defer any annuity, counting the deferment correctly.
- ☐ Write down and use $(Ia)_{\overline{n}|}$, $(Da)_{\overline{n}|}$ and their advance and accumulated versions.
- ☐ Split a cashflow in arithmetic progression into a level and an increasing part.
- ☐ Avoid the trap of deferring a *varying* annuity by subtraction.
- ☐ Tell the continuous cases apart: a step rate against a smooth ramp.

Two things were true of every annuity so far: all payments had the same size, and the first came at once. Today both assumptions go. Nothing new is needed to see it: *drag each payment back by its own discount factor and add*. The symbols are only a shorthand for the sums that come up most often.

8.1 Waiting before the payments begin

DEFERRED ANNUITY

Payment of an annuity is **deferred** by k units of time if nothing is paid until time k, after which the annuity runs as usual. Its present value is written ${}_{k|}a_{\overline{n}|}$.



The payments fall at times $k+1, \ldots, k+n$. The deferment is k, and the first payment comes one *further* period later, because the annuity itself is in arrears.

TWO ROUTES TO THE SAME NUMBER

$$ {}_{k|}a_{\overline{n}|} \;=\; a_{\overline{k+n}|} - a_{\overline{k}|} \qquad \text{(subtract the years you did not get)}, $$

$$ {}_{k|}a_{\overline{n}|} \;=\; \mathrm{v}^{k}\, a_{\overline{n}|} \qquad \text{(value late, then discount)}. $$

From time k the payments form an ordinary n-period annuity in arrears, so bring that one number back by v^k. The routes agree: $a_{\overline{k+n}|} - a_{\overline{k}|} = \frac{\mathrm{v}^k - \mathrm{v}^{k+n}}{i} = \mathrm{v}^k \frac{1-\mathrm{v}^n}{i}$. **Every** level annuity defers the same way: for $\ddot{a}_{\overline{n}|}$, $a^{(m)}_{\overline{n}|}$, $\ddot{a}^{(m)}_{\overline{n}|}$, $\bar{a}_{\overline{n}|}$ replace the symbol and keep v^k (or the difference). In practice use $\mathrm{v}^k \times$symbol.

HOW BIG IS k?

The deferment is counted to the period *before* the first payment. A first payment at time 6 means $k = 5$. A first payment "in the month after time 4" (monthly, arrears, so the first payment is a month later) means $k = 4$ with $m = 12$ payments a year.

EXAMPLE 8.1 · A deferred pension

A pension of €10,000 p.a. is payable annually in arrears for twenty years, the first payment six years from now. At $i = 5\%$, find the present value.

SOLUTION

First payment at time 6, so $k = 5$, $n = 20$. With $\mathrm{v}^5 = 0.783526$ and $a_{\overline{20}|} = 12.462210$,

$$ 10{,}000\, {}_{5|}a_{\overline{20}|} \;=\; 10{,}000 \times 0.783526 \times 12.462210 \;=\; €97{,}644.68. $$

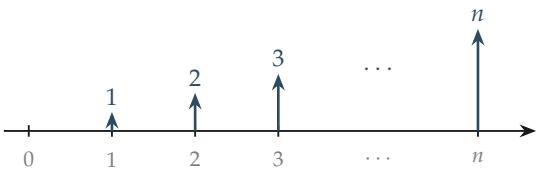
By subtraction, $a_{\overline{25}|} - a_{\overline{5}|} = 14.093945 - 4.329477 = 9.764468$: the same factor.

8.2 Payments that grow by one each period

INCREASING ANNUITY

The present value of payments $1, 2, \ldots, n$ made at times $1, 2, \ldots, n$ is written $(Ia)_{\overline{n}|}$:

$$ (Ia)_{\overline{n}|} \;=\; \mathrm{v} + 2\mathrm{v}^2 + 3\mathrm{v}^3 + \cdots + n\mathrm{v}^n. $$



THE INCREASING ANNUITY

$$\boxed{(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\mathrm{v}^n}{i}}$$

$$(I\ddot{a})_{\overline{n}|} = (1+i)(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\mathrm{v}^n}{d},$$

$$(Is)_{\overline{n}|} = (1+i)^n (Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}.$$

The rule of the previous chapters survives: advance replaces i by d, and moving the valuation date to time n replaces the numerator $\ddot{a}_{\overline{n}|} - n\mathrm{v}^n$ by $\ddot{s}_{\overline{n}|} - n$. This is the transfer principle again.

THE DERIVATION TO KNOW: SHIFT AND SUBTRACT, FOR THE SECOND TIME

$(Ia)_{\overline{n}|}$ is not a geometric progression, but the same trick works. Multiply by $1+i$, which lowers every exponent by one, and subtract the original series:

$$\begin{aligned}
(1+i)(Ia)_{\overline{n}|} &= 1 + 2\mathrm{v} + 3\mathrm{v}^2 + \cdots + n\mathrm{v}^{n-1}, \\
(Ia)_{\overline{n}|} &= \quad \mathrm{v} + 2\mathrm{v}^2 + \cdots + (n-1)\mathrm{v}^{n-1} + n\mathrm{v}^n, \\
i\,(Ia)_{\overline{n}|} &= 1 + \mathrm{v} + \mathrm{v}^2 + \cdots + \mathrm{v}^{n-1} - n\mathrm{v}^n \;=\; \ddot{a}_{\overline{n}|} - n\mathrm{v}^n.
\end{aligned}$$

Every coefficient falls by exactly one, a staircase is flattened into a level annuity, and $n\mathrm{v}^n$ removes the payment that the shift invented at the far end.

EXAMPLE 8.2 · A growing annuity

An annuity pays €1,000 at the end of the first year, €2,000 at the end of the second, and so on, the last payment of €10,000 falling at the end of the tenth year. At $i = 6\%$, find the present value.

SOLUTION

$\ddot{a}_{\overline{10}|} = 7.801692$ and $10\mathrm{v}^{10} = 5.583948$, so

$$(Ia)_{\overline{10}|} = \frac{7.801692 - 5.583948}{0.06} = 36.962408, \qquad 1000\,(Ia)_{\overline{10}|} = €36{,}962.41.$$

The same €55,000 paid level, €5,500 a year, would be worth $5500\,a_{\overline{10}|} = €40{,}480.48$. The growing annuity is worth less, because larger sums arrive later. (Likewise $(Ia)_{\overline{4}|} = 8.410622$ at 6% is the value of payments $1, 2, 3, 4$.)

ROUNDING

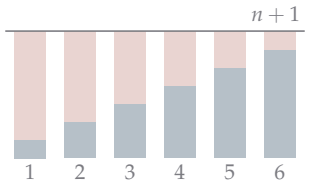
The numerator is a *difference* and then divided by i. Dividing by 0.06 multiplies whatever rounding survives the subtraction by nearly seventeen. Carry more places than the six shown.

8.3 Decreasing payments, and everything between

DECREASING ANNUITY

$(Da)_{\overline{n}|} = nv + (n-1)v^2 + \cdots + v^n$: payments $n, n-1, \ldots, 1$ at times $1, 2, \ldots, n$.

Pay both at once: at time k they hand over k and $n+1-k$, which is $n+1$ whatever k is. Together they are a level annuity of size $n+1$.



Lower block k (increasing), upper block $n+1-k$ (decreasing).

DECREASING ANNUITY

$$(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} \;=\; (n+1)\, a_{\overline{n}|}, \qquad (Da)_{\overline{n}|} \;=\; \frac{n - a_{\overline{n}|}}{i}, \qquad (D\ddot{a})_{\overline{n}|} \;=\; \frac{n - a_{\overline{n}|}}{d}.$$

So no separate derivation is needed: a decreasing annuity is a level annuity minus an increasing one. (Substitute $(Ia)_{\overline{n}|}$, use $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$ and $v^n = 1 - ia_{\overline{n}|}$; everything cancels down to $n - a_{\overline{n}|}$.) Check at $i = 6\%$, $n = 10$: $(Da)_{\overline{10}|} = (10 - 7.360087)/0.06 = 43.99855$, and $(Ia)_{\overline{10}|} + (Da)_{\overline{10}|} = 36.96241 + 43.99855 = 80.96096 = 11\, a_{\overline{10}|}$.

ARITHMETIC PROGRESSIONS

Payment at time t equal to $P + (t-1)Q$ (so P first, then Q more each time) splits as

$$P + (t-1)Q \;=\; (P - Q) + tQ \quad\Longrightarrow\quad \text{PV} \;=\; (P - Q)\, a_{\overline{n}|} + Q\,(Ia)_{\overline{n}|}.$$

A level annuity plus a multiple of an increasing one: no third symbol. For payments that *fall* by Q each time, Q is negative: $\text{PV} = (P+Q)a_{\overline{n}|} - Q\,(Ia)_{\overline{n}|}$, with Q now the size of the fall.

EXAMPLE 8.3 · A rising rent

A tenant pays €1,500 at the end of the first year and each year after that €100 more than the year before, for ten years. At $i = 6\%$, find the present value of the rent.

SOLUTION

$P = 1500$, $Q = 100$, $n = 10$, so $P - Q = 1400$:

$$\begin{aligned} 1400\, a_{\overline{10}|} + 100\,(Ia)_{\overline{10}|} &= 1400 \times 7.360087 + 100 \times 36.962408 \\ &= €10{,}304.12 + €3{,}696.24 \;=\; €14{,}000.36. \end{aligned}$$

The tenant hands over €19,500 in all. Paid level, at €1,950 a year, the same total would be worth $1950\, a_{\overline{10}|} = €14{,}352.17$, so the rising schedule is cheaper by €351.81.

WHICH SYMBOL?

1. List the payments and their dates.
2. Constant $\to a$. $1,2,3,\ldots \to (Ia)_{\overline{n}|}$. $n,\ldots,2,1 \to (Da)_{\overline{n}|}$. Arithmetic progression $\to (P-Q)a + Q(Ia)$.
3. Arrears or advance: i or d in the denominator. Valued at time 0 or n: $\ddot{a}_{\overline{n}|} - nv^n$ or $\ddot{s}_{\overline{n}|} - n$.
4. *First payment later than usual?* See the next section.

8.4 Deferring a varying annuity

DISCOUNTING WORKS DIRECTLY

Payments $1,2,\ldots,n$ at times $k+1,\ldots,k+n$ are worth $(Ia)_{\overline{n}|}$ at time k, so ${}_{k|}(Ia)_{\overline{n}|} = v^k\,(Ia)_{\overline{n}|}$.

SUBTRACTION DOES NOT

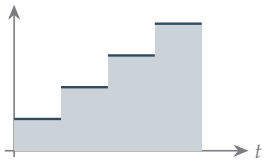
$(Ia)_{\overline{k+n}|} - (Ia)_{\overline{k}|}$ is a *different* cashflow. It pays $k+1,k+2,\ldots,k+n$ at those dates, because the long annuity does not restart its count at the deferment. In fact

$$(Ia)_{\overline{k+n}|} - (Ia)_{\overline{k}|} \;=\; v^k\big((Ia)_{\overline{n}|} + k\,a_{\overline{n}|}\big),$$

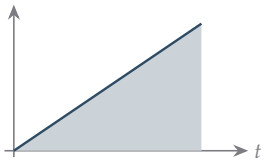
the extra term being the level k that every surviving payment carries. At $i = 6\%$, $k = 5$, $n = 10$: deferment gives $0.747258 \times 36.962408 = 27.620462$, while subtraction gives $67.266800 - 12.146912 = 55.119888$. The two are not close.

8.5 Growth in continuous time

A continuous annuity that grows can mean either of two different cashflows.



$(I\bar{a})_{\overline{n}|}$: a step rate



$(\bar{I}\bar{a})_{\overline{n}|}$: a smooth ramp

- $(I\bar{a})_{\overline{n}|}$ pays continuously at the *constant* rate k throughout the k-th period. Each year's total is still $1,2,\ldots,n$.
- $(\bar{I}\bar{a})_{\overline{n}|}$ pays continuously at rate t at time t: the rate itself never stops rising.

A bar over the a says how the money is paid; a bar over the I says how the amount grows.

THE TWO CONTINUOUS INCREASING ANNUITIES

$$(I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\mathrm{v}^n}{\delta} = \frac{i}{\delta}(Ia)_{\overline{n}|}, \qquad (\bar{I}\bar{a})_{\overline{n}|} = \int_0^n t\,\mathrm{v}^t\,\mathrm{d}t = \frac{\bar{a}_{\overline{n}|} - n\mathrm{v}^n}{\delta}.$$

The denominator says how often the money is paid; the numerator says how the amount grows ($\ddot{a}_{\overline{n}|}$ for a yearly step, $\bar{a}_{\overline{n}|}$ for a continuous one). The second is integration by parts: $\int_0^n t\,\mathrm{e}^{-\delta t}\,\mathrm{d}t = \left[-t\mathrm{e}^{-\delta t}/\delta\right]_0^n + \frac{1}{\delta}\int_0^n \mathrm{e}^{-\delta t}\,\mathrm{d}t$.

Symbol	Payment in the k-th period	Denominator	Value at 5%, $n = 8$
$(Ia)_{\overline{8}|}$	k, at the year end	i	27.4332
$(I\bar{a})_{\overline{8}|}$	k, spread through the year	δ	28.1134
$(\bar{I}\bar{a})_{\overline{8}|}$	on average $k - \frac{1}{2}$, spread	δ	24.7748

WHICH WAY SHOULD IT GO?

$(I\bar{a})_{\overline{8}|} > (Ia)_{\overline{8}|}$: the same yearly totals arriving earlier. $(\bar{I}\bar{a})_{\overline{8}|} < (Ia)_{\overline{8}|}$, and not because of discounting: the ramp simply hands over less money (rate t, so on average $k - \frac{1}{2}$ in the k-th year).

THE CHAPTER IN ONE BOX

$$ {}_{k|}a_{\overline{n}|} = \mathrm{v}^k a_{\overline{n}|} = a_{\overline{k+n}|} - a_{\overline{k}|}, \qquad (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\mathrm{v}^n}{i}, \qquad (Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i},$$

$$(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} = (n+1)a_{\overline{n}|}, \qquad P + (t-1)Q \;\to\; (P-Q)a_{\overline{n}|} + Q\,(Ia)_{\overline{n}|},$$

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - n\mathrm{v}^n}{\delta}.$$

Advance: divide by d. Continuous: divide by δ. At time n: replace $\ddot{a}_{\overline{n}|} - n\mathrm{v}^n$ by $\ddot{s}_{\overline{n}|} - n$. Deferment multiplies by v^k, but only when the payments themselves restart at 1; never subtract two varying annuities.

TRY IT YOURSELF

1. An annuity of €5,000 per month in arrears runs for ten years, the first payment falling in the month after time 4. At $i = 6\%$, find its value today.
2. Prove $(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} = (n+1)a_{\overline{n}|}$ from the definitions, and deduce the closed form of $(Da)_{\overline{n}|}$. Evaluate $(Da)_{\overline{10}|}$ at $i = 6\%$.
3. An annuity pays €800 at the end of the first year and €50 more in each year after that, for twelve years, at $i = 7\%$. Find its present value.
4. Payments are made continuously for ten years at the rate €1,000 t per annum at time t. At $i = 4\%$, find the present value.

ANSWERS

1. Total €60,000 p.a., $m = 12$, $n = 10$, $k = 4$. $i^{(12)} = 0.058411$, $\mathrm{v}^{10} = 0.558395$,

$a^{(12)}_{\overline{10}|} = 7.560360$, $v^4 = 0.792094$: $60000 \times 0.792094 \times 7.560360 = €359{,}310.80$. Other route: $a^{(12)}_{\overline{14}|} - a^{(12)}_{\overline{4}|} = 5.988513 = 0.792094 \times 7.560360$.

2. Payment at time k is k and $n+1-k$, so together $\sum_{k=1}^{n}(n+1)v^k = (n+1)a_{\overline{n}|}$. Then $(Da)_{\overline{n}|} = (n+1)a_{\overline{n}|} - \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} = \frac{n - a_{\overline{n}|}}{i}$, as above. $(Da)_{\overline{10}|} = (10 - 7.360087)/0.06 = 43.99855$ (payments $10, 9, \ldots, 1$, total 55).

3. $P = 800$, $Q = 50$, $n = 12$: $750\, a_{\overline{12}|} + 50\,(Ia)_{\overline{12}|}$. $v^{12} = 0.444012$, $a_{\overline{12}|} = 7.942686$, $\ddot{a}_{\overline{12}|} = 8.498674$, $(Ia)_{\overline{12}|} = (8.498674 - 12 \times 0.444012)/0.07 = 45.293298$. Value $5957.01 + 2264.66 = €8{,}221.68$.

4. A smooth ramp: $1000\,(\bar{I}\bar{a})_{\overline{10}|}$. $\delta = 0.0392207$, $v^{10} = 0.6755642$, $\bar{a}_{\overline{10}|} = 8.2720533$, $(\bar{I}\bar{a})_{\overline{10}|} = (8.2720533 - 6.7556417)/0.0392207 = 38.663540$: €38,663.54. (Level €5,000 p.a. continuously, the same total €50,000, would be $5000\bar{a}_{\overline{10}|} = €41{,}360.27$; the ramp's money arrives late.)

PART III

Yield and uncertainty

CHAPTER 9

The yield of a project

LECTURE 10 · *Reading the equations the other way round*

AFTER THIS CHAPTER YOU CAN

- ☐ Treat NPV as a function of i and set up the equation of value.
- ☐ Say when the yield exists and is unique.
- ☐ Bracket a root and find the yield by linear interpolation.
- ☐ Say whether the estimate is too high or too low, and why.
- ☐ Solve a two-payment yield exactly.

So far the rate was given and we found the value. Now we read the same formulas backwards: given a cashflow, ask *at what rate is the project worth exactly nothing?* That rate is the project's **yield**.

9.1 The value as a function of the rate

NET PRESENT VALUE AS A FUNCTION OF i

For a project with net cashflows c_t and $\rho(t)$, ending at time T (and $T = \infty$ is allowed),

$$\mathrm{NPV}(i) \;=\; \sum_t \mathsf{v}^t c_t + \int_0^T \mathsf{v}^t \rho(t)\,\mathrm{d}t, \qquad \mathsf{v} = \frac{1}{1+i} = \mathsf{e}^{-\delta}.$$

$\mathrm{NPV}(i)$ is defined and continuous for $i > -1$. (For $i \le -1$, $\ln(1+i)$ does not exist.)

NOW THE RATE IS A CONSTANT

In Chapters 3–4 the cashflow varied and the discounting was a function of time. Here the cashflow is fixed and NPV is a function of *one number* i: the force is constant. Otherwise there would be no unique rate to solve for.

EXAMPLE 9.1 · A project to follow through the chapter

A software company sets up a system for a client, contracting out the work. Costs (in €1,000s): 150 at the start of year 1, 250 at the start of year 2, 250 at the start of year 3. Sales of 1000 at the end of year 3. Find the net present value at 20%, the rate at which the company can lend or borrow.

SOLUTION

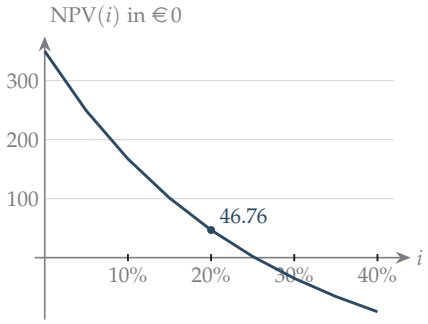
Start of year 1 is time 0; end of year 3 is time 3.

$$\mathrm{NPV}(i) \;=\; -150 - 250v - 250v^2 + 1000v^3.$$

At $i = 0.2$, $v = 0.833333$:

t	c_t	v^t	$v^t c_t$
0	−150	1.000000	−150.00
1	−250	0.833333	−208.33
2	−250	0.694444	−173.61
3	1,000	0.578704	578.70
		NPV(0.2)	46.76

So $\mathrm{NPV}(0.2) = €46{,}759 > 0$: at a borrowing rate of 20% the project is worth doing.



At $i = 0$ nothing is discounted: $\mathrm{NPV}(0) = 1000 - 650 = 350$. As i rises the late 1000 is discounted harder than the early outgo, so the curve falls, crossing zero near 25%. Above that, funding the project costs more than it earns.

9.2 The equation of value and the yield

EQUATION OF VALUE, YIELD

The **equation of value** (or **yield equation**) is $\mathrm{NPV} = 0$: $\sum_t e^{-\delta t} c_t + \int_0^T e^{-\delta t}\rho(t)\,dt = 0$, equally written in $(1+i)^{-t}$ since $e^{\delta} = 1 + i$. If it has a **unique** root δ_0 (with $i > -1$), then $i_0 = e^{\delta_0} - 1$ is the **yield per unit time**, the **force of interest implied by the transaction**. In finance it is the **internal rate of return** (IRR).

WHAT THE YIELD IS, IN WORDS

If all the transaction's inflows and outflows went through a hypothetical account, at what interest rate would the balance be exactly zero at the end? It is the effective rate of return the transaction generates, and it answers: *is this project worth doing, or would I do better in a risk-free asset?* If the yield exceeds the risk-free rate, do it.

WHEN IS THE YIELD DEFINED?

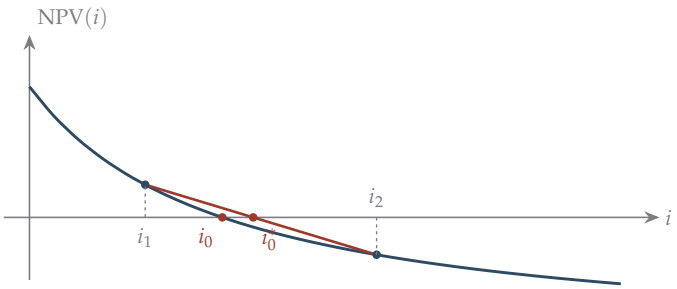
Defined *if and only if* the equation has *precisely one* root with $i > -1$. Roots at or below -1 are inadmissible. The equation may have no root, one or several. Two ways to know there is exactly one:

- **Sign change plus monotonicity.** If NPV is monotonic and $\mathrm{NPV}(i_1)\,\mathrm{NPV}(i_2) < 0$, continuity gives a root between them and monotonicity forbids a second.
- **Spend first, earn later.** A unique root exists if all the negative net cashflows precede all the positive ones (or vice versa). Many projects are like this: invest at the start, earn after.

And a quick no: **if all the cashflows have the same sign there is no solution** (terms cannot cancel).

9.3 Finding the yield

The yield equation is a polynomial in v whose degree is the term of the project. Beyond two payments there is no general formula, so we estimate.



Replace the curve by the chord between $(i_1, \mathrm{NPV}(i_1))$ and $(i_2, \mathrm{NPV}(i_2))$, which we *can* solve, and hope the two cross the axis at nearly the same place. The chord meets the axis at i_0^*, a little to the right of the true root i_0.

LINEAR INTERPOLATION

Find $i_1 < i_2$, close together, with $\mathrm{NPV}(i_1)\ \mathrm{NPV}(i_2) < 0$. Then

$$\boxed{i_0 \;\approx\; i_0^* \;=\; i_1 + \frac{\mathrm{NPV}(i_1)}{\mathrm{NPV}(i_1) - \mathrm{NPV}(i_2)}\,(i_2 - i_1)}$$

Memorable reading: the fraction $\dfrac{\mathrm{NPV}(i_1)}{\mathrm{NPV}(i_1) + |\,\mathrm{NPV}(i_2)|}$ lies in $(0,1)$, so i_0^* divides $[i_1, i_2]$ in the ratio of the two magnitudes, *nearer the end where* $|\,\mathrm{NPV}\,|$ *is smaller*. If $\mathrm{NPV}(i_0^*) \neq 0$, keep the end of the bracket with the opposite sign and repeat.

FINDING A YIELD

1. Write $\mathrm{NPV}(i)$. Check the yield exists (one sign change; outgo before income).
2. Guess a trial rate: sketch, or use $\mathrm{NPV}(0)$ and a rough size. Evaluate.
3. Move the rate in the direction that brings NPV towards 0 until the sign changes: a *narrow* bracket, say one percentage point.
4. Interpolate. State the answer with sensible accuracy.
5. If asked, check the sign of NPV at the estimate, or tighten the bracket.

EXAMPLE 9.2 · The warm-up project

An investor pays €10,000 now and receives €3,000 at the end of each of five years. At 8%, NPV $= -10000 + 3000a_{\overline{5}|} =$ €1,978.13 > 0. Find the yield. Interpolate on $[0.15, 0.16]$.

SOLUTION

All outgo precedes all income, so the yield exists. $\mathrm{NPV}(0.15) = -10000 + 3000(3.352155) = +56.47$ and $\mathrm{NPV}(0.16) = -177.12$, so

$$i_0 \;\approx\; 0.15 + \frac{56.47}{56.47 + 177.12}(0.01) \;=\; 15.24\%.$$

The 8% left a wide margin: the project pays for borrowing at anything under about 15.2%.

EXAMPLE 9.3 · The software project: the yield

Find the yield on the project of Section 9.1.

SOLUTION

The outgo (times $0, 1, 2$) precedes the income (time 3), so the root is unique; from the graph it is near 25%. $\mathrm{NPV}(0.25) = -150 - 200 - 160 + 512 = +2$. Only just positive, so try $i_2 = 0.26$ ($\mathrm{v} = 0.793651$): $\mathrm{NPV}(0.26) = -150 - 198.413 - 157.470 + 499.906 = -5.977$. Then

$$i_0 \;\approx\; 0.25 + \frac{2}{2 - (-5.977)}(0.01) \;=\; 0.2525072, \qquad \text{about } 25.25\% \text{ p.a.}$$

HOW GOOD IS THE ESTIMATE? KEEP THE BRACKET NARROW

Between i_1 and i_2 the chord lies *above* a convex NPV curve, so it reaches zero slightly later: linear interpolation *overstates* the yield. The error depends on how much the curve bends across the bracket; halving the width roughly quarters it. For the software project (true root 0.2524721):

bracket	$\mathrm{NPV}(i_1)$	$\mathrm{NPV}(i_2)$	estimate
[0.20, 0.30]	46.759	−35.071	0.2571421
[0.24, 0.28]	10.283	−21.063	0.2531221
[0.25, 0.26]	2.000	−5.977	0.2525072

Every estimate is too high, and narrowing the bracket cures it.

EXAMPLE 9.4 · Two payments: solve it exactly

An investor pays €1,000 now for €600 at the end of each of two years. Find the yield exactly.

SOLUTION

$-1000 + 600v + 600v^2 = 0$; divide by 200: $3v^2 + 3v - 5 = 0$. So

$$v = \frac{-3 + \sqrt{69}}{6} = 0.8844373 \quad \text{(positive root, as } v > 0\text{)}, \qquad i_0 = \frac{1}{v} - 1 = 0.1306624.$$

Interpolating instead on $[0.13, 0.14]$: $\mathrm{NPV}(0.13) = 0.8615$ and $\mathrm{NPV}(0.14) = -12.0037$ give 0.130670, overshooting by 8×10^{-6}. Two payments is the largest case with an exact route (three would need a cubic).

EXAMPLE 9.5 · Growth that cancels discounting

Income is paid annually in arrears for 22 years, the first payment 1.05 and each later one 5% larger. Value it at $i = 5\%$.

SOLUTION

Payment at time j is 1.05^j and $v = 1/1.05$, so every term is $1.05^j \times 1.05^{-j} = 1$. The present value is 22. Growth at exactly the rate of discounting contributes the same value every year. If the growth g and the rate differ, the terms are $\left(\frac{1+g}{1+i}\right)^j$, a geometric sum with that ratio.

THE CHAPTER IN ONE BOX

$$\mathrm{NPV}(i) = \sum_t v^t c_t + \int_0^T v^t \rho(t)\,\mathrm{d}t \quad (i > -1),$$

$$\mathrm{NPV}(i_0) = 0 \text{ with } i_0 \text{ unique} \implies i_0 = \text{yield},$$

$$i_0 \;\approx\; i_1 + \frac{\mathrm{NPV}(i_1)}{\mathrm{NPV}(i_1) - \mathrm{NPV}(i_2)}\,(i_2 - i_1).$$

A unique root is guaranteed when all the outgo precedes all the income. No root if all cashflows share a sign. Bracket with a sign change, take the chord, repeat. The estimate is usually a little high; narrow the bracket. Two payments: solve the quadratic in v.

TRY IT YOURSELF

1. A project costs €5,000 now and returns €2,000 at the end of each of three years. Find $\mathrm{NPV}(0.09)$, $\mathrm{NPV}(0.10)$, and the yield.
2. An investor pays €1,000 now for €600 at the end of each of two years. Show that the yield satisfies $3v^2 + 3v - 5 = 0$; solve it exactly, then by interpolation on $[0.13, 0.14]$.
3. Income is paid annually in arrears for 22 years, the first payment being 1.05 and each later one 5% larger. Value it at $i = 5\%$.

ANSWERS

1. $\mathrm{NPV}(i) = -5000 + 2000a_{\overline{3}|}$; outgo precedes income, so the yield exists. $a_{\overline{3}|} = 2.531295$ at 9%: NPV $= +62.59$; $a_{\overline{3}|} = 2.486852$ at 10%: NPV $= -26.30$. $i_0 \approx 0.09 + \frac{62.59}{62.59+26.30}(0.01) = 0.09704$. The true root is 0.09701, so the estimate is high by 3×10^{-5}.
2. Equation $-1000 + 600v + 600v^2 = 0$, i.e. $3v^2 + 3v - 5 = 0$: $v = \frac{-3+\sqrt{69}}{6} = 0.8844373$, $i_0 = 0.1306624$. By interpolation 0.130670.
3. Each term is 1: present value 22.

CHAPTER 10

Cashflows that may not arrive

LECTURE 11 · *Factoring risk (probability) in as a second discount*

AFTER THIS CHAPTER YOU CAN

- ☐ Attach a probability to each payment and find the expected present value.
- ☐ Use $p(t) = \mathrm{e}^{-\mu t}$, so that $\delta^* = \delta + \mu$.
- ☐ See that "discount harder" and "attach probabilities" can be the same thing.
- ☐ Check a risky project before calculating (does the expected income even cover the outlay?).
- ☐ Find the yield of a risky project with a substitution.

Every payment so far was *certain*, in full and on time. In real life risk is one of the most important factors; today it enters the model.

10.1 Allowing for uncertainty

TWO WAYS TO COMPENSATE FOR RISK

1. **Attach probabilities.** Replace the equation of value by

$$\sum_t p_t \, \mathrm{v}^t c_t + \int_0^T p(t) \, \mathrm{v}^t \rho(t) \, \mathrm{d}t \;=\; 0,$$

   where p_t and $p(t)$ are the probabilities of the cashflow at time t. Each term is: probability of arriving $\times$ discount factor $\times$ amount.
2. **Discount harder.** Use a higher rate and leave the reason implicit. This is what a "risk premium" or "risk-adjusted discount rate" means.

They sound different but, as we show next, in many cases they agree.

EXAMPLE 10.1 · A payment that arrives with probability 0.8

A payment of €1,000 is due at time 3 with probability 0.8. $i = 6\%$. (a) What is the expected present value? (b) At what rate would a *certain* €1,000 at time 3 have the same present value?

SOLUTION

(a) Discount, then weight by the chance of arriving: $1000 \times 0.8 \times 1.06^{-3} = 800 \times 0.839619 = €671.70$.
(b) $1000(1+i^*)^{-3} = 671.70$, so $(1+i^*)^3 = 1000/671.70 = 1.48877$ and $i^* = 1.48877^{1/3} - 1 = 14.185\%$. "6% with probability 0.8" has the same expected value as "14.2%, guaranteed".

PROBABILITY AS A SECOND DISCOUNT

Take $p(t) = \mathrm{e}^{-\mu t}$ with $\mu > 0$. It is natural: $p(0) = 1$; it decreases; and over $s+t$ it multiplies, $p(s+t) = p(s)p(t)$. A cashflow further away is less likely to materialise, at a constant rate; the same assumption that modelled growth in Chapter 2. (Pensions use life expectancy this way, developed in MA3992.) Since $\mathrm{e}^{-\delta t}\mathrm{e}^{-\mu t} = \mathrm{e}^{-(\delta+\mu)t}$,

$$\sum_t \mathrm{e}^{-\delta^* t} c_t + \int_0^T \mathrm{e}^{-\delta^* t} \rho(t)\,\mathrm{d}t = 0, \qquad \boxed{\delta^* = \delta + \mu}$$

Risk contributes one more term to the exponent.

FORCES ADD, EFFECTIVE RATES DO NOT

It is the *forces* that differ by exactly μ, not the effective rates. $i^* = \mathrm{e}^{\delta+\mu} - 1 \neq i + \mu$. In the example: $\mathrm{e}^{-3\mu} = 0.8$ gives $\mu = 0.0744$, so $\delta^* = 0.0583 + 0.0744 = 0.1327$ and $i^* = \mathrm{e}^{0.1327} - 1 = 14.185\%$, the same as in (b).

EXAMPLE 10.2 · Same cashflow, two analysts

(a) Discounting at $\delta = 0.06$ with each payment arriving with probability $\mathrm{e}^{-\mu t}$, $\mu = 0.02$: find δ^* and i^*. (b) One analyst discounts at $\delta = 0.05$ with probabilities $\mathrm{e}^{-0.03t}$; another at $\delta^* = 0.08$ with none. Do they agree?

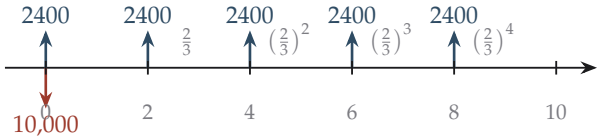
SOLUTION

(a) $\delta^* = 0.06 + 0.02 = 0.08$, $i^* = \mathrm{e}^{0.08} - 1 = 8.3287\%$, against $\mathrm{e}^{0.06} - 1 = 6.1837\%$ from the force alone: the uncertainty costs a little over two percentage points of effective rate.
(b) In the first analyst's equation every term carries $\mathrm{e}^{-0.05t}\mathrm{e}^{-0.03t} = \mathrm{e}^{-0.08t}$, exactly the second analyst's discount factor, so the equations agree term by term. The first is explicit about *why*; the second must simply believe that 8% is the right all-in rate.

10.2 A project with uncertain receipts

EXAMPLE 10.3 · The question

For an initial investment of €10,000, an investor expects to receive €2,400 paid in advance every two years for the next ten years, with probability $\left(\frac{2}{3}\right)^{k-1}$ of receiving the k-th payment. What is the expected yield per annum?



Five receipts, at times $0, 2, 4, 6, 8$. The first is certain; each one after that is two-thirds as likely as the one before.

A YIELD EQUATION IN DISGUISE

1. **Look before calculating.** Add up what the investor *expects* to receive, ignoring discounting. If it is less than the outlay, discounting only makes it worse: the yield is negative.
2. Write the equation of value, with probabilities attached. Recognise a geometric progression.
3. **Substitute** the repeated expression by one letter (here $x = \frac{2}{3}v^2$) and solve the tidier equation in x.
4. Check uniqueness (a polynomial with positive coefficients is strictly increasing), get a bracket from $f(0)$ and $f(1)$, interpolate.
5. Translate back to v, then i.

EXAMPLE 10.4 · The solution

SOLUTION

Look first: expected receipts are $2400\left(1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}\right) = 2400 \times 2.604938 =$ €6,251.85 $<$ €10,000. The yield cannot be positive: we are looking for a root with $i_0 < 0$.
Equation (in €1,000s): the bracket is a geometric progression with first term 1 and ratio $x = \frac{2}{3}v^2$,

$$0 = -10 + 2.4\left(1 + x + x^2 + x^3 + x^4\right) = -10 + 2.4\,\frac{1-x^5}{1-x} =: f(x).$$

f is a polynomial with positive coefficients, so strictly increasing for $x > 0$; $f(0) = -7.6 < 0$ and $f(1) = -10 + 2.4 \times 5 = +2 > 0$. So exactly one positive root, in $(0,1)$, and since $f(1)$ is much nearer zero than $f(0)$ it lies well over towards 1.
Trial values: $f(0.9) = -10 + 2.4 \times 4.0951 = -0.17176 < 0$ and $f(0.95) = -10 + 2.4 \times 4.52438 = +0.858515 > 0$. (The signs run the opposite way from Chapter 9's NPV, which

fell; the algorithm does not care.) Then

$$x_0 \approx 0.9 + \frac{-0.17176}{-0.17176 - 0.858515}(0.05) = 0.9 + 0.166713 \times 0.05 = 0.9083356.$$

Back to a yield: $\mathsf{v}^2 = \frac{3}{2}x$, so $1 + i = (\frac{3}{2}x)^{-1/2}$ and

$$i_0 \approx \left(\tfrac{3}{2} \times 0.9083356\right)^{-1/2} - 1 = -14.33\%.$$

(The exact root $x_0 = 0.9087055$ gives -14.35%.)

A NEGATIVE YIELD IS A LEGITIMATE ROOT

It satisfies $i_0 > -1$, so $\delta_0 = \ln(1 + i_0)$ exists. It says the investor expects to *lose* about 14% a year, unsurprising when the probabilities cut the expected receipts to €6,251.85 against €10,000 paid out. Knowing the exact projected loss can still be very helpful.

THE CHAPTER IN ONE BOX

$$\sum_t p_t \mathsf{v}^t c_t + \int_0^T p(t)\mathsf{v}^t \rho(t)\,\mathrm{d}t = 0, \qquad p(t) = \mathsf{e}^{-\mu t} \implies \delta^* = \delta + \mu.$$

Attaching probabilities can often be reduced to discounting harder, especially if p has the same form as v. Forces add; effective rates do not. Before solving a yield, compare expected income with outlay. In a yield equation, see whether a common expression can be substituted by one letter.

TRY IT YOURSELF

1. A cashflow is discounted at a force $\delta = 0.06$, each payment arriving with probability $\mathsf{e}^{-\mu t}$, $\mu = 0.02$. Find δ^* and i^*, and compare with δ alone.
2. Two analysts value one risky cashflow. One discounts at $\delta = 0.05$ with survival probabilities $\mathsf{e}^{-0.03t}$; the other at $\delta^* = 0.08$ with none. Show that they must agree.
3. An investor pays €5,000 now for €1,500 at the end of years $2, 4, 6, 8$. Find the yield, using a substitution.

ANSWERS

1. $\delta^* = 0.08$, $i^* = \mathsf{e}^{0.08} - 1 = 8.3287\%$ against $\mathsf{e}^{0.06} - 1 = 6.1837\%$.
2. Each term carries $\mathsf{e}^{-0.05t}\mathsf{e}^{-0.03t} = \mathsf{e}^{-0.08t}$: the same discount factor as the second analyst, term by term.
3. Even times only, so put $x = \mathsf{v}^2$: $g(x) = 1500(x + x^2 + x^3 + x^4) - 5000 = 0$, strictly increasing for $x > 0$; $g(0) = -5000$, $g(1) = 1000$. $g(0.9) = -357.35$, $g(0.95) = +286.57$, so $x_0 \approx 0.9 + \frac{357.35}{357.35+286.57}(0.05) = 0.92775$ and $1 + i = x_0^{-1/2}$: $i_0 \approx 3.82\%$. (Exact root 0.928386, giving 3.79%.)

PART IV

Loans

CHAPTER 11

Inside a loan repayment

LECTURES 12–13 · *Interest, capital, what is still owed, and when it tips*

AFTER THIS CHAPTER YOU CAN

- ☐ Split an instalment into interest and capital.
- ☐ Find the loan outstanding at any time, prospectively and retrospectively.
- ☐ Use the closed forms for level instalments, and the fact that the capital content grows by $(1+i)$ each period.
- ☐ Find the interest and capital paid over a block of instalments from two balances.
- ☐ Find the instalment at which capital first exceeds interest.

Until now a loan was a single cashflow: we equated the sum borrowed with the value of the repayments, and found the instalment. But how much of *each* payment was interest and how much repaid the capital? That matters for tax, for early repayment, and for understanding what you are paying for.

11.1 What a repayment is doing

EXAMPLE 11.1 · A three-year loan

A bank lends €1,000 in return for three level payments of X, one at the end of each year, at an effective rate of 7%. Find X and construct the schedule.

SOLUTION

Three payments in arrears: $1000 = Xa_{\overline{3}|}$, so $X = 1000 \times 0.07/(1 - 1.07^{-3}) = €381.05$. Each payment pays the interest due on what is *still outstanding*, then reduces the capital with the rest.

t	Payment	Interest	Capital	Outstanding
0	—	—	—	1,000.00
1	381.05	70.00	311.05	688.95
2	381.05	48.23	332.83	356.12
3	381.05	24.93	356.12	0.00
	1,143.15	143.15	1,000.00	

Interest at time 2 is 7% of €688.95 = €48.23: you pay interest only on what is still outstanding.

TWO CHECKS ON ANY SCHEDULE

The **capital column sums to the loan**, and the **balance runs exactly to zero**. The interest column then sums to the total paid less the loan ($1143.15 - 1000 = 143.15$). With level instalments the interest falls as the capital is repaid, so the capital content must rise.

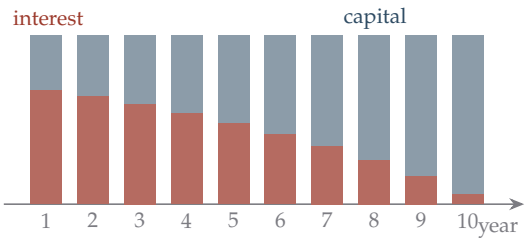
NOTATION

L_t: loan outstanding at time $t = 0, 1, \ldots, n$, *immediately after* the repayment at t. X_t: the instalment at t (not assumed level). b_t: interest paid at t; f_t: capital repaid at t. Then

$$X_t = b_t + f_t, \qquad \boxed{b_t = iL_{t-1}, \qquad f_t = X_t - iL_{t-1} = L_{t-1} - L_t}$$

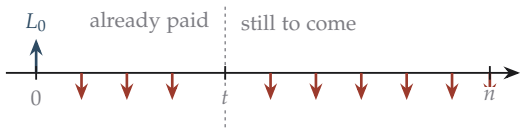
and the equation of value at time 0 is $L_0 = X_1 v + X_2 v^2 + \cdots + X_n v^n$.

Interest first, capital second. Over a period the debt grows by iL_{t-1}; the payment settles that first, and whatever is left over reduces the debt.



A €10,000 ten-year loan, level monthly instalments, 1% per month: every bar is the same height.

11.2 The loan outstanding



Standing at time t, what is owed can be said by looking forward at what is still to come, or backward at what has happened. They agree.

PROSPECTIVE AND RETROSPECTIVE

$$L_t \;=\; X_{t+1}v + X_{t+2}v^2 + \cdots + X_n v^{n-t} \;=\; \sum_{k=t+1}^{n} X_k v^{k-t} \qquad \text{(prospective)}$$

$$L_t \;=\; L_0(1+i)^t - \sum_{k=1}^{t} X_k (1+i)^{t-k} \qquad \text{(retrospective)}$$

Prospective: the outstanding loan is the discounted value at t of the future instalments. **Retrospective:** it is the accumulated value at t of the original loan, less the accumulated value of the repayments made to date. Both come from the one-step recursion

$$L_t \;=\; L_{t-1}(1+i) - X_t \quad\Longleftrightarrow\quad L_t \;=\; (L_{t+1} + X_{t+1})v.$$

WHY THEY MUST AGREE

Substitute $L_0 = \sum_{k=1}^{n} X_k v^k$ into the retrospective formula: the first t terms cancel and what is left is $(1+i)^t \sum_{k>t} X_k v^k$, the prospective sum. Both are the *same* sum, split at t. The prospective one prices the terms after t, the retrospective one the terms before it, and the equation of value says the two halves make up the whole loan. (Also: *money has no memory*. At t you could refinance L_t afresh and nothing would change.)

LEVEL INSTALMENTS

With $X_t = X$ throughout, both sums are annuities:

$$L_t \;=\; X\, a_{\overline{n-t}|} \;=\; L_0(1+i)^t - X\, s_{\overline{t}|}.$$

LOAN OUTSTANDING AT TIME t

1. Find the instalment first ($X = L_0 / a_{\overline{n}|}$ if level).
2. **Prospective** if few payments remain or the instalments are level: $L_t = X a_{\overline{n-t}|}$, one number.
3. **Retrospective** if you are near the start, *or if the future instalments are not yet known* (for example when the rate is reset each year).
4. Check: the other method gives the same number.

EXAMPLE 11.2 · Two methods, one answer

A loan of €20,000 is repaid by five level annual instalments in arrears at $i = 6\%$. Find the instalment and L_2, both ways.

SOLUTION

$a_{\overline{5}|} = 4.212364$, so $X = 20000/4.212364 = €4{,}747.93$. *Prospectively* (three payments remain): $L_2 = Xa_{\overline{3}|} = 4747.93 \times 2.673012 = €12{,}691.27$. *Retrospectively*: $L_2 = 20000(1.06)^2 - Xs_{\overline{2}|} = 22472.00 - 4747.93 \times 2.06 = €12{,}691.27$. (For the three-year loan above, likewise: $L_1 = 381.05\, a_{\overline{2}|} = €688.95$ and $L_2 = 381.05\, a_{\overline{1}|} = €356.12$, without building the schedule.)

11.3 Interest and capital in closed form

LEVEL INSTALMENTS: INTEREST AND CAPITAL

Since $b_t = iL_{t-1} = iXa_{\overline{n-t+1}|}$ and $ia_{\overline{m}|} = 1 - v^m$,

$$b_t = X\big(1 - v^{n-t+1}\big), \qquad f_t = L_{t-1} - L_t = X\big(a_{\overline{n-t+1}|} - a_{\overline{n-t}|}\big) = Xv^{n-t+1},$$

and they add to X. Consequently

$$\boxed{f_t = (1+i)\, f_{t-1} = f_1(1+i)^{t-1}}, \qquad f_1 = X - iL_0 = Xv^n.$$

The capital content grows by exactly $(1+i)$ every period, and the interest content falls to match. **Knowing any one capital repayment gives you all the others.**

WHY CAPITAL GROWS AT RATE i

Imagine paying only the interest each period and putting the rest in a savings account at the *same* rate. The loan stays put and the savings grow at i, until they reach the amount owed and clear it. The capital portions of a level repayment are exactly those savings deposits, which is why they compound. (This is a *sinking fund*; not part of this module.)

EXAMPLE 11.3 · Splitting an instalment

In the €20,000 loan above, split the third instalment into interest and capital. And for a €50,000 loan over ten level annual instalments at 5%, find the capital repaid in the first and last years and explain their ratio.

SOLUTION

$b_3 = X(1 - v^3) = 4747.93 \times 0.160381 = €761.48$ and $f_3 = Xv^3 = 4747.93 \times 0.839619 = €3{,}986.45$; they sum to X. (Check: $b_3 = 0.06\, L_2 = 0.06 \times 12691.27 = 761.48$.)
Second loan: $a_{\overline{10}|} = 7.721735$, $X = €6{,}475.23$. $f_1 = Xv^{10} = €3{,}975.23$ and $f_{10} = Xv = €6{,}166.88$. The ratio is $(1+i)^9 = 1.05^9 = 1.5513$: the capital grows by $(1+i)$ in each of the nine years between them.

11.4 A block of instalments

How much of the money handed over in between was interest, and how much was capital, over instalments $t+1, \dots, t+r$? No individual balance is needed.

INTEREST AND CAPITAL OVER A BLOCK

The capital repayments telescope: $f_k = L_{k-1} - L_k$, so

$$\sum_{k=t+1}^{t+r} f_k \;=\; L_t - L_{t+r}.$$

Since every instalment is $X_k = f_k + b_k$,

$$\boxed{\sum_{k=t+1}^{t+r} b_k \;=\; \sum_{k=t+1}^{t+r} X_k - \left(L_t - L_{t+r}\right)}$$

Interest paid = money handed over − reduction in the debt. Only the two end balances are needed. Nothing here used level instalments or a constant rate.

EXAMPLE 11.4 · A ten-year monthly loan

A loan of €10,000 is repaid by 120 monthly instalments in arrears at 1% per month effective. Find the instalment, the first month's split, and the interest and capital paid in the first and in the last year.

SOLUTION

Month as the unit: $i = 0.01$, $n = 120$, $X = 10000 \times 0.01/(1 - 1.01^{-120}) = 100/0.697005 =$ €143.47. First month: $b_1 = 0.01 \times 10000 =$ €100.00, $f_1 =$ €43.47. Total repaid $120X =$ €17,216.40.

First year. $L_{12} = Xa_{\overline{108}|} = 143.47 \times 65.857790 =$ €9,448.62, so capital repaid is $10000 - 9448.62 =$ €551.38. Money paid in a year is $12 \times 143.47 =$ €1,721.64, so interest is $1721.64 - 551.38 =$ €1,170.26: more than twice the capital.

Last year. The capital repaid is what was outstanding at its start, $L_{108} = Xa_{\overline{12}|} = 143.47 \times 11.255077 =$ €1,614.77, so interest is $1721.64 - 1614.77 =$ €106.87: the capital is fifteen times the interest.

ROUND ONCE, AT THE END

The first-year capital came out as €551.38 (via L_{12}), €551.31 (via $f_1\, s_{\overline{12}|} = 43.47 \times 12.682503$) and €551.32 at full precision. Neither route is wrong: rounding X or f_1 to the cent, or even carrying six digits in L_{12}, changes the answer in the last place. On a real loan the final instalment is adjusted by a few cents for the same reason.

11.5 When does capital overtake interest?

FIRST INSTALMENT WITH CAPITAL > INTEREST

1. Since $X = f_t + b_t$: $b_t < f_t \iff X < 2f_t$.
2. With $f_t = Xv^{n-t+1}$, X cancels: $\frac{1}{2} < (1+i)^{-(n-t+1)}$, i.e. $(1+i)^{n-t+1} < 2$.
3. Take logarithms: $n - t + 1 < \dfrac{\ln 2}{\ln(1+i)}$, so $t > n + 1 - \dfrac{\ln 2}{\ln(1+i)}$; take the next whole number.

EXAMPLE 11.5 · The crossover month

For the ten-year monthly loan ($n = 120$, $i = 0.01$): $\dfrac{\ln 2}{\ln 1.01} = 69.66$, so $t > 121 - 69.66 = 51.34$ and $t = 52$. Check another loan: fifteen annual instalments at 7%: $\ln 2 / \ln 1.07 = 10.24$, $t > 16 - 10.24 = 5.76$, so $t = 6$. Indeed $f_5 = 1564.88 < b_5 = 1728.96$ but $f_6 = 1674.42 > b_6 = 1619.42$ (for a €30,000 loan).

THE LOAN SIZE DROPS OUT

X cancelled, so no rounded quantity entered and the answer depends only on n and i: double the loan and the crossover instalment is unchanged. The curves for f_t and b_t are mirror images about $X/2$ (since $f_t + b_t = X$), crossing where each equals $X/2$.

THE CHAPTER IN ONE BOX

$$X_t = b_t + f_t, \qquad b_t = iL_{t-1}, \qquad f_t = X_t - iL_{t-1} = L_{t-1} - L_t.$$

$$L_t = \sum_{k=t+1}^{n} X_k v^{k-t} = L_0(1+i)^t - \sum_{k=1}^{t} X_k (1+i)^{t-k},$$

$$\text{level:} \quad L_t = X a_{\overline{n-t}|} = L_0(1+i)^t - X s_{\overline{t}|}.$$

$$f_t = Xv^{n-t+1} = f_1(1+i)^{t-1}, \qquad \sum_{k=t+1}^{t+r} f_k = L_t - L_{t+r}, \qquad \sum_{k=t+1}^{t+r} b_k = \sum X_k - (L_t - L_{t+r}).$$

Interest first, capital second. The two balance formulas are one sum split at t. Capital content grows by $(1+i)$ while interest content falls. A block needs only its two end balances. Crossover: $(1+i)^{n-t+1} < 2$. Round once.

TRY IT YOURSELF

1. A loan of €20,000 is repaid by five level annual instalments in arrears at $i = 6\%$. Find the instalment, and L_2 both prospectively and retrospectively. Split the third instalment into interest and capital, and check the two sum to it.
2. A loan of €50,000 is repaid by ten level annual instalments in arrears at $i = 5\%$. Find the capital repaid in the first and last years, and explain their ratio.
3. A loan of €30,000 is repaid by 15 level annual instalments in arrears at $i = 7\%$. Find the instalment and the interest and capital content of the first one.

4. For the loan of question 3, find the total capital repaid and the total interest paid over years 6 to 10 inclusive, using L_5 and L_{10} only.
5. For the same loan, find the first instalment whose capital content exceeds its interest content, and check the answer does not depend on the €30,000.

ANSWERS

1. $X = €4{,}747.93$; $L_2 = Xa_{\overline{3}|} = €12{,}691.27 = 20000(1.06)^2 - Xs_{\overline{2}|}$. $b_3 = 0.06 \times 12691.27 = €761.48$, $f_3 = Xv^3 = €3{,}986.45$; sum 4747.93.
2. $X = €6{,}475.23$, $f_1 = €3{,}975.23$, $f_{10} = €6{,}166.88$, ratio $(1.05)^9 = 1.5513$.
3. $a_{\overline{15}|} = 9.107914$, $X = €3{,}293.84$. $b_1 = 0.07 \times 30000 = €2{,}100.00$, $f_1 = €1{,}193.84$ (check $Xv^{15} = 3293.84 \times 0.362446$).
4. $L_5 = Xa_{\overline{10}|} = €23{,}134.55$, $L_{10} = Xa_{\overline{5}|} = €13{,}505.39$. Capital €9,629.16; five instalments total $5X = €16{,}469.19$; interest $16469.19 - 9629.16 = €6{,}840.03$.
5. $(1.07)^{16-t} < 2 \Rightarrow 16 - t < 10.24 \Rightarrow t > 5.76$, so $t = 6$. The loan amount cancelled: the same at €30,000 or €30.

CHAPTER 12

Loan schedules in general

LECTURE 14 · *A moving rate, a whole table, and payments between the years*

AFTER THIS CHAPTER YOU CAN

- ☐ Run the loan recursion when the rate changes from year to year.
- ☐ Write the prospective and retrospective formulas with a product of factors.
- ☐ Build a complete schedule and check it.
- ☐ Build an m-thly schedule from an annual effective rate, using $i^{(m)}/m$, *not* i/m.
- ☐ Say when $a_{\overline{n}|}^{(m)}$ is defined for a non-integer n.

Everything so far assumed one rate for the whole term and one payment per period. Today both go.

12.1 When the rate is not constant

EXAMPLE 12.1 · A rate that moves

A loan of €1,000 is repaid by two annual instalments of €400 and a final payment. The rate charged is 5% in the first year and 6% in the second. Find the loan outstanding after each of the two instalments.

SOLUTION

No closed form applies, but the recursion never asked for a constant rate. Add the interest of the term, subtract the payment, repeat:

$$L_1 = L_0(1+i_1) - X_1 = 1000(1.05) - 400 = €650.00,$$

$$L_2 = L_1(1+i_2) - X_2 = 650(1.06) - 400 = €289.00.$$

The final payment must then be $289.00\,(1+i_3)$, whatever i_3 turns out to be. This is why a lender that reserves the right to change the rate can only quote the schedule one year at a time. (No annuity formula and no assumption of equal instalments was used.)

Let the rate charged from time $r-1$ to r be i_r. The only thing that changes in the earlier chapters is that a power $(1+i)^n$ is replaced by a *product* $(1+i_1)\cdots(1+i_n)$: the principle of consistency in action.

VARYING RATES

$$L_t \;=\; L_{t-1}(1+i_t) - X_t \quad\Longleftrightarrow\quad L_t = \frac{L_{t+1}+X_{t+1}}{1+i_{t+1}},$$

$$b_t = i_t L_{t-1}, \qquad f_t = X_t - i_t L_{t-1}.$$

$$\textit{Prospective:}\quad L_t = \frac{X_{t+1}}{1+i_{t+1}} + \frac{X_{t+2}}{(1+i_{t+1})(1+i_{t+2})} + \cdots + \frac{X_n}{(1+i_{t+1})\cdots(1+i_n)},$$

$$\textit{Retrospective:}\quad L_t = L_0(1+i_1)\cdots(1+i_t) - \big(X_1(1+i_2)\cdots(1+i_t) + \cdots + X_{t-1}(1+i_t) + X_t\big).$$

Each instalment is discounted (or accumulated) by the rates of exactly the years it travels through.

WHICH RATES?

X_1, paid at time 1, is accumulated by $i_2, \ldots, i_t$ but *not* by i_1: it was not in the fund during the first year. The last instalment X_t carries no factor at all.

12.2 The schedule

The payments can be set out as a table, or **schedule**. Each row uses only the row above it, so the whole schedule is a recursion (the kind of thing done on a spreadsheet).

Year from–to	Loan o/s at r	Instalment at $r+1$	Interest due at $r+1$	Capital repaid at $r+1$	Loan o/s at $r+1$
0–1	L_0	X_1	iL_0	$X_1 - iL_0$	$L_1 = L_0 - (X_1 - iL_0)$
⋮	⋮	⋮	⋮	⋮	⋮
r–r+1	L_r	X_{r+1}	iL_r	$X_{r+1} - iL_r$	$L_{r+1} = L_r - (X_{r+1} - iL_r)$
⋮	⋮	⋮	⋮	⋮	⋮
$n-1$–n	L_{n-1}	X_n	iL_{n-1}	$X_n - iL_{n-1}$	$L_n = 0$

("o/s" means outstanding.)

THE SCHEDULE OF A LEVEL ANNUITY

For a fixed i and level payments of 1 over n periods:

Payment	Interest content	Capital repaid	Loan o/s after payment
1	$ia_{\overline{n}\vert} = 1 - v^n$	v^n	$a_{\overline{n-1}\vert}$
2	$ia_{\overline{n-1}\vert} = 1 - v^{n-1}$	v^{n-1}	$a_{\overline{n-2}\vert}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	$ia_{\overline{n-t+1}\vert} = 1 - v^{n-t+1}$	v^{n-t+1}	$a_{\overline{n-t}\vert}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$ia_{\overline{1}\vert} = 1 - v$	v	0

The last column is the identity $a_{\overline{m}\vert} - v^m = a_{\overline{m-1}\vert}$ (drop the last term of $v + \cdots + v^m$). **Scaling:** for a loan L_0 repaid by instalments $L_0/a_{\overline{n}\vert}$, every monetary amount is that table multiplied by $L_0/a_{\overline{n}\vert}$. There is really only one level-annuity schedule, and every loan is a rescaling of it. That is why the crossover instalment of the previous chapter did not depend on the loan size.

12.3 More frequent repayments

No new principle is involved; care is needed only over the interest due at each instalment date.

INSTALMENTS m TIMES A YEAR

- If the rate is *effective over the same time unit as the instalments* ("1% per month, monthly payments"), take that unit and everything goes exactly as before.
- If the rate is quoted *annually* (effective i) but payments are m-thly, the effective rate over each $\frac{1}{m}$ of a year is

$$(1+i)^{1/m} - 1 \;=\; \frac{i^{(m)}}{m}, \qquad b_{t+1/m} = \frac{i^{(m)}}{m}\, L_t, \quad f_{t+1/m} = X_{t+1/m} - \frac{i^{(m)}}{m}\, L_t.$$

- The exponent on v is always the time from the valuation date to the payment in whole units: an instalment $\frac{k}{m}$ later is discounted by $v^{k/m}$, one $\frac{k}{m}$ earlier is accumulated by $(1+i)^{k/m}$.

NOT i/m

A twelfth of 6% is 0.5%, but the monthly compounding of 6% is $1.06^{1/12} - 1 = 0.4868\%$. This is the nominal-versus-effective distinction of Chapter 1 once again.

EXAMPLE 12.2 · A half-yearly schedule

A loan of €1,000 is repayable by a level annuity payable half-yearly in arrears for three years, at 15% per annum effective. Construct the lender's schedule.

SOLUTION

The effective rate over a half-year is $i^{(2)}/2 = 1.15^{1/2} - 1 = 0.0723805$. Six payments at that rate:

$$X\,a_{\overline{6}|} = 1000 \quad\Longrightarrow\quad X = \frac{1000 \times 0.0723805}{1 - 1.0723805^{-6}} = €211.34.$$

n	o/s at start	Interest (0.0723805×)	Capital (211.34−)	o/s at end
1	1000.00	72.38	138.96	861.04
2	861.04	62.32	149.02	712.02
3	712.02	51.54	159.80	552.22
4	552.22	39.97	171.37	380.85
5	380.85	27.57	183.77	197.08
6	197.08	14.26	197.08	0.00
		268.04	1000.00	

Every row is one application of the recursion. Check: capital sums to €1,000 and the balance runs to zero. The capital grows by 1.0723805 each half-year ($138.96 \times 1.0723805 = 149.02$): the geometric progression of the previous chapter, with the half-year as the period. Total paid $6 \times 211.34 = €1{,}268.04$.

12.4 Fractional terms

In the example the annuity could also be valued in *years*, as $2X\,a^{(2)}_{\overline{3}|}$. But $a^{(m)}_{\overline{n}|}$ was defined only for positive integers n. For some non-integral n it has an obvious meaning, for others not: $a^{(4)}_{\overline{23.5}|}$ is the present value of 1 p.a. payable quarterly for 23.5 years (94 quarterly payments of 0.25); $a^{(2)}_{\overline{23.25}|}$ has no obvious meaning, because 46.5 half-years is half a payment.

WHEN $a^{(m)}_{\overline{n}|}$ IS DEFINED

Define $a^{(m)}_{\overline{n}|}$ when $n = p/m$ with p a whole number, i.e. **when nm is a whole number**: the value at time 0 of p payments of $\frac{1}{m}$ at times $\frac{1}{m}, \ldots, \frac{p}{m}$. For $i > 0$ summing the geometric progression exactly as in Chapter 7 gives the same formula (nothing in the derivation needed n to be whole):

$$a^{(m)}_{\overline{n}|} = \frac{1 - \mathsf{v}^n}{i^{(m)}}, \qquad \ddot{a}^{(m)}_{\overline{n}|} = \frac{1 - \mathsf{v}^n}{d^{(m)}}, \qquad s^{(m)}_{\overline{n}|} = \frac{(1+i)^n - 1}{i^{(m)}}, \qquad \ddot{s}^{(m)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(m)}}.$$

A change of time unit gives $a^{(m)}_{\overline{n}|}$ at rate $i = \frac{1}{m}\,a_{\overline{nm}|}$ at rate $i^{(m)}/m$. (If $i = 0$, $a^{(m)}_{\overline{n}|} = n$.)

IF nm IS NOT WHOLE

There is no universally recognised definition. Some writers add a smaller final payment: with $n = n_1 + f$, n_1 a multiple of $1/m$ and $0 < f < 1/m$, define $a^{(m)}_{\overline{n}|} := a^{(m)}_{\overline{n_1}|} + f\mathsf{v}^n$. In this module, such a symbol is simply not defined.

APR

To let consumers compare loans, lenders must state the effective annual rate of interest charged. In the UK this is the **APR**: the effective annual rate, rounded to the nearer $\frac{1}{10}$ of 1%. (Loans may also have extra costs, which it takes into account.)

THE CHAPTER IN ONE BOX

$$L_t = L_{t-1}(1 + i_t) - X_t \quad \text{(a product of factors, not a power)},$$

$$b_{t+1/m} = \frac{i^{(m)}}{m} L_t, \qquad a^{(m)}_{\overline{n}|} = \frac{1 - \mathsf{v}^n}{i^{(m)}} \quad \text{whenever } nm \in \mathbb{N}.$$

Choose the unit of time so that the rate in hand is effective, and the m-thly schedule is the annual one with the clock relabelled. Every level-annuity schedule is a rescaling of one table. Interest per sub-period is $i^{(m)}/m$, never i/m.

TRY IT YOURSELF

1. A loan of €5,000 is repayable by a level annuity payable quarterly in arrears for two years at 12% per annum effective. Find the instalment and the first two lines of the schedule.
2. For which of $a^{(2)}_{\overline{23.25}|}$, $a^{(4)}_{\overline{23.25}|}$, $a^{(12)}_{\overline{7.25}|}$, $a^{(4)}_{\overline{7.4}|}$ is the symbol defined? Give a reason in each case.
3. A loan is repaid quarterly at an effective annual rate of 8%. Find the effective quarterly rate and $i^{(4)}$, and the interest due in a quarter on a balance of €20,000.

ANSWERS

1. Quarter as the unit: $1.12^{1/4} - 1 = 0.0287373 = i^{(4)}/4$, $n = 8$, $a_{\overline{8}|} = 7.057233$, so $X = 5000/7.057233 = €708.49$. (Using X rounded to the cent:) interest 143.69, capital 564.80, o/s 4435.20; then interest 127.46, capital 581.03, o/s 3854.17. Note $143.69 = 0.0287373 \times 5000$, not 0.03×5000.
2. Defined exactly when nm is whole: $a^{(2)}_{\overline{23.25}|}$: 46.5, no. $a^{(4)}_{\overline{23.25}|}$: 93, yes. $a^{(12)}_{\overline{7.25}|}$: 87, yes. $a^{(4)}_{\overline{7.4}|}$: 29.6, no. In the undefined cases the term ends part-way through a payment interval and there is no agreed convention for the incomplete last instalment.
3. $1.08^{1/4} - 1 = 0.019427$, $i^{(4)} = 0.077706$. Interest $0.019427 \times 20000 = €388.53$. Using $i/4 = 2\%$ would give €400, an overcharge of €11.47 a quarter.

PART V

Securities

CHAPTER 13

Bonds and their prices

LECTURE 15 · *A bond is an annuity with a lump sum, and tax*

AFTER THIS CHAPTER YOU CAN

- ☐ Say what a fixed-interest bond is: nominal amount, coupon rate, redemption price.
- ☐ Work per unit nominal, and read a price as a present value.
- ☐ Price a bond with no tax, with tax at source, and with tax paid k years late.
- ☐ Count the coupons the buyer actually receives, from the purchase date.
- ☐ Read the same equation as a price (given a yield) or a yield (given a price).

Everything is in place. A contract pays €3 at the end of each of the next three years, and €100 at the end of the third. At $i = 5\%$ it is worth $3a_{\overline{3}|} + 100v^3 = 3 \times 2.723248 + 100 \times 0.863838 =$ €94.55. It is worth less than the €100 at the end, because 3% a year does not compensate for 5% interest: you are effectively the lender. **That is a bond.** Everything in this chapter is to calculate this price, with the added difficulty of tax.

13.1 Securities and fixed-interest bonds

SECURITY

A **security** is a tradeable financial instrument: a financial contract that can be bought and sold. Examples: government bonds, shares, corporate bonds, certificates of deposit.

Two questions arise, Chapter 9's two directions again. What price P should an investor pay to secure a net yield i? And if the investor pays P, what net yield is obtained?

FIXED-INTEREST SECURITY

A loan floated on a stock exchange by a government or company, issued in **bonds** of stated nominal amount. Holders receive regular fixed **coupon** payments until a specified date, then a lump sum.

- The **nominal amount** is the face value of the bond.
- The **coupon rate** D is the nominal rate per annum paid to holders (payable m-thly), as a fraction of the nominal amount.
- The **redemption price** R is paid at redemption per unit nominal. It is a *ratio*, not a stream: you hand over X nominal and get back $R \times X$.

Redemption $R = 1$ is *at par*; $R > 1$ *at a premium*; $R < 1$ *at a discount*.

THE NOMINAL AMOUNT IS A LABEL, NOT THE COST

What the bond costs is set by the market. Everything is quoted *per unit nominal*, or per €100 nominal. An investor with €900 nominal has not spent €900.

EXAMPLE 13.1 · A government bond

An investor is allocated €900 nominal of a bond issued with coupon 4.25% p.a. payable half-yearly, redeemable at par on 1 March 2041, issue price €98.50 per €100 nominal. What do they pay and receive?

SOLUTION

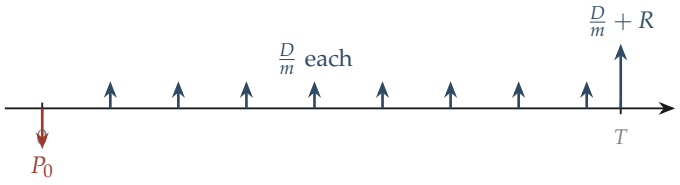
They pay $0.985 \times 900 = €886.50$, then receive $\frac{0.0425}{2} \times 900 = €19.13$ every six months for seventeen years (34 coupons), and €900 on redemption together with the final coupon. After issue the bond can be bought and sold, so its *price* becomes market-driven and hard to predict.

STRIPS, AND WHY INVESTORS DIFFER

Coupons and redemption proceeds can sometimes be traded separately: a **zero-coupon bond** (the lump sum only) and a **bond redeemable at zero** (the coupons only). Present values add, so the bond is worth the sum of its strips; otherwise an investor would buy the cheap side, sell the expensive one and pocket the difference at no risk. This is Chapter 1's no-arbitrage argument again.

Investors are also taxed differently, on coupons and on gains. If several bid for a bond, all wanting the same *net* return, the one with the *lowest* income tax can bid highest. So the same bond is genuinely worth different amounts to different buyers.

13.2 The price is a present value



PRICE, NO TAX

Coupons D per annum payable m-thly in arrears, redeemable at R at time T (so $\frac{D}{m}$ at times $\frac{1}{m},\dots,T$). Per unit nominal

$$\boxed{P_0 \;=\; D\,a^{(m)}_{\overline{T}|} + R\,\mathrm{v}^{\,T}} \qquad \text{and per €100 nominal} \qquad P = 100D\,a^{(m)}_{\overline{T}|} + 100R\,\mathrm{v}^{\,T}.$$

Coupons collapse into $Da^{(m)}_{\overline{T}|}$, the lump sum into $R\mathrm{v}^T$; both are valued at time 0, so they add. Read it in two directions: given i it *computes* the price; given P it is an equation of value whose solution i is the **gross yield** (**net yield** once the payments are adjusted for tax). There is no formula for i, so interpolate, remembering that $i^{(m)}$ itself depends on i and must be recomputed at every trial rate.

EXAMPLE 13.2 · A bond at 7%

A bond pays coupons of 6% p.a. half-yearly, redeemable at par in ten years. Find the price per €100 nominal for a gross yield of 7%.

SOLUTION

$D = 0.06$, $R = 1$, $T = 10$, $m = 2$. $i^{(2)} = 2(1.07^{1/2} - 1) = 0.0688161$, $\mathrm{v}^{10} = 0.5083493$, so $a^{(2)}_{\overline{10}|} = (1 - 0.5083493)/0.0688161 = 7.144415$ and

$$P \;=\; 6 \times 7.144415 + 100 \times 0.5083493 \;=\; €93.70.$$

Below par, as it must be: the required 7% exceeds the 6% coupon, so the shortfall is made up by buying cheaply.

13.3 Income tax

TAX AT SOURCE

If the investor pays income tax at rate τ_1 on coupons, deducted when they are paid, each coupon is $(1-\tau_1)\frac{D}{m}$ and the redemption is untouched:

$$\boxed{P_0 \;=\; (1-\tau_1)D\,a^{(m)}_{\overline{T}|} + R\,\mathrm{v}^{\,T}}$$

Here i is the *net* yield.

TAX PAID k YEARS LATE

Suppose T is a whole number of years and each year's tax bill $\tau_1 D$ is paid k years after that year's final coupon ($0 \le k \le 1$), at times $1+k, \dots, T+k$. Take the coupons *gross* and subtract the tax as a deferred annuity:

$$\boxed{P_0 \;=\; D\, a_{\overline{T}|}^{(m)} + R\, v^{T} - \tau_1 D\, v^{k}\, a_{\overline{T}|}}$$

since $v^{k+1} + \cdots + v^{k+T} = v^k a_{\overline{T}|}$ (Chapter 8's deferment). Paying the same tax later is worth more to the investor, so this price is *higher* than the tax-at-source price.

EXAMPLE 13.3 · Tax at source versus tax paid late

The 6%, ten-year par bond again, for an investor paying income tax at 40% and requiring 7% net. Price (a) with tax at source, (b) with the tax paid half a year after each year's final coupon.

SOLUTION

(a) Net coupons $(1-0.4)\times 6 = 3.6$: $P = 3.6 \times 7.144415 + 50.83493 = €76.55$.
(b) $k = 0.5$, $a_{\overline{10}|} = 7.023582$, $v^{0.5} = 0.9667365$:

$$P = 42.866 + 50.835 - 0.4 \times 6 \times 0.9667365 \times 7.023582 = €77.41.$$

Higher than €76.55, and it must be: the same tax is paid later, so its present value is smaller, and the investor can pay more and still earn 7%.

13.4 A worked example, and a method

PRICING A BOND

1. **Dates.** Measure T from the *purchase* date to redemption. A coupon paid *before* purchase is not received: the first coupon is the next one after purchase.
2. Check Tm is a whole number (Chapter 12), so $a_{\overline{T}|}^{(m)}$ is defined: that is the number of coupons.
3. List D, R, τ_1, the yield i and how the tax is paid.
4. Compute $i^{(m)}$ from i with the bridge, then the formula for the tax treatment.
5. Check the size: below par if the required yield exceeds the net coupon rate.

EXAMPLE 13.4 · Eighteen and three quarter years

A bond pays coupons of 8% p.a. quarterly, with the next coupon due on 31 December 2020, redeemable at 115% on 1 October 2039. Calculate the price on 1 January 2021 for an investor liable to income tax of 30% at source who wants an effective net yield of 10% p.a.

SOLUTION

The coupon of 31 December 2020 is paid *before* the purchase, so the buyer does not get it. Measure time from 1 January 2021: $T = 18.75$, and $18.75 \times 4 = 75$ is a whole number, so there are exactly 75 quarterly coupons and $a^{(4)}_{\overline{18.75}|}$ is defined. $D = 0.08$, $R = 1.15$, $i = 0.1$, $\tau_1 = 0.3$, $i^{(4)} = 4(1.1^{1/4} - 1) = 0.096454756$. Per €100 nominal,

$$P_0 = (1 - 0.3)\, 8\, a^{(4)}_{\overline{18.75}|} + 115\, v^{18.75} = 5.6\, \frac{1 - 1.1^{-18.75}}{0.096454756} + 115 \times 1.1^{-18.75}$$
$$= 5.6 \times 8.63150002 + 115 \times 0.16745077 \;=\; €67.59.$$

READING THE ANSWER

The investor pays €67.59 and receives €2 of coupon each quarter, €1.40 after tax, plus €115 at redemption. The 75 net coupons are worth €48.34 (71.5% of the price) and the redemption €19.26 (28.5%). The price is far below par: the required 10% is well above the 8% coupon, and tax widens the gap. Untaxed, the same bond would be worth €88.31.

THE CHAPTER IN ONE BOX

$$P_0 = D\, a^{(m)}_{\overline{T}|} + R v^T, \qquad P_0 = (1 - \tau_1) D\, a^{(m)}_{\overline{T}|} + R v^T,$$

$$P_0 = D\, a^{(m)}_{\overline{T}|} + R v^T - \tau_1 D\, v^k a_{\overline{T}|} \quad \text{(tax } k \text{ years late, } T \text{ whole).}$$

Everything per unit nominal (or per €100). Count coupons from the purchase date. Given i, the formula is a price; given a price, an equation of value for the yield. Tax lowers the price; tax paid later lowers it less.

TRY IT YOURSELF

1. A bond pays coupons of 6% p.a. half-yearly, redeemable at par in ten years. Find the price per €100 nominal for a gross yield of 7%.
2. For the same bond, price it for an investor paying income tax at 40% at source and requiring 7% net; then again with the tax paid half a year after each year's final coupon.
3. A bond is stripped into its coupons and its redemption payment. Why must the two strips be worth the bond?

ANSWERS

1. €93.70 (below par, as the required yield exceeds the coupon).
2. Source: $3.6 \times 7.144415 + 50.83493 = €76.55$. Late ($k = 0.5$): €77.41, higher, because the same tax is paid later.
3. Present values add: coupon strip $D a^{(m)}_{\overline{T}|}$ plus redemption strip $R v^T$ is exactly the bond's price. If the strips sold for less than the bond, buy both, recombine, sell the bond and pocket the difference, with no capital and no risk; if for more, run it

backwards. Such trades continue until the prices agree: no-arbitrage.

CHAPTER 14

Tax on income, tax on gains

LECTURE 16 · *Late income tax, and a price that sits on both sides of its own equation*

AFTER THIS CHAPTER YOU CAN

- ☐ Price a bond when income tax is paid late and the last tax year is a part year.
- ☐ Test whether there will be a capital gain.
- ☐ Price with capital gains tax, including the circular equation.
- ☐ Check an answer by closing the circle, and avoid the shortcut that is wrong.

Warm-up. The bond of the last chapter paid 8% quarterly, redeemable at 115% in 18.75 years. For an investor paying *no tax* who requires 10%: $P = 8a^{(4)}_{\overline{18.75}|} + 115\mathrm{v}^{18.75} = 69.05 + 19.26 =$ €88.31. Taxed at 30% at source it was worth €67.59: income tax alone costs this investor €20.72, about a quarter.

14.1 Income tax paid late

EXAMPLE 14.1 · Tax each 31 March

The same bond (8% quarterly, redeemable at 115% on 1 October 2039). An investor pays tax at 30% on 31 March on income received in the previous calendar year. What price should the bond have on 1 January 2021 for an effective net yield of 10%?

SOLUTION

Last chapter's formula $-\tau_1 D\mathrm{v}^k a_{\overline{T}|}$ assumed T was a whole number of years, so every tax year was full. Here $T = 18.75$, so the last year is not full. Measure time from 1 January 2021. Coupons arrive at $t = 0.25, 0.5, \ldots, 18.75$; tax is paid each 31 March, at $t = 1.25, 2.25, \ldots$, a quarter after each year end.

- Years $1, \ldots, 18$ give income D, taxed at $t = k + 0.25$: the bill is $\tau_1 D$ at $t = 1.25, \ldots, 18.25$.
- The last year (2039) gives income $\frac{3}{4}D$, taxed at $t = 19.25$.

Their present value is

$$\tau_1 D\left(\mathrm{v}^{1.25} + \cdots + \mathrm{v}^{18.25}\right) + \tfrac{3}{4}\tau_1 D\,\mathrm{v}^{19.25} \;=\; \tau_1 D\,\mathrm{v}^{0.25}\left(a_{\overline{18}|} + \tfrac{3}{4}\mathrm{v}^{19}\right).$$

So, per €100 nominal,

$$P_0 = 8\,a^{(4)}_{\overline{18.75|}} + 115\,v^{18.75} - 0.3 \times 8\,v^{0.25}\left(a_{\overline{18|}} + \tfrac{3}{4}v^{19}\right)$$
$$= 69.0520 + 19.2568 - 2.4 \times 8.12804592 \;=\; €68.80.$$

TAX PAID LATE: GETTING THE DATES RIGHT

1. Put the purchase date at time 0. List the coupon dates.
2. Work out which *tax year* each coupon belongs to, and when that year's bill is paid. A full year has income D; a part year has the fraction of coupons actually received.
3. Factor the lag out as v^{lag} and write the full years as an annuity; patch the part year separately.
4. Subtract the present value of the tax from the gross price.

THREE TAX TREATMENTS OF ONE BOND

Tax treatment	price per €100 nominal
No tax at all	88.31
Deducted at source, with each coupon	67.59
Paid each 31 March	68.80

Deferring the tax bill by a few months is worth €1.21 per €100 nominal: money paid later is cheaper.

14.2 Capital gains tax: will there be a gain?

CAPITAL GAINS TAX

If the price paid for a bond is less than the redemption price (or the sale price, if sold earlier), the investor has made a **capital gain** and **capital gains tax** (rate τ_2) is levied on that gain. In contrast to income tax this is normally payable *once only*, at the date of sale or redemption.

Before pricing, decide whether there *will* be a gain. Ignoring capital gains tax, let P be the price the investor would pay (here with tax at source). Then, using $1 - v^T = i^{(m)}a^{(m)}_{\overline{T|}}$,

$$R - P \;=\; R\big(1 - v^T\big) - (1-\tau_1)Da^{(m)}_{\overline{T|}} \;=\; \left(i^{(m)} - (1-\tau_1)\frac{D}{R}\right)R\,a^{(m)}_{\overline{T|}}.$$

THE CAPITAL GAINS TEST

Since $R\,a^{(m)}_{\overline{T}|} > 0$, the sign of $R - P$ is the sign of the bracket:

$$\boxed{P < R \iff i^{(m)} > (1-\tau_1)\frac{D}{R}}$$

$(1-\tau_1)\frac{D}{R}$ is the **net running yield**: net coupon income per unit of *redemption*. If the required rate $i^{(m)}$ exceeds what the coupons alone deliver, the shortfall must be made up by buying below R: a capital gain. The test involves no annuity values at all. (This form is for income tax at source; the approach is identical in general.)

14.3 Pricing with capital gains tax

A PRICE ON BOTH SIDES

Let P be the price with income tax *only* (however it is levied) and P_0 the price once capital gains tax is paid too. At redemption there is an extra liability $\tau_2(R - P_0)$, worth $\tau_2(R - P_0)v^T$ today, which the investor deducts from what they offer:

$$P_0 \;=\; P - \tau_2(R - P_0)v^T.$$

The gain depends on the price and the price on the gain, but P_0 appears only to the first power, so the equation is linear. Gather the P_0 terms:

$$\boxed{P_0 \;=\; \frac{P - \tau_2 R v^T}{1 - \tau_2 v^T}} \qquad R - P_0 = \frac{R - P}{1 - \tau_2 v^T}.$$

Since $1 - \tau_2 v^T \in (0,1)$, when $P < R$ we get $P_0 < P$: capital gains tax lowers the price, so the gain $R - P_0$ is larger than $R - P$. Also $R - P_0$ and $R - P$ always share a sign, so the test needs only P.

INCOME TAX AT SOURCE, AND NO GAIN

For tax at source, $P - \tau_2 R v^T = (1-\tau_1) D a^{(m)}_{\overline{T}|} + (1-\tau_2) R v^T$, so

$$\boxed{P_0 \;=\; \frac{1-\tau_1}{1-\tau_2 v^T}\, D\, a^{(m)}_{\overline{T}|} + \frac{1-\tau_2}{1-\tau_2 v^T}\, R v^T} \qquad \text{if there is a gain;} \qquad P_0 = P \;\text{ otherwise.}$$

ALWAYS RUN THE TEST FIRST

If $P \geq R$ there is no gain and no liability at redemption: $P_0 = P$. The gain formula would give $R - P_0 \leq 0$ and $P_0 \geq P$: it treats a loss as a tax refund and pays *more* than P.

EXAMPLE 14.2 · Income tax at 30% and capital gains tax at 40%

The 8% quarterly bond, redeemable at 115% on 1 October 2039. Price on 1 January 2021 for an investor liable to income tax of 30% at source and capital gains tax of 40%, for an effective net yield of 10%.

SOLUTION

$T = 18.75$, $D = 0.08$, $R = 1.15$, $i = 0.1$, $\tau_1 = 0.3$, $\tau_2 = 0.4$, $i^{(4)} = 0.096454756$. *Test:* the net running yield is $\dfrac{0.7 \times 0.08}{1.15} = 0.048696$, and $i^{(4)} = 0.096455 > 0.048696$, so there *is* a gain, comfortably. With $a^{(4)}_{\overline{18.75}|} = 8.63150002$, $\mathrm{v}^{18.75} = 0.16745077$ and $1 - 0.4\mathrm{v}^{18.75} = 0.933020$:

$$\begin{aligned} P_0 &= \frac{0.7}{0.933020} \times 8 \times 8.63150002 + \frac{0.6}{0.933020} \times 115 \times 0.16745077 \\ &= €64.19 \quad \text{per } €100 \text{ nominal.} \end{aligned}$$

CLOSING THE CIRCLE

The gain is $115 - 64.19 = €50.81$; tax due at redemption $0.4 \times 50.81 = €20.32$, with present value $20.32 \times 0.16745 = €3.40$. And $67.59 - 3.40 = €64.19$. The price is consistent with the tax it implies: a *fixed point* of the circular equation.

P_0 GIVES P; P DOES NOT GIVE P_0 SO EASILY

The tax is on the gain $R - P_0$, which needs the *final* price. Using the wrong one, $(115 - 67.59) \times 0.4 \times 0.16745 = 3.18$ and $67.59 - 3.18 = 64.41$, which is *not* 64.19. The relation is $P = P_0 + \tau_2(R - P_0)\mathrm{v}^T$, not $P_0 = P - \tau_2(R - P)\mathrm{v}^T$.

THE CHAPTER IN ONE BOX

$$i^{(m)} > (1 - \tau_1)\frac{D}{R} \iff \text{capital gain,}$$

$$P_0 = \frac{(1 - \tau_1) D\, a^{(m)}_{\overline{T}|} + (1 - \tau_2) R \mathrm{v}^T}{1 - \tau_2 \mathrm{v}^T} \quad \text{(gain, tax at source);} \qquad P_0 = P \ \text{(no gain).}$$

Test first, then price. Income tax paid late: factor the lag out as v^k and patch any part year separately. Capital gains tax: the price is on both sides, so gather and divide.

TRY IT YOURSELF

1. A bond pays coupons of 5% p.a. half-yearly, redeemable at 110% in twelve years, for an investor paying 25% income tax and requiring 8% net. Show there is a capital gain.
2. For the same bond and investor, now also paying capital gains tax at 35%, find the price per €100 nominal and compare it with the price ignoring that tax.
3. A bond pays coupons of 9% p.a. half-yearly, redeemable at par in ten years, for an investor paying 20% income tax and requiring 6% net. Show there is *no* gain, and price it.

ANSWERS

1. $i^{(2)} = 2(1.08^{1/2} - 1) = 0.078461$; net running yield $0.75 \times 0.05/1.10 = 0.034091$. Since $0.078461 > 0.034091$, there is a gain.
2. $v^{12} = 0.397114$, $a^{(2)}_{\overline{12}|} = 7.683900$: $P_0 = \dfrac{0.75 \times 5 \times 7.683900 + 0.65 \times 110 \times 0.397114}{1 - 0.35 \times 0.397114} =$ €66.44. Ignoring capital gains tax: $0.75 \times 5 \times 7.6839 + 110 \times 0.397114 =$ €72.50, so the tax costs over €6 of price.
3. $i^{(2)} = 0.059126 < 0.8 \times 0.09 = 0.072$: no gain, so $P_0 = P = 0.8 \times 9 \times a^{(2)}_{\overline{10}|} + 100v^{10} =$ €109.62, above par, consistent with no gain. Applying the τ_2 formula would give a lower price still, taxing a loss.

CHAPTER 15

Bonds redeemable at the borrower's option

LECTURES 17–18 · *Pricing a bond whose redemption date is not yet decided, and three problems on one rule*

AFTER THIS CHAPTER YOU CAN

- ☐ Find the net yield on a bond when capital gains tax applies and the price is given.
- ☐ Say what "redeemable at the borrower's option" means for the investor.
- ☐ Decide the *worst case* redemption date: latest for a gain, earliest for a loss.
- ☐ Find the maximum price for a minimum yield, or the minimum yield for a price.
- ☐ Work a full problem with three taxes on three different dates, in the right order.

15.1 Finding the yield when capital gains tax applies

Given every other number, solve for the yield: Chapter 9's direction, with the machinery of the last two chapters.

EXAMPLE 15.1 · Writing the equation of value

An investor liable to income tax at 30% at source and capital gains tax at 40% buys, for €70 per €100 nominal, a bond with coupons of 8% p.a. quarterly, redeemable at 115% in 18.75 years. Write down the equation of value for the net yield.

SOLUTION

Per €100 nominal, with $P_0 = 70$ *given*:

$$70 = (1-0.3)\,8\,a^{(4)}_{\overline{18.75}|} + 115v^{18.75} - 0.4(115-70)v^{18.75} = 5.6\,a^{(4)}_{\overline{18.75}|} + 97v^{18.75},$$

since $115 - 0.4 \times 45 = 115 - 18 = 97$. Because P_0 is known, the circularity of Chapter 14 vanishes: the tax of €18 is a definite number, and simply reduces the redemption proceeds from €115 to €97. No gain test is needed: with the price given, compare it with R directly ($70 < 115$, a gain).

NET YIELD FROM A PRICE

1. Work out the capital gains tax as a number ($\tau_2\times$ gain), and so the effective redemption proceeds.
2. Write $\text{NPV}(i) = (1-\tau_1)Da_{\overline{T}|}^{(m)} + (\text{net redemption})\text{v}^T - P_0$.
3. Evaluate at two trial rates with a sign change. **Compute $i^{(m)}$ afresh at each trial rate** ($i^{(m)}$ depends on i).
4. Interpolate (Chapter 9).

EXAMPLE 15.2 · Two trial rates

Continue: $\text{NPV}(i) = 5.6\,a_{\overline{18.75}|}^{(4)} + 97\text{v}^{18.75} - 70$.

SOLUTION

$i_1 = 0.09$: $i^{(4)} = 0.0871127$, $\text{NPV}(0.09) = +0.7859$. $i_2 = 0.095$: $i^{(4)} = 0.0917917$, $\text{NPV}(0.095) = -2.4279$. The sign changes, and the root is much nearer the lower end:

$$i_0 \approx 0.09 + \frac{0.7859}{0.7859 + 2.4279}(0.005) = 0.0912,$$

about 9.1% net (the root is 0.09119…). It is below the 10% of the previous chapter, as it should be: €70 is more than the €64.19 that a 10% yield required. Paying more for the same cashflow means earning less.

15.2 Securities with no fixed redemption date

REDEMPTION AT THE BORROWER'S OPTION

The terms of issue may allow the borrower to redeem at any interest date on or after some specified date; or on or between two specified dates; or on any one of a series of dates. The latest possible date is the **final redemption date**; with none, the bond is **undated**. (A loan may be redeemable at the *lender's* option, but this is less common. Throughout, the choice belongs to the borrower.)

An investor cannot know when the borrower will repay, so **cannot know the yield**. But using the capital gains test, the investor can find either

- the **maximum price** to pay for a specified *minimum* net yield, or
- the **minimum net yield** for a specified price.

Both are guarantees against the worst case: the investor gives up knowing the answer exactly in exchange for a bound that holds whatever the borrower does.

Set-up. Coupons D p.a. payable m-thly in arrears, redemption R, and outstanding term T chosen by the borrower with $T_1 \le T \le T_2$ (integer multiples of $\frac{1}{m}$). An investor liable to income tax at τ_1 wants a net yield of **at least** i_0. What is the maximum price?

15.3 Which redemption date is the worst case?

At effective rate i, the present value per unit nominal of the net payments is

$$P(i,T) \;=\; (1-\tau_1)D\,a_{\overline{T}|}^{(m)} + R\mathsf{v}^{\,T} \;=\; R - \Big(i^{(m)} - (1-\tau_1)\frac{D}{R}\Big)R\,a_{\overline{T}|}^{(m)}.$$

Two consequences. **(1) In i:** P is a *decreasing* function of i (if $i_1 > i_2$ then $\mathsf{v}_1 < \mathsf{v}_2$, so $a_{\overline{T}|}^{(m)}$ and v^T are smaller, and so is P). Read backwards: less interest, higher earning for the same cashflow. **(2) In T:** the dependence is governed by the sign of the bracket, i.e. the net running yield, since $a_{\overline{T}|}^{(m)}$ rises with T.

CASE 1: A CAPITAL GAIN, $i_0^{(m)} > (1-\tau_1)\frac{D}{R}$

The bracket is positive and free of T, so $P(i_0,T)$ is a *decreasing* function of T (a larger positive quantity is subtracted from R). Suppose the investor pays $P(i_0,T')$. If the borrower redeems *earlier* than T', the bond is worth more than was paid, so the yield is $i > i_0$; later, $i < i_0$. **The sooner a gain arrives the better, so the danger is a late redemption.** To guarantee at least i_0, assume the worst case $T = T_2$ and pay $P_0 = P(i_0,T_2)$: the lowest of the candidate prices.

CASE 2: A CAPITAL LOSS, $i_0^{(m)} < (1-\tau_1)\frac{D}{R}$

Write $P(i_0,T) = R + \big((1-\tau_1)\frac{D}{R} - i_0^{(m)}\big)Ra_{\overline{T}|}^{(m)}$: now a positive quantity is *added*, so P *increases* with T. Every inequality reverses: a loss should be deferred as long as possible, so the worst case is the *earliest* redemption. Assume $T = T_1$ and pay $P_0 = P(i_0,T_1)$.

CASE 3: NEITHER, AND THE TABLE

If $i_0^{(m)} = (1-\tau_1)\frac{D}{R}$ the bracket is zero and $P(i_0,T) = R$ for *every* T: the coupons alone deliver the required rate, and the date does not matter.

Test	At redemption	Assume
$i_0^{(m)} > (1-\tau_1)\frac{D}{R}$	Capital gain	Latest date $T = T_2$
$i_0^{(m)} < (1-\tau_1)\frac{D}{R}$	Capital loss	Earliest date $T = T_1$
$i_0^{(m)} = (1-\tau_1)\frac{D}{R}$	Neither	Any date; $P_0 = R$

A gain should arrive early and a loss late. The worst cases are a *late gain* or an *early loss*; in both, price at whichever date is worse and pay the lower price.

IF THE PRICE IS GIVEN INSTEAD

To find the *minimum* net yield, again assume the worst case, comparing P_0 with R directly:

- $P_0 < R$: capital gain; worst case is redemption at the *latest* date.
- $P_0 > R$: capital loss; worst case is redemption at the *earliest* date.
- $P_0 = R$: neither, and the yield is $i_0^{(m)} = (1-\tau_1)\frac{D}{R}$ whenever it is redeemed.

DOES CAPITAL GAINS TAX CHANGE ANYTHING?

No. A gain before capital gains tax is still a gain after it, since the investor keeps the fraction $1 - \tau_2 > 0$. The gain is smaller but the worst case is still the latest redemption, and the reverse for a loss. **The order of operations is fixed:** run the test, choose T_1 or T_2, and only then apply Chapter 14's τ_2 machinery at the chosen date.

MAXIMUM PRICE FOR A MINIMUM YIELD

1. Compute $i_0^{(m)}$ from the required i_0 and the net running yield $(1 - \tau_1)D/R$.
2. Compare them. Gain $\to T_2$; loss $\to T_1$; equal $\to$ price at par.
3. Price at the chosen date as in Chapters 13–14. Check by pricing the other end and confirming that price is *higher* (and so unsafe).

EXAMPLE 15.3 · A gain: the latest date

A stock pays 6% half-yearly coupons, redeemable at par at the borrower's option between 10 and 15 years. An investor pays 25% tax and needs 8% net. Find the maximum price per €100.

SOLUTION

$i^{(2)} = 2(1.08^{1/2} - 1) = 0.078461$ against $(1 - \tau_1)D/R = 0.75 \times 0.06 = 0.045$. Since $0.078461 > 0.045$ there is a gain, so the worst case is the latest redemption, $T_2 = 15$. With $v^{15} = 0.315242$ and $a^{(2)}_{\overline{15}|} = \dfrac{1 - 0.315242}{0.078461} = 8.727375$:

$$P_0 = 0.75 \times 6 \times 8.727375 + 100 \times 0.315242 = €70.80.$$

At $T_1 = 10$ the price would be €77.11, higher and therefore unsafe: pay that and a redemption at year 15 leaves the investor short of 8%.

EXAMPLE 15.4 · A loss: the earliest date

The same, but 10% coupons, redeemable between 10 and 20 years, 20% tax, 5% net.

SOLUTION

$i^{(2)} = 2(1.05^{1/2} - 1) = 0.049390$ against $0.8 \times 0.10 = 0.08$. Since $0.049390 < 0.08$ there is a capital *loss*: the net coupons alone deliver more than required, so the bond is bought above par and the excess lost at redemption. A loss is best deferred, so the worst case is the *earliest* redemption, $T_1 = 10$. With $v^{10} = 0.613913$ and $a^{(2)}_{\overline{10}|} = 7.817079$:

$$P_0 = 0.8 \times 10 \times 7.817079 + 100 \times 0.613913 = €123.93,$$

above par, as a loss requires.

EXAMPLE 15.5 · A given price

An investor pays €95 per €100 nominal for a stock redeemable at par between 8 and 12 years. Which date gives the least yield?

SOLUTION

$P_0 = 95 < 100 = R$: a capital gain. A gain is worth more the sooner it arrives, so the worst case is the latest redemption, $T = 12$. The yield on that basis is the minimum; any earlier redemption gives more.

EXAMPLE 15.6 · The guarantee, checked

The 8% quarterly bond, redeemable at 115% at the borrower's option between 18.75 and 22.75 years, for an investor paying 30% income tax at source who wants at least 10% net. There is a gain ($i^{(4)} = 0.0965 > 0.0487$), so pay $P(0.10, 22.75) = €64.57$ per €100. Whatever the borrower does, the yield is at least 10%:

redeemed at T	18.75	19.75	20.75	21.75	22.75
net yield	10.50%	10.35%	10.22%	10.10%	10.00%

Equality only in the worst case $T = T_2$; a bonus of up to half a percentage point if redemption comes early.

15.4 The worst case in practice

The theory is complete. What remains is to identify the cashflows and take care over the dates, since three different taxes may fall on three different days.

THE ORDER OF WORK NEVER CHANGES

1. Convert every date into a time and **draw the cashflow**.
2. Run the capital gains test (or, if the price is given, compare P_0 with R) and fix T at T_1 or T_2.
3. Write the equation of value at that T, with every tax at its own date.
4. Solve for P_0 if the price is unknown, or interpolate if the rate is unknown.

One of the biggest difficulties in the exam will be finishing on *time*. Practise.

EXAMPLE 15.7 · Dates to T_1 and T_2

A stock bought on 1 January 2021 is redeemable at 115% at the borrower's option on 1 October of any year between 2039 and 2043. Write down T_1 and T_2.

SOLUTION

From 1 January 2021 the earliest redemption is 1 October 2039 and the latest 1 October 2043: $T_1 = 18 + \frac{9}{12} = 18.75$, $T_2 = 22 + \frac{9}{12} = 22.75$. Both are integer multiples of $\frac{1}{4}$, so

$a^{(4)}_{\overline{18.75}|}$ and $a^{(4)}_{\overline{22.75}|}$ are defined for quarterly coupons. (Redemption on 1 November would give $T = 18.8\overline{3}$, and $4T$ is not an integer.)

EXAMPLE 15.8 · The maximum price, with capital gains tax

A stock pays 8% p.a. quarterly, redeemable at 115% at the borrower's option on any 1 October between 2039 and 2043. Find the price on 1 January 2021 for an investor liable to income tax at 30% and capital gains tax at 40%, for an effective net yield of *at least* 10%.

SOLUTION

$i_0 = 0.1$, $D = 0.08$, $R = 1.15$, $m = 4$, $\tau_1 = 0.3$, $\tau_2 = 0.4$, $T_1 = 18.75$, $T_2 = 22.75$. *Test:* $i^{(4)} = 0.096455 > 0.048696 = \dfrac{0.7 \times 0.08}{1.15}$: a gain, so assume $T = T_2 = 22.75$. Only now does the second tax enter. With $\mathrm{v}^{22.75} = 0.114371$ and $a^{(4)}_{\overline{22.75}|} = \dfrac{1 - 0.114371}{0.096455} = 9.181806$, the price before capital gains tax is

$$P(i_0, T_2) = 1.15 - 0.054923 \times 9.181806 = 0.645708,$$

and with $1 - 0.4\mathrm{v}^{22.75} = 0.954252$,

$$P_0 = \frac{0.645708 - 0.4 \times 1.15 \times 0.114371}{0.954252} = \frac{0.593097}{0.954252} = 0.6215,$$

that is €62.15 per €100 nominal. **The test picks the date, and only then is the formula for P_0 applied at that date**: the other order is a mistake.

WHAT INSURING AGAINST THE WORST CASE COSTS

Fixed to redeem at $T = 18.75$, the same stock was priced at €64.19 (Chapter 14): the investor must pay €2.04 less to allow for redemption up to four years later. €62.15 is the *maximum* price consistent with the guarantee. What the investor actually earns, by redemption year:

Redeemed	2039	2040	2041	2042	2043
Net yield	10.34%	10.24%	10.15%	10.07%	10.00%

EXAMPLE 15.9 · The minimum yield for a price of €70

The same stock, same investor, acquired on 1 January 2021 for €70 per €100 nominal. Find the minimum net yield.

SOLUTION

$R = 115 > 70 = P_0$: a gain, so the worst case is again $T_2 = 22.75$. Then $70 = 5.6\, a^{(4)}_{\overline{22.75}|} + 97\,\mathrm{v}^{22.75}$ (as $115 - 0.4 \times 45 = 97$). $\mathrm{NPV}(i) = 5.6\, a^{(4)}_{\overline{22.75}|} + 97\mathrm{v}^{22.75} - 70$, recomputing $i^{(4)}$ each time: $i_1 = 0.085$: $i^{(4)} = 0.082418$, $\mathrm{v}^{22.75} = 0.156305$, NPV $= +2.4879$; $i_2 = 0.09$:

$i^{(4)} = 0.087113$, $v^{22.75} = 0.140782$, NPV $= -1.1097$. So

$$i_0 \approx 0.085 + \frac{2.4879}{2.4879 + 1.1097}(0.005) = 0.0885,$$

about 8.8% (exact 0.08841). On a bond fixed to redeem in 2039 the same price gave 9.1%: the borrower's four extra years cost the investor about 0.3% of guaranteed yield.

15.5 A past exam question

EXAMPLE 15.10 · Question 2, 2022/23

On 1 October 2022 a company issued a bond, redeemable at par on any date from 1 October 2032 to 1 October 2042 inclusive, at the company's option. It pays coupons at 5% p.a. half-yearly in arrears. The investor pays income tax at 25% on 31 December on coupons received in that calendar year, and capital gains tax at 45% on 31 March on any gain earned over the previous calendar year. (i) What price per €100 nominal gives a yield of at least 9% p.a.? (ii) If the price were €45, what minimum yield would be obtained?

SOLUTION

Data (time in years from 1 October 2022): $D = 0.05$, $m = 2$, $R = 1$, $i_0 = 0.09$, $\tau_1 = 0.25$, $\tau_2 = 0.45$, $T_1 = 10$, $T_2 = 20$. Three different lags are in play, and each must be converted into a time: **sketch the timeline first**.

- *Coupons* of $\frac{D}{2}$ at $t = 0.5, 1, 1.5, \ldots, 20$. The first, on 1 April 2023, is not in calendar 2022, so 2022 attracts no tax.
- *Income tax* $\tau_1 D$ on 31 December of each calendar year, a quarter after the second coupon: twenty payments at $t = 1.25, \ldots, 20.25$. It is $\tau_1 D$, *not* $2\tau_1 D$: D is the annual coupon, and the year's two payments of $\frac{D}{2}$ already add to D.
- *Capital gains tax*, once: the gain arises at redemption (1 October 2042) and is taxed on 31 March 2043, at $t = 20.5$.

Test: $i^{(2)} = 2(1.09^{1/2} - 1) = 0.0880613 > 0.75 \times 0.05 = 0.0375$, so a gain: assume $T = T_2 = 20$. (The test as proved assumes tax at the coupon dates; deferring it makes the coupons slightly more valuable, so the test is only approximate here. The margin is more than twofold, so the conclusion is safe; were it close, do the full calculation.)
(i) Equation of value. Both annuities appear: the coupons are half-yearly, but the tax on them is annual, one instalment covering both of that year's coupons:

$$P_0 = \underbrace{5\, a^{(2)}_{\overline{20}|}}_{\text{coupons}} + \underbrace{100\, v^{20}}_{\text{redemption}} - \underbrace{v^{0.25}(0.25 \times 5)\, a_{\overline{20}|}}_{\text{income tax}} - \underbrace{0.45(100 - P_0)v^{20.5}}_{\text{capital gains tax}},$$

with $a^{(2)}_{\overline{20}|} = \dfrac{1 - 1.09^{-20}}{0.0880613} = 9.329514$ and $a_{\overline{20}|} = 9.128546$:

$$P_0 = 46.647568 + 17.843089 - 11.167475 - 7.690761 + 0.0769076\, P_0,$$

so $0.9230924\, P_0 = 45.632421$ and $P_0 = €49.43$. (Equivalently: price 53.32 before capital gains tax, then the formula of Chapter 14 with $v^{20.5}$, since the tax is paid at $t = 20.5$.)

(ii) A price of €45. Sanity first: $45 < 49.43$, the price that gave exactly 9%, so we expect $i_0 > 9\%$. With the price known the circularity is gone and the capital gains tax is a definite $0.45 \times 55 = €24.75$:

$$\text{NPV}(i) = 5a^{(2)}_{\overline{20}|} + 100v^{20} - 1.25\,v^{0.25}a_{\overline{20}|} - 24.75\,v^{20.5} - 45.$$

$\text{NPV}(0.09) = 46.6476 + 17.8431 - 11.1675 - 4.2299 - 45 = +4.0933$ (everything already computed); at $i_2 = 0.10$ all four terms must be recomputed, including $i^{(2)}$: $\text{NPV}(0.10) = 43.6067 + 14.8644 - 10.3914 - 3.5077 - 45 = -0.4281$. So

$$i_0 \approx 0.09 + \frac{4.0933}{4.0933 + 0.4281}(0.01) = 0.0991,$$

about 9.9% (exact 0.0990). A minimum, as redemption before 2042 gives more.

THE CHAPTER IN ONE BOX

$i_0^{(m)} > (1-\tau_1)\frac{D}{R} \Rightarrow$ gain $\Rightarrow$ assume T_2, $\quad i_0^{(m)} < (1-\tau_1)\frac{D}{R} \Rightarrow$ loss $\Rightarrow$ assume T_1.

If the price is given, compare P_0 with R. $P(i,T)$ falls as i rises; as T rises it falls if there is a gain and rises if there is a loss. A gain should come early and a loss late, so the worst case is a late gain or an early loss. Capital gains tax changes none of it. With a given price and capital gains tax, the tax is a number and the yield comes by interpolation (recompute $i^{(m)}$ at each trial rate). Order of work: dates and cashflow; test and fix T; equation of value with each tax at its own date; solve (or interpolate). Coupons m-thly but tax annual: $a^{(m)}_{\overline{T}|}$ for the coupons, $v^{\text{lag}}a_{\overline{T}|}$ for the tax.

TRY IT YOURSELF

1. A stock pays 6% half-yearly coupons, redeemable at par at the borrower's option between 10 and 15 years. An investor pays 25% tax and needs 8% net. Find the price.
2. The same, but 10% coupons, redeemable between 10 and 20 years, 20% tax, 5% net. Show the worst case is now the earliest date, and price it.
3. An investor pays €95 per €100 nominal for a stock redeemable at par between 8 and 12 years. Which date gives the least yield?
4. In the €62.15 example the yield was exactly 10% at T_2 and higher at every earlier date. Why must equality occur at T_2 and nowhere else?
5. In the exam question, why does $a_{\overline{20}|}$ appear in the income-tax term but $a^{(2)}_{\overline{20}|}$ in the coupon term?
6. In the exam question €49.43 gave exactly 9% and €45 gave 9.9%. Without calculating, what would €55 give?

ANSWERS

1. Gain ($0.078461 > 0.045$), worst case $T_2 = 15$: €70.80 (at $T_1 = 10$ it would be €77.11,

unsafe).

2. Loss ($0.049390 < 0.08$), worst case $T_1 = 10$: €123.93.
3. $P_0 < R$: a gain, so the least yield is at the latest date, $T = 12$.
4. The price was chosen so the investor earns exactly i_0 at T_2. For $T < T_2$, with a gain, the price at a given rate decreases as T increases (the previous section), so at any earlier redemption the bond is worth strictly more than was paid; since P is strictly decreasing in i, the restoring rate is strictly greater than i_0.
5. The coupons are paid twice a year, forty payments of $\frac{D}{2}$: $a^{(2)}_{\overline{20}|}$. The tax is paid *once* a year, one instalment for both coupons: an ordinary annual annuity of twenty payments $a_{\overline{20}|}$, shifted by $v^{0.25}$.
6. Less than 9%: the cashflow is unchanged and only the price rises; paying more for the same stream earns less. (A higher price shrinks the gain, hence the tax, pushing the yield up a little, but far less than the price effect: about 8.05%.)

CHAPTER 16

Equities and property

LECTURE 19 · *Income that never stops, and grows*

AFTER THIS CHAPTER YOU CAN

- ☐ Say what an equity is, and why it is harder to value than a bond.
- ☐ Value an equity with constant dividend growth, and say when the model breaks.
- ☐ Value an equity bought ex dividend, with tax paid at a different date.
- ☐ Value a property with rent reviews every n years.

Our securities so far always ended. Here are two with no T and no R at all. The capital is never returned, so the whole value must come from an income that never stops: **warm-up.** At 9%, an income of €1.80 a year in perpetuity, in arrears, is worth $1.80/0.09 = €20.00$ (Chapter 6: €20 in the bank earns €1.80 a year, withdrawn for ever with the capital intact). An equity or a property is exactly this, except that the income *grows*.

16.1 Equities

EQUITY

Equities, or **ordinary shares**, are securities issued by companies that entitle holders to all the net profits after interest on loans and fixed-interest stocks has been paid. Shareholders own the company and have voting rights.

Two features set an equity apart from every security so far:

- **Redemption.** There is no redemption rate and no date. Capital is "redeemed" by selling the shares to someone else, and the profit or loss is the sale price less the purchase price.
- **Income.** There is no fixed rate. Part of the net profit is distributed annually as **dividends**; the rest is kept as reserves.

OUR MODEL IS NOT THE REAL PICTURE

Dividends fluctuate, payment is at the directors' discretion (it can be zero), there is a risk of default (capital lost entirely), and investors buy expecting dividends and capital values to rise. Future cashflows are unknowable and can only properly be modelled *stochastically*, which is outside this module. We assume a growth rate for the dividends, which gives a *deterministic* model instead.

16.2 The dividend growth model

Let P be the value just after the latest dividend. Assume dividends are paid annually, the dividend at time t is estimated at D_t, and they continue in perpetuity with no explicit allowance for default:

$$P = \sum_{t=1}^{\infty} D_t \, \mathrm{v}^t.$$

CONSTANT DIVIDEND GROWTH

If $D_t = D(1+g)^t$, where D is the dividend just paid,

$$P = D \sum_{t=1}^{\infty} \left(\frac{1+g}{1+i}\right)^t.$$

The trick. $\tilde{\mathrm{v}} = \frac{1+g}{1+i}$ is the discount factor for a *new* rate j with $1+j = \frac{1+i}{1+g}$:

$$j = \frac{i-g}{1+g}, \qquad \boxed{P = D\,a_{\overline{\infty}|} \text{ at rate } j = \frac{D}{j} = D\,\frac{1+g}{i-g}}$$

Growth g and discounting i combine into discounting at the single net rate j. The series converges exactly when $\tilde{\mathrm{v}} < 1$, i.e. $j > 0$, i.e. $\boxed{-1 < g < i}$.

IF $g \geq i$ THE PRICE IS INFINITE

This is not an arithmetic failure. A dividend stream growing for ever as fast as the discount rate, or faster, *is* an unbounded amount of value. You could compare two such streams, but you could not value one: the assumption must be revisited.

EXAMPLE 16.1 · Growth is worth a great deal

A share has just paid €1.80; dividends grow at $g = 3\%$; the investor requires $i = 9\%$. Then $j = 0.06/1.03 = 0.0582524$ and $P = 1.80/0.0582524 = €30.90$ (or $1.80 \times \frac{1.03}{0.06}$), against €20.00 with no growth.
Similarly $D = 2.40$, $g = 4\%$, $i = 10\%$: $j = 0.06/1.04 = 0.0576923$ and $P = 2.40/0.0576923 = €41.60$, the same as $2.40 \times \frac{1.04}{0.06}$. With no growth it would be $2.40/0.10 = €24$: growth for ever adds nearly three quarters.

EXAMPLE 16.2 · Tax deducted at each dividend

The same share ($D = 2.40$, $g = 4\%$, $i = 10\%$) for an investor paying income tax at 20%, deducted at each dividend date.

SOLUTION

Every dividend is multiplied by $1 - \tau = 0.8$, and a constant factor comes out of the sum: $P = 0.8 \times 41.60 = €33.28$. Exactly 80% because the tax is a constant proportion of every payment, deducted the moment the payment arrives, so the whole cashflow, and its present value, scale by 0.8.

NOT WHEN THE TAX IS PAID ON A DIFFERENT DATE

The 0.8 shortcut fails as soon as the tax falls on a *different* date from the dividend, as in the next example. There the factor is $1-\tau\sqrt{\mathrm{v}}$, not $1-\tau$.

16.3 Bought ex dividend

EX-DIVIDEND

Close to a dividend date an equity may be offered for sale *without* the next dividend (there may not be time for the company to update its records). It is then **ex-dividend** (**xd**). No new principle: omit the dividend the buyer will not receive and value what is left.

EXAMPLE 16.3 · An ex-dividend purchase

On 30 April 2020 an investor purchased 1000 shares *ex dividend*. Dividends are annual and the first, of €2.50 per share, was paid five months after the purchase. Dividends are expected to rise by 4% each calendar year. The investor pays tax at 30% on 31 March on all income received during the previous calendar year. Find the price paid if the expected net yield is 10% p.a.

SOLUTION

Bought ex dividend, the investor does *not* receive the first dividend of $D=2.50$ at $t=\frac{5}{12}$: it goes to the seller. The first they do receive is a year later. For $k=1,2,\dots$ the dividend per share at $t=k+\frac{5}{12}$ is $D(1+g)^k$, $g=0.04$, and the tax $\tau=0.3$ on it is paid on the following 31 March, six months later, at $t=k+\frac{11}{12}$. So per share

$$\begin{aligned} P_0 &= \sum_{k\ge 1} D(1+g)^k \mathrm{v}^{k+5/12} - \sum_{k\ge 1} \tau D(1+g)^k \mathrm{v}^{k+11/12} \\ &= D\big(1-\tau\sqrt{\mathrm{v}}\big)\mathrm{v}^{5/12}\sum_{k\ge 1}\big((1+g)\mathrm{v}\big)^k = D\big(1-\tau\sqrt{\mathrm{v}}\big)\mathrm{v}^{5/12}\,\frac{1+g}{i-g}, \end{aligned}$$

since $\mathrm{v}^{k+11/12}=\mathrm{v}^{k+5/12}\mathrm{v}^{1/2}$: the tax terms differ from the dividend terms by the constant factor $\tau\sqrt{\mathrm{v}}$. Numerically

$$2.5\times\big(1-0.3(1.1)^{-0.5}\big)\times(1.1)^{-5/12}\times\frac{1.04}{0.06}=29.73376,$$

so the price for 1000 shares is €29,733.76.

A GROWING PERPETUITY WITH A LAG

1. Which dividend is the first the buyer actually receives, and when?
2. Write the k-th dividend, its tax, and the time of each.
3. Factor out everything common: the first dividend's discount factor, and the extra lag factor on the tax.
4. What is left is $\sum((1+g)\mathrm{v})^k$, the growing perpetuity $\frac{1+g}{i-g}$ (valid only if $g < i$).

16.4 Property

Property is an investment in land and buildings. Valuing it by discounting future income follows very similar principles: both need an assumption about the growth of future income, both have income broadly linked to inflation in the long term, and both use a deterministic model. Two differences:

1. Rents are generally fixed for a number of years at a time and reviewed periodically, so changes happen at reviews (discrete times), not continually.
2. Some contracts are fixed term: after a certain period the income ceases and the property goes back to the original owner.

RENT REVIEWED EVERY n YEARS

Rent is paid monthly in arrears, reviewed every n years; the latest review has just happened and the rent for the next n years is X per month. Rents are expected to increase by g compound at each future review, in perpetuity. Between reviews the rent is level, so each block is an m-thly annuity with $m = 12$:

$$P = 12Xa^{(12)}_{\overline{n}|} + 12X(1+g)\mathrm{v}^n a^{(12)}_{\overline{n}|} + 12X(1+g)^2\mathrm{v}^{2n} a^{(12)}_{\overline{n}|} + \cdots = 12Xa^{(12)}_{\overline{n}|} \sum_{k=0}^{\infty} \left((1+g)\mathrm{v}^n\right)^k.$$

The ratio is $(1+g)\mathrm{v}^n$, and the series converges iff $(1+i)^n > 1+g$. Then

$$\boxed{P \;=\; 12X\, a^{(12)}_{\overline{n}|}\, \frac{(1+i)^n}{(1+i)^n - (1+g)}} \qquad \text{valid for } (1+i)^n > 1+g.$$

Compare the equity formula $P = D\frac{1+g}{i-g}$: both are growing perpetuities, but the growth occurs every n years here, so $(1+i)^n$ replaces $(1+i)$ and the rent block $12Xa^{(12)}_{\overline{n}|}$ replaces the single dividend.

WHEN DOES IT CONVERGE?

For $n \geq 2$ and $i > 0$, by the binomial theorem $(1+i)^n > 1+ni$; and $i \geq g/n$ gives $ni \geq g$, so $(1+i)^n > 1+g$. The condition $i \geq g/n$ therefore suffices. At $n = 1$, though, $(1+i) = 1+ni$ exactly, and $i \geq g$ allows $i = g$: the ratio is 1 and the series diverges, the equity's failure again.

EXAMPLE 16.4 · A five-year rent review

A property has just been reviewed and will pay €2,000 per month for five years, with reviews every five years and growth of 15% at each. Find the price to yield 7% p.a.

SOLUTION

$X = 2000$, $n = 5$, $g = 0.15$, $i = 0.07$. Convergence: $1.07^5 = 1.402552 > 1.15$. $i^{(12)} = 0.067850$, $v^5 = 0.712986$, $a_{\overline{5}|}^{(12)} = \frac{1-0.712986}{0.067850} = 4.230139$. So

$$P = 12 \times 2000 \times 4.230139 \times \frac{1.402552}{1.402552 - 1.15} = 101{,}523 \times 5.553523 = €563{,}812.$$

The rent is €24,000 a year, so the property sells for over twenty-three years' rent. This only makes sense because the rent is assumed to keep rising.

THE CHAPTER IN ONE BOX

$$\text{Equity:} \quad P = \frac{D}{j} = D\,\frac{1+g}{i-g}, \qquad j = \frac{i-g}{1+g}, \qquad -1 < g < i.$$

$$\text{Property:} \quad P = 12X\,a_{\overline{n}|}^{(12)}\,\frac{(1+i)^n}{(1+i)^n - (1+g)}, \qquad (1+i)^n > 1+g.$$

No redemption date and no price: the value is the whole income stream. Growth and discounting combine into one net rate; if it is not positive the model returns infinity. Ex dividend: omit the next dividend. Tax on a different date: factor out the lag.

TRY IT YOURSELF

1. A share has just paid a dividend of €2.40, growing at 4% p.a., for an investor requiring 10%. Find the price, first via j and then via $D\frac{1+g}{i-g}$.
2. For the same share, the investor now pays income tax at 20% deducted at each dividend date. Find the price, and explain why it is exactly 80% of the previous answer.
3. A property has just been reviewed and will pay €2,000 per month for five years, with reviews every five years and growth of 15% at each. Find the price to yield 7% p.a.

ANSWERS

1. $j = 0.06/1.04 = 0.0576923$, $P = 2.40/0.0576923 = €41.60$; directly $2.40 \times 1.04/0.06 = €41.60$. The two expressions differ only by clearing a fraction.
2. $P = 0.8 \times 41.60 = €33.28$: the tax is a constant proportion of every payment, deducted at the same moment it arrives, so the whole cashflow scales by 0.8.
3. €563,812 (over twenty-three years' rent).

CHAPTER 17

Inflation and real yields

LECTURE 20 · *What an investor is paid, and what it can buy*

AFTER THIS CHAPTER YOU CAN

- ☐ Deflate a cashflow with an index, and find a real yield.
- ☐ Use $1+i=(1+j)(1+i_r)$ in both directions.
- ☐ Say when indexation and deflation cancel, and when they do not.
- ☐ Work an index-linked bond with a time lag: money yield and real yield.

Thus far we moved money through time and compared the amounts. But a euro at time t does not buy what a euro at time 0 buys. Every yield so far has been a **money yield** (or **nominal yield**).

Warm-up. Invest 100 at time 0, receive 120 at time 1; prices rise 5%. Goods costing 1 at time 0 cost 1.05 at time 1, so 120 buys what $\frac{120}{1.05}=114.286$ would have bought at time 0. Against a cost of 100 purchasing power has risen by 14.286%. The face value is 20%, but $20\%-5\%=15\%$ is *not* the answer.

17.1 Real and money yields

REAL RATE OF INTEREST

The **real rate of interest** of a transaction is the rate after allowing for the effect of inflation on a payment series. We find it by first adjusting all payments for inflation, so that they are expressed in purchasing power *at the same date*.

Where the rates of inflation are *known* (completed transactions) adjust by reference to an index $Q(t)$: a payment c_t is worth $c_t\,\dfrac{Q(0)}{Q(t)}$ in time-0 money (prices rose by $\frac{Q(t)}{Q(0)}$). $Q(t)$ *deflates* the payment to time 0; then v_{i_r} *discounts* it at the real rate.

THE REAL YIELD EQUATION

$$\sum_k c_{t_k}\frac{Q(0)}{Q(t_k)}\,\mathrm{v}_{i_r}^{t_k}=0 \qquad\Longleftrightarrow\qquad \sum_k \frac{c_{t_k}}{Q(t_k)}\,\mathrm{v}_{i_r}^{t_k}=0.$$

$Q(0)$ vanishes, so the real yield does not depend on which date the payments are expressed in.

EXAMPLE 17.1 · An 8% bond in real terms

A three-year par bond with an 8% coupon (money yield 8%):

Time t	0	1	2	3
Cashflow c_t	-100	8	8	108
Index $Q(t)$	150	156	166	175
$Q(0)/Q(t)$	1	$150/156$	$150/166$	$150/175$
In time-0 money	-100	7.6923	7.2289	92.5714

The real yield solves $-100 + 7.6923v + 7.2289v^2 + 92.5714v^3 = 0$; by interpolation (Chapter 9), $i_r = 2.63\%$. An 8% bond returned barely 2.6% in purchasing power.

17.2 Constant inflation

Suppose inflation is at a constant rate j p.a., so c at time t is worth $c(1+j)^{-t}$ in time-0 terms. The real and money equations are $\sum c_{t_k} v_j^{t_k} v_{i_r}^{t_k} = 0$ and $\sum c_{t_k} v_i^{t_k} = 0$: same coefficients, same exponents, so the same solutions precisely when $v_i = v_j v_{i_r}$.

MONEY, REAL AND INFLATION

$$1 + i = (1+j)(1+i_r), \qquad \boxed{i_r = \frac{i-j}{1+j}} \qquad \text{and conversely} \qquad \boxed{i = i_r + j(1+i_r)}$$

$i_r \neq i - j$

Subtracting inflation is only a first approximation. In the warm-up $i = 0.20$, $j = 0.05$ give $i_r = \frac{0.15}{1.05} = 0.142857$, not 0.15: the approximation *overstates* the real return by 0.71 percentage points.

EXAMPLE 17.2 · Two years, four percent inflation

€500 returns €650 after two years, with inflation at 4%. Find the money and real yields, and check $i_r = (i-j)/(1+j)$.

SOLUTION

Money: $500(1+i)^2 = 650$, $i = (650/500)^{1/2} - 1 = 14.0175\%$. Real: €650 at time 2 is worth $650/1.04^2 = €600.96$ in time-0 money, so $i_r = (600.96/500)^{1/2} - 1 = 9.6323\%$. Check: $\frac{0.140175 - 0.04}{1.04} = 0.096323$. ($i - j = 10.02\%$ would have overstated it.)

17.3 Payments linked to inflation

Some contracts specify that the cashflows *will* be adjusted for inflation by a given index, so the actual cashflows are unknown until the index values are. For a nominal amount c_t, base value $Q(0)$ and relevant value $Q(t)$, the actual cashflow is $c_t \frac{Q(t)}{Q(0)}$: adjusted *upwards*. (Contrast the previous section, where payments were fixed in money terms and we deflated them.)

WHEN THE TWO INDICES AGREE

If the real yield is calculated with the *same* index used to inflate the cashflows,

$$\sum_k \Big(c_{t_k} \frac{Q(t_k)}{Q(0)} \Big) \frac{Q(0)}{Q(t_k)} \mathrm{v}_{i_r}^{t_k} = 0 \iff \sum_k c_{t_k} \mathrm{v}_{i_r}^{t_k} = 0.$$

The indexation is exactly undone by the deflation: **solve with the nominal amounts**, which already *are* the real ones. (For the government bond of Chapter 13 with coupons $19.13 \frac{Q(t)}{Q(0)}$, the real value is 19.13 as before.)

NOT ALWAYS, AND NOT WITH A LAG

It is *not* always the case that the index used to inflate the cashflows is the one used to compute the real yield. And even the same index fails to cancel if it is applied with a *time lag*.

17.4 Index-linked bonds and the time lag

An **index-linked bond** has interest and redemption payments linked to an inflation index. Computing a price or yield needs an estimate of the index, since future values are unknown. A time lag s is needed from the date of the index value used, to allow for its publication and for the coupon to be calculated in advance: *there is no inflation protection during the lag period.*

THE LAG

With lag s, a coupon c at time t becomes $c \frac{Q(t-s)}{Q(-s)}$, with real value $c \dfrac{Q(t-s)}{Q(-s)} \cdot \dfrac{Q(0)}{Q(t)}$: four index values that no longer cancel. For a one-year lag,

$$c \frac{Q(t-1)}{Q(-1)} \cdot \frac{Q(0)}{Q(t)} = c \underbrace{\frac{Q(0)}{Q(-1)}}_{1+j_{\text{base}}} \cdot \underbrace{\frac{Q(t-1)}{Q(t)}}_{1/(1+j_t)} = c \frac{1+j_{\text{base}}}{1+j_t},$$

where j_t is the inflation over the year ending at t. **The lag protects the coupon against inflation of the base year, not the current one.** Steady inflation nearly cancels; when it jumps, the real coupon falls.

EXAMPLE 17.3 · A 3.5% stock, one-year lag

A 3.5% coupon stock issued February 2014, redeemed February 2019, annual coupons linked to an index with a one-year lag; base month February 2013.

Year	2013	2014	2015	2016	2017	2018	2019
Index value	540	562	584	607	632	657	788
Inflation %		4.07	3.91	3.94	4.12	3.96	19.94

SOLUTION

Take February 2014 as time 0: coupons at $t = 1, \dots, 5$, base $t = -1$, $s = 1$; the coupon at t is $3.50\frac{Q(t-1)}{Q(-1)}$ and its real value in February 2014 terms is that times $\frac{Q(0)}{Q(t)}$. E.g. $t = 2$ (2016): $3.5 \times \frac{584}{540} = 3.79$, real $3.79 \times \frac{562}{607} \approx 3.50$. The whole table:

Year	2015	2016	2017	2018	2019
Coupon per €100	3.64	3.79	3.93	4.10	4.26
Real value, Feb 2014 terms	3.51	3.50	3.50	3.50	3.04

(Differences of a cent from rounding the coupon first.) With no lag the real value would be exactly 3.50 every year. With it, it stays within a cent for four years, and then collapses: inflation jumped to 19.94% in 2019 against a base year of 4.07%, so the real coupon is $3.5 \times \frac{1.0407}{1.1994} = 3.04$.

WHAT LIFTS A REAL COUPON ABOVE ITS NOMINAL SIZE?

The real value $3.5\,\frac{1+j_{\text{base}}}{1+j_t}$ exceeds 3.50 exactly when $j_t < j_{\text{base}}$: inflation in the current year is *lower* than in the base year. Falling inflation benefits the holder of a lagged index-linked bond, rising inflation hurts them. This is the opposite of what such a bond is usually assumed to do: the protection is real but delayed, by exactly one lag period.

AN INDEX-LINKED YIELD QUESTION

1. Put the issue date at $t = 0$ and list the payment times. For each payment c_t, find $Q(t-s)$ (the lagged value), the base $Q(-s)$, and $Q(t)$, $Q(0)$.
2. *Money yield:* inflate each payment by $\frac{Q(t-s)}{Q(-s)}$; solve for i.
3. *Real yield:* inflate *and* deflate by $\frac{Q(0)}{Q(t)}$; solve for i_r.
4. If all the times are even, put $x = v^2$ and solve a quadratic; discard the negative root.

EXAMPLE 17.4 · A four-year index-linked security

A four-year index-linked security, issued on 1 April 2015, pays nominal coupons of 8% every two years in arrears and is redeemed at 110%. Coupons and capital are inflated by reference to an index Q six months before payment. (a) Find the effective rate i p.a. earned over the life of the security, if the price is €100 per €100 nominal on 1 April 2015. (b) Find the real rate of return i_r.

Date	1/10/14	1/4/15	1/10/16	1/4/17	1/4/18	1/10/18	1/4/19
Q	601	631	662	695	730	788	877
Time	-0.5	0	1.5	2	3	3.5	4

SOLUTION

$s = 0.5$, time 0 is 1/4/15. Coupons of 8 fall at $t = 2, 4$ and the redemption of 110 at $t = 4$ (so 118 at $t = 4$). Note which index values appear: the payment at $t = 2$ uses the index six months *earlier*, $t = 1.5$, and the base is $Q(-0.5)$, not $Q(0)$.

(a) $100 = 8\mathrm{v}^2\frac{662}{601} + 118\mathrm{v}^4\frac{788}{601} = 8.81\mathrm{v}^2 + 154.72\mathrm{v}^4$. Put $x = \mathrm{v}^2$: $154.72x^2 + 8.81x - 100 = 0$, roots $x = 0.775979$ and -0.8329; discard the negative. $i = \frac{1}{\sqrt{0.775979}} - 1 = 13.52\%$.

(b) In real terms,

$$100 = 8\mathrm{v}_{i_r}^2\frac{662 \times 631}{601 \times 695} + 118\mathrm{v}_{i_r}^4\frac{788 \times 631}{601 \times 877} = 8\mathrm{v}_{i_r}^2 + 111.32\mathrm{v}_{i_r}^4,$$

(the first coefficient came out at almost exactly 8: over that stretch the lag cost almost nothing). Solving, $\mathrm{v}_{i_r}^2 = 0.912541$ and $i_r = \frac{1}{\sqrt{0.912541}} - 1 = 4.68\%$.

The investor earned 13.52% in money terms but 4.68% in purchasing power. Consistent: Q rose from 631 to 877 over four years, 8.58% p.a., and $(1.0468)(1.0858) - 1 = 13.66\%$, close to 13.52%, the small gap being the effect of the lag.

THE CHAPTER IN ONE BOX

$$\sum_k c_{t_k}\frac{Q(0)}{Q(t_k)}\mathrm{v}_{i_r}^{t_k} = 0, \qquad 1 + i = (1 + j)(1 + i_r), \qquad i_r = \frac{i - j}{1 + j}.$$

Deflate first, then discount. The base date cancels. Indexing with the same index used to deflate makes the two cancel, and the nominal amounts *are* the real ones. A time lag breaks that cancellation: it protects against base-period inflation rather than current.

TRY IT YOURSELF

1. €500 returns €650 after two years, with inflation at 4%. Find the money and real yields, and check $i_r = (i - j)/(1 + j)$.
2. A bond has $Q(0) = 200$, $Q(3) = 260$, $Q(-1) = 190$, $Q(2) = 245$ and a coupon of nominal €4. Find the coupon at $t = 3$ and its real value, with no lag and with a one-year lag.
3. In the 3.5% stock example the real coupons stayed near 3.50. What would push one above it?

ANSWERS

1. $i = 14.0175\%$; $i_r = 9.6323\% = \frac{0.140175 - 0.04}{1.04}$.
2. No lag: $4 \times \frac{260}{200} = €5.20$, real $5.20 \times \frac{200}{260} = €4.00$ (the two factors are reciprocals). One-year lag: $4 \times \frac{245}{190} = €5.16$, real $5.1579 \times \frac{200}{260} = €3.97$: the indexation used $Q(2)/Q(-1)$ but the deflation $Q(0)/Q(3)$, different periods. The shortfall is the inflation of the year from $t = 2$ to $t = 3$, against which the coupon was never protected.

3. $3.5\frac{1+j_{\text{base}}}{1+j_t} > 3.5 \iff j_t < j_{\text{base}}$: inflation in the current year lower than in the base year.

PART VI

Project appraisal

CHAPTER 18

Project appraisal

LECTURE 21 · *Escalating cashflows, accumulated profit and the IRR*

AFTER THIS CHAPTER YOU CAN

- ☐ Escalate a project's cashflows and value them at the adjusted rate $\frac{i-j}{1+j}$.
- ☐ Compute the accumulated profit $A(T)$ of a project, discrete and continuous.
- ☐ Use $A(T) = (1+i)^T \,\mathrm{NPV}(i)$ and know when two projects may be compared.
- ☐ Decide profitability from the yield and the slope of NPV.

A security is bought, or a *project* is undertaken. How do we appraise one? **Warm-up.** A venture needs €1,000 now and returns €400, €500, €600 at the ends of the next three years, borrowing and lending at 10%:

$$\mathrm{NPV}(0.10) = -1000 + \frac{400}{1.1} + \frac{500}{1.1^2} + \frac{600}{1.1^3} = -1000 + 1227.65 = €227.65.$$

Positive, so it is worth doing at 10%. But those figures are in *today's* prices; the money that actually changes hands in three years' time will have different buying power.

18.1 Escalating cashflows

INFLATION AND ESCALATION

Inflation is the rise in the general price level of a broad basket of goods: the overall decline in purchasing power. **Escalation** is the rise in the price of a *specific item or cost*, driven by factors unique to it (supply and demand, regulation, scarcity), so it can be higher or lower than general inflation. For this module we treat them mostly interchangeably.

If every item of cashflow is subject to the same escalation rate j per unit time, then with c_t and $\rho(t)$ estimated *without* any allowance for inflation (today's prices),

$$c_t^e = (1+j)^t c_t, \qquad \rho^e(t) = (1+j)^t \rho(t),$$

what actually changes hands at time t. The whole project is scaled up, date by date.

NET PRESENT VALUE WITH ESCALATION

$$\begin{aligned}\mathrm{NPV}_j(i) &= \sum_t c_t(1+j)^t(1+i)^{-t} + \int \rho(t)(1+j)^t(1+i)^{-t}\,\mathrm{d}t \\ &= \sum_t c_t(1+i_{\mathrm{e}})^{-t} + \int \rho(t)(1+i_{\mathrm{e}})^{-t}\,\mathrm{d}t,\end{aligned}$$

where $(1+j)^t(1+i)^{-t} = \left(\frac{1+i}{1+j}\right)^{-t}$, so

$$\boxed{1+i_{\mathrm{e}} = \frac{1+i}{1+j}, \qquad i_{\mathrm{e}} = \frac{i-j}{1+j}}$$

(written i_0 in the lecture slides). **Escalating cashflows discounted at i behave like *level* cashflows discounted at the lower rate i_{e}.**

This is the algebra of Chapter 17 again: there the money cashflows were *fixed* and then deflated, giving $i_r = \frac{i-j}{1+j}$; here they are *escalated* and then discounted, and we get the same expression. In both cases $(1+j)$ and $(1+i)$ meet as a ratio and inflation is absorbed into one adjusted rate.

THE APPROXIMATION $i - j$

For small j, $i_{\mathrm{e}} \approx i - j$: $i = 10\%$, $j = 5\%$ gives 5% against the exact 4.762%. A useful check, but it discounts every term slightly too hard (the exact rate is lower).

EXAMPLE 18.1 · Escalation can transform a project

The warm-up venture $(-1000, +400, +500, +600$ at $t = 0, 1, 2, 3)$, financed at $i = 10\%$, with every cashflow escalating at $j = 5\%$.

SOLUTION

$i_{\mathrm{e}} = 0.05/1.05 = 4.762\%$, so

$$\mathrm{NPV}_j(0.10) = -1000 + \frac{400}{1.04762} + \frac{500}{1.04762^2} + \frac{600}{1.04762^3} = €359.24,$$

against €227.65: more than 50% more profit. Projects that look unprofitable when interest rates are high can become highly profitable once even a modest allowance is made for inflation, because escalating income financed at i behaves like level income financed at the lower i_{e}.

18.2 Accumulated profit

To appraise a project we need a cashflow model: outgo $(-)$, income $(+)$, and the uncertainty and risks that go with them (delays, default, force majeure). Cashflow estimation and risk assessment take experience, judgement and assumptions; the arithmetic here is the easy part.

ACCUMULATED PROFIT

The **accumulated profit** of a project ending at time $T < \infty$ is the accumulated value of its net cashflows at the time of the last payment, at effective rate i:

$$A(T) = \sum_t (1+i)^{T-t} c_t + \int_0^T (1+i)^{T-t} \rho(t)\,\mathrm{d}t.$$

Appraisal: choose the project with the highest accumulated profit.

A DEFINITE END DATE IS NEEDED

$A(T)$ requires $T < \infty$. A perpetual cashflow has no accumulated profit. And note $T - t$, not $t - T$: we are *accumulating* to time T, not discounting to time 0.

EXAMPLE 18.2 · A venture with a long continuous inflow

An investor estimates that a new venture needs an initial outlay of €20,000 and a further outlay of €10,000 after one year. Beginning in three years, an inflow of €3,000 p.a. payable continuously for ten years is expected, and a final inflow of €6,000 at the end of the project in thirteen years. Find the accumulated profit at an effective rate of 1.2%.

SOLUTION

$c_0 = -20000$, $c_1 = -10000$, $c_{13} = +6000$, and $\rho(t) = 3000$ for $3 \le t \le 13$:

$$A(13) = -20000(1.012)^{13} - 10000(1.012)^{12} + 6000 + \int_3^{13} 3000(1.012)^{13-t}\,\mathrm{d}t.$$

The integral is an accumulated continuous annuity: $u = 13 - t$ turns it into $\int_0^{10} 3000(1.012)^u\,\mathrm{d}u = 3000\,\bar{s}_{\overline{10}|}$. With $\delta = \ln 1.012$,

$$3000\,\bar{s}_{\overline{10}|} = 3000\,\frac{1.012^{10} - 1}{\ln 1.012} = 3000 \times 10.620868 = €31{,}862.60,$$

while the discrete flows accumulate to $-20000(1.167741) - 10000(1.153895) + 6000 = -€28{,}893.77$. Therefore

$$A(13) = -28893.77 + 31862.60 = €2{,}968.83 > 0.$$

18.3 Present value or accumulated profit?

Every summand of $A(T)$ is $(1+i)^T$ times the corresponding summand of $\mathrm{NPV}(i)$: A values at the *end*, NPV at the *start*.

TWO DATES, ONE CASHFLOW

$$\boxed{A(T) = (1+i)^T \cdot \mathrm{NPV}(i)} \qquad (T < \infty).$$

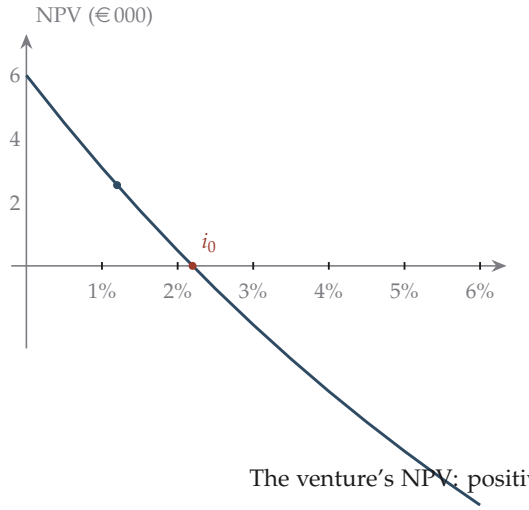
This is Chapter 4's transfer principle. As $(1+i)^T > 0$, $A(T)$ and NPV have the same sign, and the highest accumulated profit and the highest NPV are the same project. Profitable means $\mathrm{NPV}(i) > 0$.

THE DATES MUST MATCH

Comparing $A(T_1)$ with $A(T_2)$ for $T_1 \neq T_2$ compares two different moments and can rank projects wrongly. Convert to a common date, or just use NPV.

For the venture above, $\mathrm{NPV}(0.012) = \dfrac{A(13)}{1.012^{13}} = \dfrac{2968.83}{1.167741} = €2{,}542.37 > 0$: profitable at 1.2%, as the positive accumulated profit already said.

18.4 Profitability and the internal rate of return



The venture's NPV: positive at 1.2%, zero at $i_0 = 2.197\%$.

Let i be the rate at which the investor may lend or borrow, and assume the yield i_0 exists and $\mathrm{NPV}(i)$ changes sign there. The crossing may go either way, so the slope must be specified.

THE PROFITABILITY RULE

$$\boxed{\text{profitable} \iff (i - i_0) \cdot \text{NPV}'(i_0) > 0}$$

where $\text{NPV}'(i_0)$ is the derivative of NPV with respect to i at the yield. For the usual *decreasing* NPV ($\text{NPV}' < 0$) this reads $i < i_0$: profitable when you can borrow below the project's yield. If NPV is *increasing* ($\text{NPV}' > 0$, a project that takes money in first and pays out later, i.e. a borrowing) it reads $i > i_0$: a higher rate is good news.

INTERNAL RATE OF RETURN

The **IRR**, or *yield to redemption*, is the accountant's and economist's name for the yield i_0 when it exists: the perceived return if the project is *not* financed by a loan, i.e. financed internally.

EXAMPLE 18.3 · Reading the rule

For the venture, $i_0 = 2.197\%$ and NPV is decreasing:

Rate available	$i - i_0$	$\text{NPV}(i)$	Verdict
$i = 0.5\%$	< 0	$+€4{,}503$	Profitable
$i = 1.2\%$	< 0	$+€2{,}542$	Profitable
$i = 3\%$	> 0	$-€1{,}854$	Loses money

The venture earns 2.197% internally. Borrow more cheaply and you keep the difference; borrow more dearly and the loan costs more than the venture makes.

APPRAISING A PROJECT

1. Build the cashflow, escalated if required (replace i by $\frac{i-j}{1+j}$ for constant escalation).
2. $\text{NPV}(i)$ at the investor's rate (or $A(T) = (1+i)^T \text{NPV}$): positive means worth doing.
3. Or find the yield and apply $(i - i_0)\,\text{NPV}'(i_0) > 0$, checking the slope.

THE CHAPTER IN ONE BOX

$$A(T) = (1+i)^T \text{NPV}(i), \qquad i_e = \frac{i-j}{1+j} \text{ (escalation at } j\text{)},$$

$$\text{profitable} \iff (i - i_0)\ \text{NPV}'(i_0) > 0.$$

Escalating cashflows discounted at i behave like level ones discounted at i_e. The IRR is a property of the project alone; whether it is worth doing depends on the rate the investor gets.

TRY IT YOURSELF

1. A project costs €5,000 now and returns €2,000 at the end of each of three years, at $i = 6\%$. Find $\text{NPV}(0.06)$ and $A(3)$, and check $A(3) = (1.06)^3 \text{NPV}(0.06)$.

2. The same project's cashflows now escalate at $j = 3\%$, still financed at 6%. Find i_e and the new net present value, and compare with the approximation $i_e \approx i - j$.
3. A project has NPV *increasing* in i, with yield $i_0 = 8\%$. Is it profitable at 5%? At 11%?

ANSWERS

1. $a_{\overline{3}|} = 2.673012$: NPV $= -5000 + 2000 \times 2.673012 = €346.02$. $A(3) = -5000(1.06)^3 + 2000(1.06^2 + 1.06 + 1) = -5955.08 + 6367.20 = €412.12$, and $346.02 \times 1.191016 = 412.12$. Both positive: profitable.
2. $i_e = 0.03/1.03 = 0.029126$, $a_{\overline{3}|} = 2.833370$ at that rate: NPV $= -5000 + 2000 \times 2.833370 = €666.74$, nearly double. The approximation 0.03 is high by 0.00087 and costs about €9.52 here; the error grows with the term.
3. $\text{NPV}'(i_0) > 0$, so profitable iff $i > i_0$: *not* at 5%, but yes at 11%. The reverse of the usual pattern, hence the slope in the rule: the project takes money in first and pays it out later, a borrowing for which a higher rate is good news.

CHAPTER 19

Comparing projects

LECTURE 22 · *Why the higher yield need not win*

AFTER THIS CHAPTER YOU CAN

- ☐ Rank two projects by NPV at the investor's rate, not by yield.
- ☐ Find the cross-over rate and say which project wins on each side of it.
- ☐ Value a project with a continuous outgo, and a loan with a final return of capital.
- ☐ Use the par-bond shortcut for the yield of a loan.

That settles whether *one* project is worth doing. It does not settle which of *two* is better.

Warm-up. Two ventures have $\mathrm{NPV}_P(i) = -100 + 130v$ and $\mathrm{NPV}_Q(i) = -100 + 145v^2$. Setting each to zero: $i_P = 1.3 - 1 = 30\%$, and $i_Q = \sqrt{1.45} - 1 = 20.4\%$. P has a much higher yield, and the natural answer to "which would you rather have?" is P. Read on.

19.1 Comparing two projects

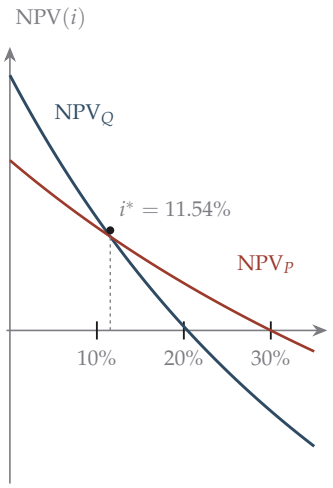
Assume the investor's borrowing power is **not limited**. Let NPV_A, NPV_B be the net present value functions and i_A, i_B the yields (assumed to exist).

THE RIGHT CRITERION

Project A is more profitable than project B at the rate i_1 at which the investor may actually lend or borrow if

$$\mathrm{NPV}_A(i_1) > \mathrm{NPV}_B(i_1).$$

Equivalently, compare profits at time T, the date when the later project ends. $i_A > i_B$ does *not* imply $\mathrm{NPV}_A(i_1) > \mathrm{NPV}_B(i_1)$: two decreasing curves can cross.



Q has the *lower* yield (20.4% against 30%) but, for every rate below the **cross-over rate** i^*, the higher NPV.

EXAMPLE 19.1 · The warm-up, resolved

The two ventures cross where $130\mathrm{v} = 145\mathrm{v}^2$, that is $\mathrm{v} = \frac{130}{145}$: $i^* = \frac{145}{130} - 1 = 11.54\%$. Below i^*, Q is better; above it, P. At 10%:

$$\mathrm{NPV}_P(0.10) = -100 + \frac{130}{1.1} = 18.18, \qquad \mathrm{NPV}_Q(0.10) = -100 + \frac{145}{1.21} = 19.83.$$

If the investor's rate is 10%, Q wins despite $i_P = 30\%$ against $i_Q = 20.4\%$.

YIELD BELONGS TO THE PROJECT ALONE

The yield does not know the investor. The *profit* depends on the project *and* the rate at which the investor can borrow or lend. For an unrestricted investor a higher yield might do; in general the investor is constrained, and below the cross-over rate the project with the *lower* yield wins.

19.2 Two ways to build a system

EXAMPLE 19.2 · Contract out or do it in-house?

A software company is to set up a new computer system for a client. It considers whether to contract out the work (A) or do it in-house (B). It can borrow or lend at 20% p.a. Would you advise it to invest in either project, and if so which is more profitable? (Amounts in €1,000s; "through" means a continuous cashflow.)

Project A			**Project B**		
Contractor fees	Start Y1	−150	New equipment	Start Y1	−325
Contractor fees	Start Y2	−250	Staff costs	Through Y1	−75
Contractor fees	Start Y3	−250	Staff costs	Through Y2	−90
Sales	End Y3	1000	Staff costs	Through Y3	−120
			Sales	End Y3	1000

SOLUTION

Project A is the project of Chapter 9: $\mathrm{NPV}_A(i) = -150 - 250v - 250v^2 + 1000v^3$, $\mathrm{NPV}_A(0.2) = 46.76$, yield $i_A \approx 0.2525$.

Project B. The staff costs are paid *through* each year, so they are continuous streams at rates $75, 90, 120$. Each year's stream is worth $\bar{a}_{\overline{1}|}$ times its rate, valued at the *start* of that year, so

$$\mathrm{NPV}_B(i) = -325 - \left(75 + 90v + 120v^2\right)\bar{a}_{\overline{1}|} + 1000v^3, \qquad \bar{a}_{\overline{1}|} = \frac{1-v}{\delta} = \frac{d}{\delta}.$$

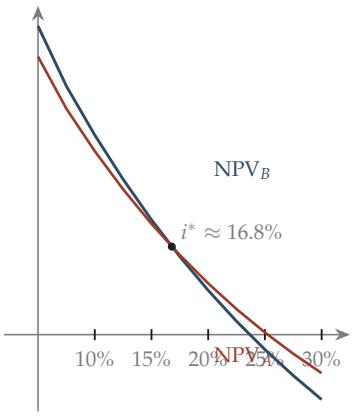
At 20%: $\delta = \ln 1.2 = 0.1823216$, $\bar{a}_{\overline{1}|} = 0.914136$, and $75 + 90(0.8\overline{3}) + 120(0.8\overline{3})^2 = 233.3333$, so

$$\mathrm{NPV}_B(0.2) = -325 - 233.3333 \times 0.914136 + 1000(0.8\overline{3})^3 = 40.41.$$

Both projects are profitable at 20%, and A is more profitable: $46.76 > 40.41$. For B's yield, $\mathrm{NPV}_B(0.25) = -325 - 223.8 \times 0.896284 + 512 = -13.59 < 0$; interpolating,

$$i_B \approx 0.2 + \frac{40.41}{40.41 + 13.59}(0.05) = 0.237$$

(a wide bracket, so rough; the true value is 0.2366). So $0.2 < i_B < i_A$: both yields exceed the borrowing rate, confirming both are worthwhile.



The curves cross at about 16.8%. At $i = 10\%$: $\mathrm{NPV}_B = 182.1$ against $\mathrm{NPV}_A = 167.4$, so **B** is better despite its *lower* yield. At 15% (also below i^*), 104.8 against 101.1: B again. At 24%, where $i_B < 0.24 < i_A$, B makes a loss and A a profit. The rate at which the company can borrow or lend is the decisive factor. Why? B's outgo is spread through the three years while A's falls in three lumps at the year starts, so B suffers less when money is cheap and more when it is dear.

COMPARING PROJECTS

1. Compute the NPV of each project from its cashflow model.
2. Compute the yield of each, and check profitability at the borrowing/lending rate.
3. Choose the project with the highest NPV *at that rate*.

The first two answer whether the projects are worth doing at all. Only the third ranks them.

19.3 Two loans

EXAMPLE 19.3 · Loan X or loan Y?

An investor considers two loans. X: for €10,000, €1,000 p.a. payable quarterly in arrears for 15 years. Y: for €11,000, €605 annually in arrears for 18 years, and the initial investment returned at the end. The investor may borrow or lend at 4%. Is either worth investing in, and if so which is more profitable? (Work in €1,000s.)

SOLUTION

Loan X: $\mathrm{NPV}_X(i) = -10 + a^{(4)}_{\overline{15}|}$. At $i = 0.04$, $i^{(4)} = 0.039414$ and $v^{15} = 0.555265$, so $a^{(4)}_{\overline{15}|} = 11.2838$ and $\mathrm{NPV}_X(0.04) = 1.2838$. Interpolating between 5% and 6%: $i_X \approx 0.0589$.
Loan Y: $\mathrm{NPV}_Y(i) = -11 + 0.605a_{\overline{18}|} + 11v^{18}$. At 0.04, $a_{\overline{18}|} = 12.659297$, $v^{18} = 0.493628$: $\mathrm{NPV}_Y(0.04) = 2.08878$. Interpolating gives 0.0552, but *here we can do better*: Y returns exactly the capital it cost, so it is a bond redeemable *at par*, and its yield is its coupon rate $0.605/11 = 0.055$. Indeed at 0.055, $0.605a_{\overline{18}|} = 11(1 - v^{18})$, so $\mathrm{NPV}_Y(0.055) = -11 + 11 - 11v^{18} + 11v^{18} = 0$ exactly: $i_Y = 5.5\%$, and the interpolated value was high by two ten-thousandths.
Verdict.

	Loan X	Loan Y
Yield	0.0589	0.0550
NPV at 4%	1.2838	2.0888

Both are worth investing in (each yield exceeds 4%), and Y **is more profitable**, even though its yield is the lower. Again the borrowing rate decides: at 5.7%, $i_Y < 0.057 < i_X$, $\mathrm{NPV}_X = 0.115$ while $\mathrm{NPV}_Y = -0.244$.

THE PAR-BOND SHORTCUT

If a loan returns exactly its purchase price at the end, it is a bond redeemable at par and **its yield is the coupon rate** (Chapter 13): at that rate the income is exactly the interest due on the capital, which is returned intact. For €5,000 lent for twenty years at €300 a year, the yield is $300/5000 = 6\%$, without any calculation.

THE CHAPTER IN ONE BOX

$$A \text{ more profitable than } B \text{ at } i_1 \iff \mathrm{NPV}_A(i_1) > \mathrm{NPV}_B(i_1),$$
$$i_A > i_B \not\Rightarrow \mathrm{NPV}_A(i_1) > \mathrm{NPV}_B(i_1).$$

Rank by profit, not by yield. The yield is a property of the project alone; the ranking brings the investor in. Below the cross-over rate the project with the lower yield wins. Continuous outgo through a year: rate $\times \bar{a}_{\overline{1}|}$, valued at the start of that year. A loan returning its price is a par bond: yield = coupon rate.

TRY IT YOURSELF

1. A loan costs €5,000 and pays €300 annually in arrears for 20 years, with the €5,000 returned at the end. Write down its yield without calculating, and justify it.
2. In the software example, verify that NPV_A and NPV_B agree at $i^* = 0.168$, and say which project you would choose at $i = 15\%$.
3. Two projects have the same yield i_0. Can one still be preferable to the other? Justify your answer with a sketch.

ANSWERS

1. 6%: the loan is a par bond, and at 0.06, $300a_{\overline{20}|} = 5000(1 - \text{v}^{20})$, so $\text{NPV} = -5000 + 5000 - 5000\text{v}^{20} + 5000\text{v}^{20} = 0$. The income is exactly the interest on €5,000, so the capital is neither repaid nor eroded.
2. At 0.168 both are about 80.3 per €0. At 15%, below i^*: $\text{NPV}_B = 104.8 > \text{NPV}_A = 101.1$, so choose **B**, even though $i_A = 0.2525 > 0.2366 = i_B$.
3. Yes. The yield fixes only where a curve crosses the axis, nothing about the curve anywhere else. Two curves through the same point on the axis can differ elsewhere, and at every rate below i_0 the upper one is worth more. Equal yields mean equally *marginal*, not equally *good*.

CHAPTER 20

Borrowing dearer than lending

LECTURES 23–24 · *Tracking the balance, and finding the date it turns*

AFTER THIS CHAPTER YOU CAN

- ☐ Say why NPV and the yield lose their meaning when $j_1 \neq j_2$.
- ☐ Roll a balance forward year by year, at whichever rate applies.
- ☐ Treat a fixed-term loan with no early repayment correctly.
- ☐ Define and find the discounted payback period, and say what it depends on.
- ☐ Compare projects that finish on different dates.
- ☐ Find the payback date by dividing by $(1+j_1)^t$: continuous income, a blip, two rates, and annual payments (the integer case).

Every appraisal so far rested on a *single* i, borrowed and lent alike. In reality an investor faces two, and today we drop that assumption.

Warm-up. A company can borrow at 8% and lend at 5%. It borrows €50,000 to fund a project returning €20,000 at the end of each of three years. Repayments and lendings are made at year ends. What is the balance after three years? (Work in €0s.)

	interest	receipt	balance owed
End Y1	4.000	20	$50+4-20=34.000$
End Y2	2.720	20	$34+2.72-20=16.720$
End Y3	1.338	20	$16.72+1.338-20=-1.942$

The debt turns into a credit during the third year, a profit of €1.94 (thousand). Only the *borrowing* rate was used: the balance was never in credit while interest was being charged. The debt changed sign somewhere between years 2 and 3; finding that date is this chapter's second half.

20.1 Two rates

In practice the investor pays a higher rate j_1 on borrowings than the rate j_2 received on investments. The gap depends on the investor's credit-worthiness and the expense of raising a loan.

NO SINGLE i, NO NPV(i), NO YIELD

The concepts of net present value and yield are in general no longer meaningful: with no single i there is no function NPV(i) whose root could be a yield.

WHAT TO DO INSTEAD

Accumulate the net cashflows *from first principles*, the rate depending on whether the account is in credit.

1. Charge j_1 when the balance is negative; credit j_2 when it is positive.
2. If loan repayments may be made at any year end, use each positive net cashflow *at once* to repay the loan (since $j_1 > j_2$, that beats investing it).
3. If the loan is *fixed term, with no early repayment*, a positive cashflow cannot reduce the loan: only the interest can be paid, and any surplus is *accumulated at* j_2 until the loan ends.
4. Compare projects at a *common date*, even when they finish at different times.

20.2 Three projects

EXAMPLE 20.1 · Projects C, D and E

Consider Projects C, D and E over five years, financed by loans. Loan repayments for C may be made at the end of each year; for D and E only annual interest repayments may be made until the projects end. The borrowing rate is 6.25%; the deposit rate 4%. All outlays are at year *starts*, all proceeds at year *ends* (€0s). Which are profitable, and which is the most profitable?

	C		**D**		**E**	
Year	Outlay	Proceeds	Outlay	Proceeds	Outlay	Proceeds
Y1	70	10	100		80	10
Y2	30	20			20	30
Y3	20	40			5	87
Y4	10	80				
Y5				140		

SOLUTION

"Start of year $r+1$" and "end of year r" are the same date, so those cashflows combine. **Project C** (repayments allowed: roll the loan forward). With $j_1 = 0.0625 > 0.04$, use positive net cashflows to repay immediately.

Date	Net cashflow incl. interest	Action	Loan after
End Y1	$-70(0.0625) - 30 + 10 = -24.375$	borrow	94.375
End Y2	$-94.375(0.0625) - 20 + 20 = -5.898$	borrow	100.273
End Y3	$-100.273(0.0625) - 10 + 40 = +23.733$	repay	76.540
End Y4	$-76.540(0.0625) + 80 = +75.216$	repay	1.324

A balance of 1.324 remains: the company is **unable to repay the loan**; by the end of year 5 it has grown to $1.324(1.0625) = 1.407$. **C is not profitable.**

Project D. No income before the end, so the loan simply accumulates at j_1 for five years, interest included. After repaying it,

$$-100(1.0625)^5 + 140 = -135.408 + 140 = 4.592,$$

so D is profitable. j_2 never appears: no positive balance exists before $t = 5$, so the lending rate has nothing to act on.

Project E. Borrow to cover the negative flows; since no capital repayments are permitted, positive flows must be invested at $j_2 = 4\%$.

$$\begin{aligned}
&\text{End Y1:} && -80(0.0625) - 20 + 10 = -15 && \Rightarrow \text{ borrow; loan 95,}\\
&\text{End Y2:} && -95(0.0625) - 5 + 30 = 19.063 && \Rightarrow \text{ invest (cover the interest from gains),}\\
&\text{End Y3:} && -95(0.0625) + 87 = 81.063 && \Rightarrow \text{ repay the loan, invest the balance.}
\end{aligned}$$

Balance at the end of Y3: $19.063(1.04) + 81.063 - 95 = 5.888$, positive, so it earns j_2: accumulated profit at the end of Y5 is $5.888(1.04)^2 = 6.368$.

	Accumulated profit at end Y5	Verdict
Project C	—	loan not repaid; **not profitable**
Project D	4.592	profitable
Project E	6.368	**most profitable**

E FINISHES IN YEAR 3, NOT YEAR 5

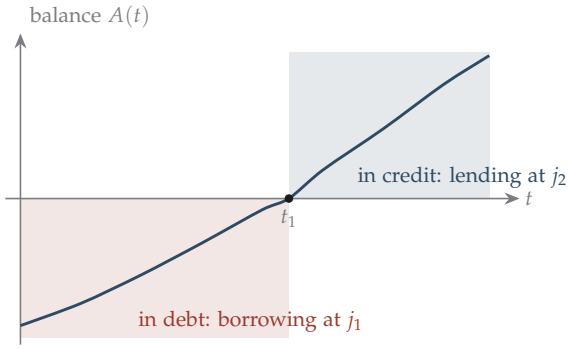
The loan for Project E is repaid at the end of Y3, when its proceeds arrive. The other projects end in Y5 and we compare all at Y5, so E's surplus then earns j_2 for two more years. An easy detail to miss in an exam.

WHAT CHANGED SINCE CHAPTER 18

Both rates appear in Project E: 6.25% on the debt, 4% on the surplus. We compare on a *common* date despite the projects finishing at different times. What has changed from Chapter 18 is that we can no longer get there by discounting at a single rate; the balance must be rolled forward year by year, at whichever rate applies.

20.3 The discounted payback period

In many practical problems the net cashflow changes sign only once, from negative to positive. The balance in the investor's account then changes sign at a unique time t_1, or is always negative (in which case the project is not viable).



DISCOUNTED PAYBACK PERIOD

The **discounted payback period** (DPP), if it exists, is the smallest t_1 such that

$$A(t_1) = \sum_{s \le t_1} (1+j_1)^{t_1-s} c_s + \int_0^{t_1} (1+j_1)^{t_1-s} \rho(s)\,\mathrm{d}s \;\ge\; 0.$$

WHAT THE DPP DEPENDS ON

t_1 depends only on j_1, the rate at which the investor *borrows*: until t_1 the account was never in credit, so the lending rate had nothing to act on. If $A(T) < 0$, where T is the end of the project (equivalently $\mathrm{NPV}(j_1) < 0$), the project has no discounted payback period and is not profitable. Once the period has elapsed the balance grows at j_2:

$$A(T) = (1+j_2)^{T-t_1} A(t_1) + \sum_{t \ge t_1} (1+j_2)^{T-t} c_t + \int_{t_1}^{T} (1+j_2)^{T-t} \rho(t)\,\mathrm{d}t.$$

The final profit therefore depends on *both* rates: on j_1 through t_1 and the balance reached there, on j_2 through the accumulation after. All else being equal, the project with the *shortest* DPP is generally recommended: profit sooner, less at risk for less time.

THE UNDISCOUNTED PAYBACK PERIOD

Set $j_1 = 0$ in the definition and the result is just the **payback period**. Its use instead of the DPP often leads to erroneous results and is **not recommended**: it pretends the borrowed money costs nothing, so it always reports a shorter, and sometimes a spurious, payback.

RAISING EACH RATE

Raising j_1 pushes t_1 later and lowers the profit. Raising j_2 leaves t_1 untouched and raises the profit.

20.4 Finding the payback date

The definition puts the unknown t_1 in three places at once: the exponent, the bounds of the integral and the range of the sum. The trick makes it appear in one.

DIVIDE BY $(1+j_1)^t$, EVERY TIME

A single outlay C, met by income at rate R p.a. payable continuously, with borrowing at j_1:

$$A(t) = -C(1+j_1)^t + R\,\bar{s}_{\overline{t}|} \geq 0 \qquad \Longleftrightarrow \qquad \boxed{\bar{a}_{\overline{t}|} \geq \frac{C}{R}} \quad \text{at rate } j_1.$$

Dividing by $(1+j_1)^t > 0$ reverses no inequality, and $(1+j_1)^{-t}\bar{s}_{\overline{t}|} = \bar{a}_{\overline{t}|}$ by the transfer principle (Chapter 4). Now t appears only inside $\bar{a}_{\overline{t}|}$, which *increases* with t, so the equation can be inverted with logarithms. For annual payments in arrears the same line gives $R\,s_{\overline{t}|} \geq C(1+j_1)^t \iff a_{\overline{t}|} \geq \frac{C}{R}$ at j_1.

PAYBACK DATE

1. **Net** income and expenditure that run over the same interval.
2. **Ignore what cannot matter:** a cost falling after the crossing cannot change when the crossing happens. Ignore a *blip*, a temporary excursion into credit before a large outlay: look for the first crossing *after* the last big outlay.
3. **Accumulate any discrete flows to a common date**, and restart the clock there (measure $u = t -$ that date).
4. Divide by $(1+j_1)^u$, solve $\bar{a}_{\overline{u}|} = C'/R$ (or $a_{\overline{u}|} = C'/R$) by logs.
5. **Annual payments:** t_1 must be a whole number: round *up*.

EXAMPLE 20.2 · Continuous income, $j_1 = 10\%$

An investor borrows at an effective 10% p.a. to invest in a six-year project with an internal rate of return of approximately 18.9%. Outlay 25,000; income 10,000 p.a. and expenditure 2,000 p.a., both continuous in the first five years; decommissioning expense 5,000 at the end of the project. Find the discounted payback period.

SOLUTION

Net the income and expenditure: $\rho = 8$ p.a. (in thousands) for $0 \leq t \leq 5$, and we expect the crossing within those five years. The decommissioning cost falls after any crossing, so it cannot change *when* the sign turns (though it affects the final profit and the quoted IRR). So $A(t) = -25(1.1)^t + 8\,\bar{s}_{\overline{t}|}$, and

$$\bar{a}_{\overline{t}|} = \frac{25}{8} = 3.125 \iff (1.1)^{-t} = 1 - 3.125\ln 1.1 = 0.702156$$
$$\iff t = \frac{\ln 0.702156}{-\ln 1.1} = 3.71.$$

The DPP is 3.71 years (about three years and eight and a half months). The balance crosses zero at 3.71 (check: $A(3.5) = -1.66$, $A(4) = +2.35$), climbs while the income lasts, and after year 5 merely accrues interest.

EXAMPLE 20.3 · A cashflow with a blip

A new company has a five-year lease for a local bus service (amounts in €000s). Set-up -100 at the start of year 1; advertising income $+200$ at the end of month 1; vehicle purchases -2000 at the end of month 3; passenger fares $+1000$ p.a. and staff costs -400 p.a., both continuous from month 3; asset resale value $+500$ at the end of year 5. Find the DPP with a flexible loan at 10% p.a.

SOLUTION

The net cashflow changes sign more than once. After month 1 the balance is $-100(1.1)^{1/12} + 200 = +99.20$, *positive*, and stays so until the vehicles are bought at month 3. That is a **blip**: look for the first time *after the vehicle purchases* with $A(t) \geq 0$. For $t \geq \frac{3}{12}$,

$$\begin{aligned} A(t) &= \left(-100(1.1)^{3/12} + 200(1.1)^{2/12} - 2000\right)(1.1)^{t-3/12} + 600\,\bar{s}_{\overline{t-3/12}|} \\ &= -1899.21\,(1.1)^{t-3/12} + 600\,\bar{s}_{\overline{t-3/12}|}. \end{aligned}$$

The three discrete flows are accumulated to $t = \frac{3}{12}$, total -1899.21; from there it is the general case with $u = t - \frac{3}{12}$, net income $1000 - 400 = 600$:

$$\bar{a}_{\overline{u}|} = \frac{1899.21}{600} = 3.16535 \iff (1.1)^{-u} = 1 - 3.16535 \ln 1.1 = 0.698310 \iff u = 3.768,$$

so $t = 3.768 + 0.25 = 4.018$ years: just over four years, inside the lease.

EXAMPLE 20.4 · Two rates: payback, then profit

A project is financed by a €500,000 bank loan at 9%; surplus funds can be invested at 6%. It has an initial outlay of €500,000 and income of €70,000 p.a. payable continuously for twenty years. Find the DPP and the accumulated profit after twenty years.

SOLUTION

Payback (thousands): while in debt the income repays a 9% loan, so $A(t) = -500(1.09)^t + 70\,\bar{s}_{\overline{t}|}$ (everything accumulated at 9%), and

$$\bar{a}_{\overline{t}|} = \frac{1 - (1.09)^{-t}}{\ln 1.09} = \frac{500}{70} \implies t = 11.093 \text{ years (DPP)}.$$

Profit. $A(\text{DPP}) = 0$, so the balance restarts from zero at 11.093 and the remaining 8.907 years of income accumulate at $j_2 = 6\%$:

$$A(20) = 70\,\frac{(1.06)^{8.907} - 1}{\ln 1.06} = 817.34,$$

that is €817,340.

EVEN AT THE PAYBACK DATE THE BALANCE NEED NOT BE ZERO

Here the income was continuous, so the balance crossed zero smoothly and $A(\text{DPP}) = 0$. A large lump sum can instead turn the balance straight from debt into a positive credit, so that $A(t_1) > 0$: this is why the general formula carries an $A(t_1)$ term, and why you must not drop it. The next example shows it in action.

20.5 A level annual annuity: the integer case

ANNUAL PAYMENTS

A single investment C for payments of R annually in arrears for n years: t_1 is the smallest *integer* t with $A(t) = -C(1+j_1)^t + R\,s_{\overline{t}|} \geq 0$, i.e.

$$\boxed{a_{\overline{t}|} \geq \frac{C}{R} \quad \text{at rate } j_1}$$

Viable if $t_1 \leq n$, and the profit after n years is

$$A(t_1)(1+j_2)^{n-t_1} + R\,s_{\overline{n-t_1}|} \quad \text{at rate } j_2.$$

THE BALANCE OVERSHOOTS

Because t_1 is a whole number, $A(t_1)$ is generally strictly positive, not zero: the balance passed zero partway through the year and kept climbing. Unlike the continuous case, the first term does not vanish, and dropping it is a common error. Also: dividing by $-\ln(1+j_1) < 0$ reverses the inequality, and the answer is rounded *up*, since payments arrive only at integer times.

EXAMPLE 20.5 · Twenty-five years of €10,500

An investment of €100,000 produces €10,500 annually in arrears for twenty-five years. Borrowing costs 9%; money can be invested at 7%. Find the DPP and the accumulated profit after twenty-five years.

SOLUTION

$a_{\overline{t}|} \geq \frac{100000}{10500} = 9.5238$ at 9%:

$$(1.09)^{-t} \leq 1 - 9.5238 \times 0.09 = 0.142857 \iff t \geq \frac{\ln 0.142857}{-\ln 1.09} = 22.58,$$

so $t_1 = 23$. Two years remain, and $A(23) = -100000(1.09)^{23} + 10500 s_{\overline{23}|} = €4{,}297.91 > 0$:

$$\text{Profit} = A(23)(1.07)^2 + 10500\,s_{\overline{2}|}\ (\text{at } 7\%) = 4920.67 + 21735 = €26{,}655.67.$$

Ignoring the surplus $A(23)$ would understate the profit by nearly €5,000.

THE CHAPTER IN ONE BOX

$$A(t_1) \geq 0 \text{ first at } t_1 \quad (\text{computed at } j_1 \text{ alone}),$$

$$A(T) = (1+j_2)^{T-t_1} A(t_1) + \sum_{t \geq t_1} (1+j_2)^{T-t} c_t + \int_{t_1}^{T} (1+j_2)^{T-t} \rho(t)\, \mathrm{d}t.$$

With $j_1 \neq j_2$, NPV and the yield lose their meaning: roll the balance forward year by year at whichever rate applies and compare accumulated profits on a common date. Where a single sign change exists, the payback date is set by the borrowing rate and everything after it by the lending rate.

$$R\,\bar{s}_{\overline{t}|} \geq C(1+j_1)^t \iff \bar{a}_{\overline{t}|} \geq \frac{C}{R},$$

$$R\,s_{\overline{t}|} \geq C(1+j_1)^t \iff a_{\overline{t}|} \geq \frac{C}{R} \quad (\text{integer } t, \text{ round up}).$$

Net, ignore blips and late costs, accumulate discrete flows to a common date, divide by $(1+j_1)^t$. With whole-year payments keep $A(t_1)$ in the profit.

TRY IT YOURSELF

1. Repeat the warm-up assuming no capital repayments are permitted, so that surpluses must be deposited at 3%. Compare with the warm-up and explain the difference.
2. Why did the lending rate j_2 play no part in Project D, and what would have to change for it to matter?
3. Explain why the discounted payback period depends on j_1 but not on j_2, and why the accumulated profit depends on both.
4. An outlay of €200,000 produces €30,000 p.a. in arrears for twelve years, borrowing at 8%. Find the DPP, and the profit after twelve years if surplus funds earn 5%.
5. A project has an outlay of €40,000 and income of €6,000 p.a. continuously, borrowing at 7%. Find the DPP, and say what happens if the income were only €2,500.
6. In the 18.9% IRR example the decommissioning cost was ignored. Justify this, and say when such a cost *would* change the answer.

ANSWERS

1. The debt of 50 accumulates at 8% throughout: $50(1.08)^3 = 62.986$; the receipts are deposited at 3%: $20(1.03)^2 + 20(1.03) + 20 = 61.818$. Since $61.818 < 62.986$ the project makes a *loss* of 1.168, against a profit of 1.942 in the warm-up. The cashflows are identical: the whole difference is the financing terms, i.e. holding cash at 3% while paying 8%.
2. D has no positive balance at any time: a single outlay at the start and a single receipt at the very end, used at once to clear the loan. For j_2 to matter there must be a surplus on deposit at some point: an earlier receipt, large enough to exceed the interest then due (a receipt that merely covers the interest leaves the balance unchanged). Project E is exactly that case: its 30 at the end of year 2 exceeds 5.9375 of interest plus the 5 outlay, so 19.063 goes on deposit.

3. Before t_1 the account has never been in credit, so no money has been lent and j_2 has nothing to act on: the balance up to t_1 depends on j_1 alone, hence so does the date it first reaches zero. After t_1 the account grows at j_2, so the final profit depends on both rates.
4. Integer case: $a_{\overline{t}|} \geq 6.6667$ at 8%: $(1.08)^{-t} \leq 1 - 6.6667 \times 0.08 = 0.466667$, $t \geq 9.903$, so $t_1 = 10$. $A(10) = -200000(1.08)^{10} + 30000 s_{\overline{10}|} = €2{,}811.87 > 0$ and two years remain: profit $2811.87(1.05)^2 + 30000 s_{\overline{2}|}$ at 5% $= 3100.09 + 61500 = €64{,}600.09$.
5. $A(t) = -40(1.07)^t + 6\bar{s}_{\overline{t}|}$ gives $\bar{a}_{\overline{t}|} = \frac{40}{6} = 6.6667$: $(1.07)^{-t} = 1 - 6.6667 \ln 1.07 = 0.548942$, $t = 8.86$ years. With €2,500 we would need $\bar{a}_{\overline{t}|} = 16$, i.e. $(1.07)^{-t} = 1 - 16 \ln 1.07 = -0.0825 < 0$, impossible: the income never covers the interest on the outstanding debt, so the balance never turns: no DPP and no profit.
6. The cost falls at the end ($t = 6$), after the balance has turned positive, and $A(t)$ for $t < 6$ involves no payment made at $t = 6$. It would change the answer if it fell *before* the crossing (it would be a term in $A(t)$ and delay it), or if it were large enough to push the balance back below zero afterwards, when the first crossing is no longer the last: the blip of the bus example, signs reversed.

PART VII

The term structure

CHAPTER 21

Spot rates and forward rates

LECTURE 25 · *When the rate depends on the term*

AFTER THIS CHAPTER YOU CAN

- ☐ Move between zero-coupon prices P_n and spot rates y_n.
- ☐ Price a coupon bond strand by strand from spot rates.
- ☐ Find forward rates $f_{t,r}$ from spot rates or prices, and the reverse.
- ☐ Chain one-year forward rates into spot rates, and read a spot rate as a geometric mean.

Until now one i served every date: $\text{PV} = \sum c_t v^t$. Today each date gets its own rate.

Warm-up. A contract paying €1 in one year costs €0.95; one paying €1 in two years costs €0.89; one paying €1 in three years costs €0.82. Each is a single payment, so $P = (1+y)^{-n}$ and $y = P^{-1/n} - 1$:

$$\frac{1}{0.950} - 1 = 5.2632\%, \qquad 0.890^{-1/2} - 1 = 6.00\%, \qquad 0.820^{-1/3} - 1 = 6.8387\%.$$

Three rates, from the same market on the same day, and no two equal. There is no single i here, and in practice there never is.

21.1 Zero-coupon bonds and spot rates

ZERO-COUPON BOND

A **unit zero-coupon bond** of term n is an agreement to pay €1 at the end of n years with no coupons (a **pure discount bond**). Its price at issue is P_n. It isolates *one* date, so its price says what the market charges for money at that horizon and nothing else: the atoms of the market.

SPOT RATE

The yield on a unit zero-coupon bond of term n is the **n-year spot rate** y_n. It measures the average interest rate over the n-year period.

$$P_n = \frac{1}{(1+y_n)^n}, \qquad y_n = P_n^{-1/n} - 1.$$

Since rates differ by term, in general $y_t \neq y_s$ for $s \neq t$. The variation by term is the **term structure of interest rates**; P_n and y_n carry identical information.

EVERY BOND IS A PORTFOLIO OF ZERO-COUPON BONDS

A bond paying coupons D for n years with redemption R at time n is n zero-coupon bonds of maturity value D at terms $1, \ldots, n$, plus one of nominal value R at term n. With $\mathsf{v}_t = (1+y_t)^{-1}$,

$$\boxed{P \;=\; D\,(P_1 + \cdots + P_n) + R\,P_n \;=\; D\big(\mathsf{v}_1 + \mathsf{v}_2^2 + \cdots + \mathsf{v}_n^n\big) + R\,\mathsf{v}_n^n}$$

The portfolio has the same payouts as the bond, so the prices must agree: no arbitrage (Chapters 1 and 13). Otherwise an investor would buy the cheap side, sell the dear one and keep the difference, risk-free.

EVERY DATE CARRIES ITS OWN FACTOR

The notation changed: previously all discounting used a single v; now the factor for date t is $\mathsf{v}_t^t = (1+y_t)^{-t}$. No annuity symbol can appear: $a_{\overline{n}|}$ would need one rate for all dates, and there is no such rate.

EXAMPLE 21.1 · Pricing from spot rates

Price a five-year fixed-interest security, redeemable at par, with 6% annual coupons, when the annual term structure is $y_1 = 7\%$, $y_2 = 7\frac{1}{4}\%$, $y_3 = 7\frac{1}{2}\%$, $y_4 = 7\frac{3}{4}\%$, $y_5 = 8\%$.

SOLUTION

Per 100 nominal, with each strand at its own spot rate:

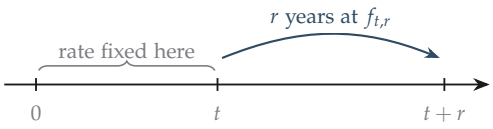
t	1	2	3	4	5	total
P_t	0.934579	0.869372	0.804961	0.741875	0.680583	
coupon $6P_t$	5.607	5.216	4.830	4.451	4.083	24.19
redemption					68.06	68.06
						92.25

$P = 6\big(\frac{1}{1.07} + \frac{1}{1.0725^2} + \cdots + \frac{1}{1.08^5}\big) + \frac{100}{1.08^5} = €92.25$. We could have said at once that the price is below par: the 6% coupon is below every spot rate. And no single "yield" appeared: each cashflow was discounted at the rate for *its own* date.

21.2 Forward rates

DISCRETE-TIME FORWARD RATE

The **forward rate** $f_{t,r}$ is the *annual* rate of interest *agreed at time* 0 for an investment made at time $t > 0$ for a period of r years. An investor who agrees at time 0 to invest €100 at time t for r years has $100(1+f_{t,r})^r$ at time $t+r$.



The rate is settled *now*, but the money moves *later*, which is why the definition is needed.

THE CONVERSION FORMULA

One unit invested at 0 for $t+r$ years can be followed three ways, and by consistency (Chapter 1) the proceeds cannot depend on the route:

$$(1+y_t)^t(1+f_{t,r})^r \;=\; (1+y_{t+r})^{t+r} \;=\; \frac{1}{P_{t+r}}.$$

Rearranging,

$$\boxed{(1+f_{t,r})^r \;=\; \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} \;=\; \frac{P_t}{P_{t+r}}}$$

Any *one* of the spot rates, the forward rates or the zero-coupon prices determines the other two: three descriptions of one term structure. The P form is the most natural: P_t/P_{t+r} compares the price of money at two horizons.

EXAMPLE 21.2 · Forward rates from a market

For the warm-up market, find the one-year forward rates f_1, f_2, and verify $(1+y_3)^3 = (1+f_0)(1+f_1)(1+f_2)$.

SOLUTION

The cleanest route is through the prices: $1+f_1 = \frac{P_1}{P_2} = \frac{0.950}{0.890} = 1.067416$ and $1+f_2 = \frac{P_2}{P_3} = \frac{0.890}{0.820} = 1.085366$, so $f_1 = 6.7416\%$, $f_2 = 8.5366\%$; and $f_0 = y_1 = 5.2632\%$. The product telescopes: $\frac{1}{P_1}\cdot\frac{P_1}{P_2}\cdot\frac{P_2}{P_3} = \frac{1}{P_3} = \frac{1}{0.820} = 1.219512$, and $(1+y_3)^3 = 1.068387^3 = 1.219512$ too.

21.3 One-year forward rates, and the spot rate as an average

CHAINING

Write $f_t = f_{t,1}$ and define $f_0 = y_1$. Investing 1 for t years at the spot rate gives the same as investing one year at a time at the one-year forward rates:

$$(1+y_t)^t = (1+f_0)(1+f_1)(1+f_2)\cdots(1+f_{t-1}),$$

$$1+y_t = \sqrt[t]{(1+f_0)(1+f_1)\cdots(1+f_{t-1})}.$$

So $1+y_t$ is the **geometric mean** of the t one-year growth factors: the right average when quantities *compound* rather than add. This makes precise the remark that y_n "measures the average interest rate over the n-year period".

ROLLING AVERAGES

When forwards rise, every spot rate lags behind the forward just reached, since it averages all the years so far. Check at $y_2 = 4\%$, $y_5 = 5\%$: the first two years average 4%, so the last three must exceed 5% to lift the five-year average to 5%. Indeed

$$(1+f_{2,3})^3 = \frac{(1.05)^5}{(1.04)^2} = \frac{1.276282}{1.081600} = 1.180000, \qquad f_{2,3} = 5.6720\%,$$

above both y_2 and y_5.

SPOT, FORWARD, PRICE: WHICH ROUTE?

1. Know P_n? Get $y_n = P_n^{-1/n} - 1$; forwards are $\frac{P_t}{P_{t+r}}$ to the power $1/r$.
2. Know spots, want $f_{t,r}$? Use $\frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t}$ to the power $1/r$.
3. Know one-year forwards, want a spot? Multiply the factors, then take the t-th root.
4. Coupon bond? Discount *each* payment at the spot rate for its own date; never use an annuity symbol.

THE CHAPTER IN ONE BOX

$$P_n = \frac{1}{(1+y_n)^n}, \qquad (1+f_{t,r})^r = \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} = \frac{P_t}{P_{t+r}},$$

$$(1+y_t)^t = \prod_{k=0}^{t-1}(1+f_k), \qquad P = D\left(\mathsf{v}_1 + \cdots + \mathsf{v}_n^n\right) + R\,\mathsf{v}_n^n.$$

Any one of P_n, y_n, $f_{t,r}$ determines the other two. Every coupon bond is a portfolio of zero-coupon bonds, priced strand by strand.

TRY IT YOURSELF

1. For the warm-up market, find the one-year forward rates f_1 and f_2, and verify $(1+y_3)^3 = (1+f_0)(1+f_1)(1+f_2)$.

2. A three-year bond redeemable at par pays annual coupons of 5%. Using the same three prices, find its price per €100 nominal.
3. With $y_2 = 4\%$ and $y_5 = 5\%$, find the forward rate $f_{2,3}$.

ANSWERS

1. $f_1 = 6.7416\%$, $f_2 = 8.5366\%$, $f_0 = 5.2632\%$; product $= 1/0.820 = 1.219512 = (1.068387)^3$.
2. $P = 5(P_1 + P_2 + P_3) + 100P_3 = 5(2.660) + 82.00 =$ €95.30. Below par, as the 5% coupon is below every spot rate (5.26%, 6.00%, 6.84%). No annuity symbol: there is no single rate.
3. $(1 + f_{2,3})^3 = \frac{1.05^5}{1.04^2} = 1.180000$, so $f_{2,3} = 1.18^{1/3} - 1 = 5.6720\%$.

CHAPTER 22

One number for a whole curve

LECTURE 26 · *Redemption yield, par yield, and two exam questions*

AFTER THIS CHAPTER YOU CAN

- ☐ Build the missing forward and spot rates from partial data, and know how far the data reach.
- ☐ Price a bond from spot rates, then find its redemption yield by interpolation.
- ☐ Say what a redemption yield is a weighted average of, and where it will sit.
- ☐ Find a par yield and the coupon bias.
- ☐ Solve a term-structure puzzle from a par yield, a forward rate and a price.

$$P = D\left(\mathrm{v}_1 + \mathrm{v}_2^2 + \cdots + \mathrm{v}_n^n\right) + R\mathrm{v}_n^n$$

is correct and inconvenient: n rates. A trader wants *one* number. Today, two ways of getting one, and what each keeps and what it throws away. A single rate standing for a stretch of curve is an *average*, and an average tells you nothing about the spread.

22.1 The redemption yield

YIELD TO MATURITY

The **yield to maturity**, or **redemption yield**, of a coupon bond is the effective rate at which the discounted proceeds equal its price: the single i solving $P = D\,a_{\overline{n}|} + R\mathrm{v}^n$. **Gross** ignores tax; **net** allows for it.

It is widely used but depends on the *coupon rate* of the bond, so it gives no simple model of term and yield. **Two bonds of the same term but different coupons have the same spot rates behind them yet different redemption yields: the curve belongs to the market, the redemption yield to the bond.**

REDEMPTION YIELD FROM A SPOT CURVE

1. Get any missing forwards and spots (chain the one-year rates).
2. Price the bond strand by strand, each cashflow at its own spot rate.
3. Solve $D\,a_{\overline{n}|} + R\mathrm{v}^n = P$ for one i: the usual interpolation (Chapter 9), recomputing $a_{\overline{n}|}$ at every trial rate.
4. **Look first:** the yield must lie among the spot rates, near the date where most of the money sits.

EXAMPLE 22.1 · CM1A April 2022

The annual effective forward rate over t to $t+r$ is $f_{t,r}$. Given $f_{0,1} = 4.2\%$, $f_{1,1} = 4.8\%$, $f_{2,1} = 5.3\%$, $f_{2,2} = 5.9\%$: (i) find $f_{3,1}$ to four decimal places; (ii) find all spot rates the information allows; (iii) using linear interpolation, find the annual effective gross redemption yield of a four-year bond redeemable at 105% with a 2.5% annual coupon in arrears.

SOLUTION

(i) Two routes carry money from time 2 to time 4: the one-year forwards $f_{2,1}$ then $f_{3,1}$, or two years at $f_{2,2}$. Note $f_{2,2}$ is an *annual* rate over two years, so we must *square* it. By consistency $(1+f_{2,2})^2 = (1+f_{2,1})(1+f_{3,1})$, so

$$f_{3,1} = \frac{(1.059)^2}{1.053} - 1 = 0.065034 = 6.5034\%.$$

(ii) Chain the one-year forwards, $(1+y_t)^t = (1+f_0)\cdots(1+f_{t-1})$:

$$\begin{aligned} y_1 &= 1.042 - 1 = 4.2000\%, \\ y_2 &= \sqrt{(1.042)(1.048)} - 1 = 4.4996\%, \\ y_3 &= \sqrt[3]{(1.042)(1.048)(1.053)} - 1 = 4.7657\%, \\ y_4 &= \sqrt[4]{(1.042)(1.048)(1.053)(1.065034)} - 1 = 5.1975\%. \end{aligned}$$

Four one-year rates give four spot rates and no more: nothing in the data reaches beyond year four, so y_5 cannot be produced. Part (i) was needed to supply the fourth link of the chain.

(iii) A gross redemption yield equates price and proceeds, so we need the price, which comes from the spot rates. Per €100 nominal, with $\mathrm{v}_t^t = (1+y_t)^{-t} = 0.959693,\ 0.915738,\ 0.869646,\ 0.816543$:

$$P = 2.5(3.561620) + 105(0.816543) = 8.904 + 85.737 = 94.6411.$$

Now find i with $g(i) = 2.5\,a_{\overline{4}|} + 105\mathrm{v}^4 - 94.6411 = 0$: $g(0.05) = 95.2486 - 94.6411 = +0.6076$, $g(0.06) = 91.8326 - 94.6411 = -2.8085$, so

$$i \approx 0.05 + \frac{0.6076}{0.6076 + 2.8085}(0.01) = 0.051779 = 5.1779\%.$$

The bracket is a full percentage point, so this is rough: narrowing it to $(0.05, 0.055)$ gives 5.1758%; the true value is 5.1745%.

WHERE THE REDEMPTION YIELD MUST LAND

5.1779% sits just below $y_4 = 5.1975\%$. The redemption yield is a *weighted average of the spot rates*, weighted by where the money sits: here 90.6% of the price is the redemption payment at $t = 4$, and the coupons are $2.5\%, 2.4\%, 2.3\%$ and 2.2% of it at $t = 1, 2, 3, 4$. The small early coupons, discounted at lower spot rates, drag it slightly below y_4 but cannot move it far. Knowing where the main payments fall tells you roughly where the answer will be, which is very useful when choosing an interpolation bracket.

22.2 Par yields

PAR YIELD

The **n-year par yield** yc_n is the coupon per €1 nominal which would make a bond of term n, redeemed at par, sell at par under the current term structure:

$$1 = yc_n\left(\mathsf{v}_1 + \mathsf{v}_2^2 + \cdots + \mathsf{v}_n^n\right) + \mathsf{v}_n^n, \qquad \text{so} \qquad \boxed{yc_n = \frac{1 - \mathsf{v}_n^n}{\mathsf{v}_1 + \mathsf{v}_2^2 + \cdots + \mathsf{v}_n^n}}$$

Informally: *what coupon would make this bond sell at par?* It is a fourth way of quoting the same term structure, alongside P_n, y_n and $f_{t,r}$. The redemption yield is defined by a bond and moves with its coupon; the par yield is intrinsic to the market, and **read off the curve**.

COUPON BIAS

The difference between the n-year par yield and the n-year spot rate, $yc_n - y_n$, is the **coupon bias**. It is not zero because a par bond pays coupons at $1, \ldots, n$, each discounted at its *own* spot rate, whereas y_n discounts a single payment at time n.

EXAMPLE 22.2 · A force among the spots

Find the three-year par yield if the annual term structure is $y_1 = 5\%$, $y_2 = 4\%$, and the third rate is a *force*, $Y_3 = 3\%$.

SOLUTION

By Chapter 2, $\mathsf{v}_3^3 = \mathrm{e}^{-3(0.03)}$, so

$$yc_3 = \frac{1 - \mathrm{e}^{-0.09}}{(1.05)^{-1} + (1.04)^{-2} + \mathrm{e}^{-0.09}} = 0.03084,$$

a par yield of 3.084%. The discrete three-year spot rate is $y_3 = \mathrm{e}^{0.03} - 1 = 3.046\%$, so the coupon bias is only 0.04 percentage points.

EXAMPLE 22.3 · CM1A September 2021

A bond issued at $t = 0$ at €107.60 per €100 nominal pays 6% coupons annually in arrears and is redeemed at par in three years. The two-year par yield at $t = 0$ is 6.5%, and the one-year forward rate at $t = 1$ is 4.5% p.a. effective. Assuming no arbitrage, find the implied one-, two- and three-year spot rates.

SOLUTION

The par yield and the forward rate give y_1 and y_2; the price then fixes y_3.
y_1: by definition $100 = 6.5(\mathrm{v}_1 + \mathrm{v}_2^2) + 100\mathrm{v}_2^2 = 6.5\mathrm{v}_1 + 106.5\mathrm{v}_2^2$. Since $f_{0,1} = y_1$, $(1+y_2)^2 = (1+y_1)(1+f_{1,1})$, so $\mathrm{v}_2^2 = (1+y_1)^{-1}(1.045)^{-1}$ and *every term now carries the single unknown factor* $(1+y_1)^{-1}$:

$$100 = (1+y_1)^{-1}\big(6.5 + 106.5(1.045)^{-1}\big) \implies 1 + y_1 = \frac{108.413876}{100}, \quad y_1 = 8.41388\%.$$

y_2: $y_2 = \sqrt{(1.0841388)(1.045)} - 1 = 6.43895\%$, the geometric mean of the two one-year factors, between them.
y_3: no arbitrage, so the bond is worth the sum of its zero-coupon parts:

$$107.60 = 6\big((1.0841388)^{-1} + (1.0643895)^{-2}\big) + 106(1+y_3)^{-3}.$$

The bracket is $6(0.922391 + 0.882671) = 10.83037$, so $(1+y_3)^{-3} = \frac{107.60 - 10.83037}{106} = 0.912921$ and $y_3 = (0.912921)^{-1/3} - 1 = 3.08345\%$.
Checks worth doing: the par yield comes back, $6.5\mathrm{v}_1 + 106.5\mathrm{v}_2^2 = 100.00$, and the price, $6(\mathrm{v}_1 + \mathrm{v}_2^2 + \mathrm{v}_3^3) + 100\mathrm{v}_3^3 = 107.60$. The curve 8.41%, 6.44%, 3.08% is steeply *falling*, which is why a 6% coupon sells above par: it beats the later spot rates by a wide margin.

THE CHAPTER IN ONE BOX

Redemption yield: $P = D\,a_{\overline{n}|} + R\mathrm{v}^n$ solved for one i, $\qquad yc_n = \dfrac{1 - \mathrm{v}_n^n}{\mathrm{v}_1 + \mathrm{v}_2^2 + \cdots + \mathrm{v}_n^n}$.

The redemption yield belongs to the bond and moves with its coupon: a weighted average of the spot rates. The par yield is intrinsic to the market: a fourth language for the term structure with P_n, y_n, $f_{t,r}$. Their gap from y_n is the coupon bias. Chain the one-year rates for spots, and remember $f_{t,r}$ is an *annual* rate over r years, so it is raised to the power r.

TRY IT YOURSELF

1. With $y_1 = 5\%$, $y_2 = 6\%$, find the two-year par yield and the coupon bias.
2. A two-year bond redeemable at par pays annual coupons at the par yield you found. Verify that it prices at 100 per €100 nominal, and find its redemption yield.
3. Two four-year bonds have coupons of 2% and 9%. Under the spot curve of CM1A April 2022, which has the higher redemption yield? Answer without calculating.

ANSWERS

1. $v_1 = 1.05^{-1} = 0.952381$, $v_2^2 = 1.06^{-2} = 0.889996$: $yc_2 = \frac{0.110004}{1.842377} = 5.9707\%$ (about 5.971%). Coupon bias $yc_2 - y_2 = -0.029$ percentage points: slightly negative, as it must be, since the par bond pays part of its money at time 1, where the rate is only 5%, so it needs a coupon a little below y_2 to sell at par.
2. $5.9707(0.952381) + 105.9707(0.889996) = 5.6864 + 94.3136 = 100.00$. A bond priced at par with coupon c has redemption yield exactly c (Chapter 19: the coupon services the capital, which is returned intact), so the redemption yield is 5.9707%, equal to the par yield. That is the point of the par yield: it is the one coupon for which the two summaries agree.
3. The 2% bond. The April 2022 curve rises (4.20% to 5.20%). A low coupon puts almost all the value in the redemption at $t = 4$, so the average sits near the highest rate y_4; a high coupon shifts weight onto early dates, where rates are lower, and drags the average down. On a rising curve the lower-coupon bond has the higher redemption yield, and on a falling curve (like September 2021) the other way round. The redemption yield belongs to the bond, not to the market.

CHAPTER 23

Yield curves and interest rate risk

LECTURE 27 · *The shape of the curve, what moves it, and how much a portfolio stands to lose*

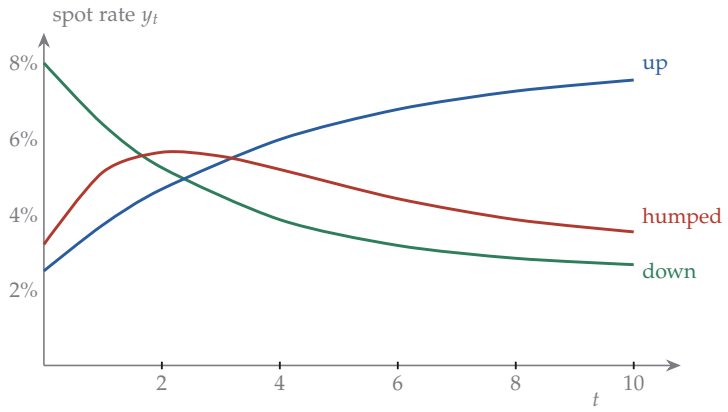
AFTER THIS CHAPTER YOU CAN

- ☐ Name the three shapes of a yield curve and the three theories offered for them.
- ☐ State the interest rate risk of an institution holding assets against liabilities.
- ☐ Compute the effective duration (volatility) $\nu(i)$ of any cashflow series.
- ☐ Use $\text{NPV}(i+h) \approx \text{NPV}(i)(1-h\nu)$, and know which way it errs.

Two chapters built the term structure and summarised it. Neither asked what *shape* it has, why, or what happens to a portfolio when the whole thing shifts.

Warm-up. A three-year bond redeemable at par pays annual coupons of 5%. Its price per €100 nominal is $P(0.06) = 97.3270$ at a yield of 6% and $P(0.061) = 97.0651$ at 6.1%: a fall of 0.2691%. A tenth of a percentage point on the yield costs a quarter of a percentage point of the price; the ratio, about 2.7, is the number defined in Section 23.3.

23.1 The shape of the curve



The curve of spot rates y_t, $t \geq 0$, is a **yield curve**. It may be **downward sloping** (long-term bonds more *expensive* than short-term ones), **upward sloping** (long-term bonds *cheaper*) or **humped** (bonds of intermediate term cheaper than both very short and long ones). Price and yield move in opposite directions: a low yield at some term means a high price at that term.

SUPPLY AND DEMAND

A dip in the curve marks a term investors want; a bulge marks one they do not. If investors believe there will be a higher overall return from bonds of a certain term, higher demand pushes up the price and hence *down* the yield compared with other terms.

Three theories say why:

1. EXPECTATIONS THEORY

The relative attraction of short- and long-term investments varies with expectations of future rate movements. If rates are expected to *fall*, long-term investments become more attractive (they lock in today's higher rates), demand for long bonds rises and their yields fall: the downward-sloping green curve.

2. LIQUIDITY PREFERENCE

Long-term bonds are more sensitive to interest-rate movements (and other risks), so risk-averse investors require higher yields to compensate: the upward-sloping blue curve. "More sensitive" is made precise in Section 23.3: a longer bond has a larger effective duration.

3. MARKET SEGMENTATION

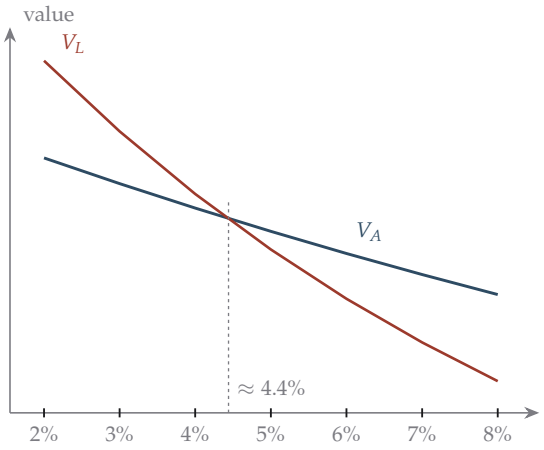
Bonds of different terms attract different investors, who choose assets similar in term to their liabilities. Banks have short-term liabilities (withdrawals at short notice) and invest short; pension funds have long-term liabilities and invest long. Government and company issuance may also differ from investors' requirements. The term structure emerges from these forces of supply and demand.

THE CURVE MOVES

We have described the *initial* term structure, with everything fixed at time 0. In practice it changes rapidly: the five-year spot rate tomorrow may differ from today's. More sophisticated treatments give every variable a second argument ($y_{s,t}$: the t-year spot rate at time s). That is beyond this module, but we still ask what such a movement does to a price.

23.2 Interest rate risk

From now on assume the term structure is *flat*: $y_t = f_{t,r} = i$ for all t, r. Suppose an institution holds assets of value V_A to meet liabilities of value V_L. Both are discounted values of future cashflows, hence both are sensitive to i, and the institution is healthy at time 0: $V_A > V_L$. If rates fall both values rise; if rates rise both fall. **The danger is not the movement itself but the two values moving by different amounts.**



V_L is the *steeper* curve. At $i_0 = 5\%$ the institution is healthy ($V_A = 100 > V_L = 96$), but a fall below about 4.4% reverses that: the liabilities gain more than the assets.

THE INTEREST RATE RISK

That following a *downward* movement in interest rates the value of the assets increases by *less* than the value of the liabilities; or that following an *upward* movement the value of the assets decreases by *more* than the value of the liabilities.

What we need is a number measuring *how steep* each curve is, one that does not depend on the size of the fund.

23.3 Effective duration

EFFECTIVE DURATION, OR VOLATILITY

For cashflows c_k at times t_k, $\mathrm{NPV}(i) = \sum_k c_k v^{t_k}$. If $\mathrm{NPV}(i) \neq 0$,

$$\nu(i) \;=\; -\frac{\mathrm{NPV}'(i)}{\mathrm{NPV}(i)} \;=\; -\frac{\mathrm{d}}{\mathrm{d}i}\ln \mathrm{NPV}(i) \;=\; \frac{\sum_k t_k c_k v^{t_k+1}}{\sum_k c_k v^{t_k}}$$

The last form comes from differentiating term by term: $\mathrm{NPV}'(i) = \sum c_k(-t_k)(1+i)^{-t_k-1} = -\sum t_k c_k v^{t_k+1}$.

HOW STEEP, NOT HOW BIG

$\nu(i)$ is "minus the slope" divided by the "height". Slope alone would double when the cashflows double; dividing by NPV cancels that, so ν measures *sensitivity*, not size. It is a logarithmic derivative, the same idea as $\delta(t) = V_t'/V_t$ in Chapter 2: slope divided by level removes the size and leaves a rate.

PROPERTIES

- If all $c_k > 0$, NPV decreases as i rises, so $\text{NPV}' < 0$ and $\nu(i) > 0$ for all i.
- ν is independent of the size of NPV.
- **First-order estimate:** for small h,

$$\text{NPV}(i+h) \;\approx\; \text{NPV}(i)\big(1 - h\,\nu(i)\big),$$

so a rise of 1% in i cuts the value by about $\nu(i)$ %.

A FIRST-ORDER ESTIMATE ONLY

Since NPV is convex, the tangent lies below the curve on both sides, so the estimate *under-states the value* for both signs of h: it exaggerates a fall and under-states a gain. Chapter 25 repairs this with the second derivative.

EXAMPLE 23.1 · Two durations

(a) The warm-up bond (three years, 5% coupons) at $i = 6\%$. (b) A ten-year zero-coupon bond at $i = 6\%$.

SOLUTION

(a) $\text{NPV}(0.06) = 97.3270$ and

$$\nu(0.06) = \frac{1(5)\mathrm{v}^2 + 2(5)\mathrm{v}^3 + 3(105)\mathrm{v}^4}{97.3270} = 2.6956,$$

so a rise of 0.1% costs about $0.001 \times 2.6956 = 0.2696\%$, against the exact 0.2691%.

(b) A single cashflow R at time n has $\nu = \dfrac{nR\mathrm{v}^{n+1}}{R\mathrm{v}^n} = n\mathrm{v}$: R cancels. Here $\nu = 10\mathrm{v} = 9.434$: far more sensitive. Liquidity preference made quantitative: the longer bond really is the riskier one.

EXAMPLE 23.2 · A coupon bond is less sensitive than a zero of the same term

At $i = 6\%$: the five-year zero has $\nu = 5(0.943396) = 4.7170$ and the three-year zero 2.8302: **duration is proportional to term**. A five-year bond redeemable at par with 6% annual coupons has NPV $= 100$ (the coupon equals the yield: par, by Chapter 19) and

$$\nu(0.06) = \frac{\sum_{k=1}^{4} k(6)\mathrm{v}^{k+1} + 5(106)\mathrm{v}^6}{100} = 4.2124,$$

less than 4.7170, though both run five years. The coupons return money early, so the average date of payment is earlier than five years. **Term and duration are not the same thing**; Chapter 24 makes the "average date" reading exact.

THE CHAPTER IN ONE BOX

$$\nu(i) = -\frac{\mathrm{NPV}'(i)}{\mathrm{NPV}(i)} = -\frac{\mathrm{d}}{\mathrm{d}i}\ln \mathrm{NPV}(i) = \frac{\sum_k t_k c_k \mathrm{v}^{t_k+1}}{\sum_k c_k \mathrm{v}^{t_k}}, \qquad \mathrm{NPV}(i+h) \approx \mathrm{NPV}(i)(1-h\nu).$$

A curve may slope down, slope up or hump; expectations, liquidity preference and market segmentation each offer a reason. The risk is not that rates move but that assets and liabilities move by different amounts: duration says by how much. Zero-coupon bond: $\nu = n\mathrm{v}$.

TRY IT YOURSELF

1. Find the effective duration at $i = 6\%$ of a five-year and of a three-year zero-coupon bond. What is the pattern?
2. A five-year bond redeemable at par pays 6% annual coupons, at $i = 6\%$. Find NPV and ν, and compare with the five-year zero.
3. An institution's assets have duration 6 and its liabilities duration 9, with $V_A = V_L$ at present. Which way must rates move to make it insolvent, and why?

ANSWERS

1. $\nu = n\mathrm{v}$, independent of the payment: $\nu_5 = 4.7170$, $\nu_3 = 2.8302$. Duration is proportional to term: the precise form of liquidity preference.
2. Coupon equals yield, so NPV $= 100$ without calculation; $\nu = 4.2124$, *less* than the zero's 4.7170, because the coupons bring the average payment date forward.
3. Rates must *fall*. With $V_A = V_L$, a change h gives $V_A(1-6h) - V_L(1-9h) = V_A(3h)$, negative exactly when $h < 0$. The liabilities are the steeper curve, so they gain more when rates fall. This is why matching durations, not merely values, is what an institution needs.

PART VIII

Immunisation

CHAPTER 24

The discounted mean term

LECTURE 28 · *Forward forces, and the centre of mass of a cashflow*

AFTER THIS CHAPTER YOU CAN

- ☐ Find the instantaneous forward rate F_t from a price P_t, and the reverse.
- ☐ Compute the discounted mean term $\tau(i)$ of a zero-coupon or coupon bond.
- ☐ Use $\tau = (1+i)v$, and say how τ moves with i, with the coupon and with the term.
- ☐ See why duration predicts price changes to first order only.

Warm-up. For a zero-coupon bond of term n redeemable at R, NPV $= Rv^n$ and $\nu = \dfrac{nRv^{n+1}}{Rv^n} = nv$ (4.7170 at $n = 5$, $i = 6\%$). R cancels: duration depends only on *when* the payment falls. And $(1+i)v = n$ exactly. That stray factor of v is the subject of Section 24.2.

24.1 The instantaneous forward rate

Each of the spot and forward rates has a continuous counterpart: take the logarithm of the growth factor, as in Chapter 2.

SPOT FORCE, FORWARD FORCE

$$Y_t = \ln(1+y_t), \qquad F_{t,r} = \ln(1+f_{t,r}).$$

Since $(1+y_t)^t = 1/P_t$ and $(1+f_{t,r})^r = P_t/P_{t+r}$, taking logarithms gives

$$Y_t = -\frac{1}{t}\ln P_t, \qquad F_{t,r} = \frac{1}{r}\ln\left(\frac{P_t}{P_{t+r}}\right).$$

INSTANTANEOUS FORWARD RATE

$F_t = \lim_{r\to 0} F_{t,r}$, the forward force applying in the instant $t \to t+\mathrm{d}t$. Since $F_{t,r} = -\frac{\ln P_{t+r} - \ln P_t}{r}$ is a difference quotient,

$$\boxed{F_t = -\frac{\mathrm{d}}{\mathrm{d}t}\ln P_t = -\frac{1}{P_t}\frac{\mathrm{d}P_t}{\mathrm{d}t}}$$

a *logarithmic derivative*: a rate of change divided by a level, like $\delta(t) = V_t'/V_t$ (Chapter 2) and $\nu(i)$ (Chapter 23).

RECOVERING THE PRICES

Integrating from 0 to t with $P_0 = 1$ (a bond maturing immediately is worth its face value):

$$\int_0^t F_s \, \mathrm{d}s = -\ln P_t, \qquad \boxed{P_t = \mathrm{e}^{-\int_0^t F_s \, \mathrm{d}s}}$$

Compare Chapter 2: a varying force of interest gave $\mathsf{v}(t) = \mathrm{e}^{-\int_0^t \delta(s)\,\mathrm{d}s}$. The same formula, with the instantaneous forward rate in place of the force of interest. As $r \to 0$ the chord of the $\ln P$ curve becomes the tangent: $F_{t,r} \to F_t$.

EXAMPLE 24.1 · A given price function

The price at time 0 of a zero-coupon bond of term $2 < t < 4$ is $P_t = 100 - 2t^2$. Find F_3.

SOLUTION

$F_t = -\dfrac{P_t'}{P_t} = \dfrac{4t}{100 - 2t^2}$, so $F_3 = \dfrac{12}{82} = 0.146$: an instantaneous forward rate of 14.6% at time 3.

24.2 The discounted mean term

MACAULAY DURATION, OR DISCOUNTED MEAN TERM

The **discounted mean term** (DMT) is the average date of the cashflows, each date weighted by the *present value* of what is paid then:

$$\boxed{\tau(i) = \frac{\sum_k t_k c_k \mathsf{v}^{t_k}}{\mathrm{NPV}(i)} = \frac{\sum_k t_k c_k \mathsf{v}^{t_k}}{\sum_k c_k \mathsf{v}^{t_k}}} \qquad \text{and} \qquad \boxed{\tau(i) = (1+i)\,\nu(i)}$$

The second identity is immediate on comparing the formulas: every term of ν carries one extra factor of v.

WHY $\tau = (1+i)\nu$

Also, with $i = \mathrm{e}^{\delta} - 1$, $\dfrac{\mathrm{d}i}{\mathrm{d}\delta} = 1 + i$, so by the chain rule

$$-\frac{1}{\mathrm{NPV}}\frac{\mathrm{d}\,\mathrm{NPV}}{\mathrm{d}\delta} = -\frac{1}{\mathrm{NPV}}\frac{\mathrm{d}\,\mathrm{NPV}}{\mathrm{d}i}\frac{\mathrm{d}i}{\mathrm{d}\delta} = \nu(i)(1+i) = \tau(i).$$

The DMT is the volatility measured against the *force* of interest rather than the effective rate: $\tau(i) = -\frac{\mathrm{d}}{\mathrm{d}\delta}\ln \mathrm{NPV}(i)$.

τ IS A CENTRE OF MASS

Place a weight $c_k \mathsf{v}^{t_k}$ at each payment date t_k: τ is the point at which the whole arrangement balances. For a ten-year 8% bond at $i = 5\%$ the redemption at $t = 10$ carries 54% of the weight (66.30 of 123.17) and pulls τ far to the right.

24.3 Basic durations

ZERO-COUPON BOND

One cashflow, so $\tau(i) = \dfrac{nRv^n}{Rv^n} = n$: intuitively obvious, the average term of a single payment is its date. The extreme case: all the weight at one date. This is the largest duration any bond of that term can have.

COUPON BOND

An n-year bond paying coupons D annually in arrears, redeemable at R. The coupons' numerator is $D\sum_{t=1}^{n} tv^t = D\,(Ia)_{\overline{n}|}$ (Chapter 8: the time-weighted companion of $a_{\overline{n}|}$), the redemption contributes nRv^n, and the price is $Da_{\overline{n}|} + Rv^n$. So

$$\boxed{\tau(i) = \frac{D\,(Ia)_{\overline{n}|} + nRv^n}{D\,a_{\overline{n}|} + Rv^n}}$$

EXAMPLE 24.2 · A ten-year 8% bond at three rates

Find the DMT of a ten-year bond with annual 8% coupons, redeemable at par, at $i = 5\%, 10\%, 15\%$.

SOLUTION

With $D = 8$, $R = 100$, $n = 10$:

$$\tau(0.05) = \frac{928.90}{123.17} = 7.54, \qquad \tau(0.10) = \frac{617.83}{87.71} = 7.04, \qquad \tau(0.15) = \frac{423.17}{64.87} = 6.52.$$

A larger τ means more sensitive to interest rate changes. τ *falls* as i rises: a higher rate discounts the late cashflows more heavily, so weight shifts to the front and the balance point moves left. At $i = 0$ nothing is discounted and $\tau = \frac{8(55)+10(100)}{180} = 8.0$; as i grows the redemption loses weight and τ falls back towards the coupon dates.

EXAMPLE 24.3 · Does duration predict the fall?

How does the price of a bond paying annual 3% coupons, redeemable at par, change when the interest rate goes from 7% to 8%, if the term is (a) five years, (b) twenty-five years?

SOLUTION

$P = 3a_{\overline{n}|} + 100v^n$:

	at 7%	at 8%	fall	$\tau(0.07)$	$\nu(0.07)$
5-year bond	83.60	80.04	4.26%	4.69	4.38
25-year bond	53.39	46.63	12.66%	14.94	13.96

The change in rate has a much greater effect on the longer bond, as duration says. The first-order prediction $\varepsilon\nu$ for $\varepsilon = 0.01$ is 4.38% and 13.96%: good for the short bond, but

too pessimistic for the long one, overshooting the actual fall by 2.8% and 10.3% of itself. Duration is the *slope* of the price curve at 7%; the curve bends away above its tangent, and the 25-year bond bends far more. The next chapter corrects this with the second derivative.

SAME WORD, THREE QUANTITIES

ν (effective duration, *volatility*) and $\tau = (1+i)\nu$ (Macaulay duration, *discounted mean term*) are different numbers; the exam may use either word. The term of the bond is a third. Redington's second condition (Chapter 25) may be stated with either ν or τ, since the common factor $(1+i_0)$ cancels.

THE CHAPTER IN ONE BOX

$$F_t = -\frac{\mathrm{d}}{\mathrm{d}t}\ln P_t, \qquad P_t = \mathrm{e}^{-\int_0^t F_s\,\mathrm{d}s},$$

$$\tau(i) = \frac{\sum_k t_k c_k \mathrm{v}^{t_k}}{\sum_k c_k \mathrm{v}^{t_k}} = (1+i)\nu(i) = -\frac{\mathrm{d}}{\mathrm{d}\delta}\ln \mathrm{NPV}(i).$$

$\tau = n$ for a zero-coupon bond; $\tau = \frac{D(Ia)_{\overline{n}|} + nR\mathrm{v}^n}{Da_{\overline{n}|} + R\mathrm{v}^n}$ for a coupon bond. τ is the centre of mass of the discounted cashflows. It falls as i rises, falls as the coupon rises, and predicts price movements to first order only.

TRY IT YOURSELF

1. Find τ and ν at $i = 8\%$ for a four-year bond redeemable at par with 6% annual coupons, and check $\tau = (1+i)\nu$.
2. Show that the duration of a level perpetuity in arrears is $\frac{1+i}{i}$, and say what happens to it as i falls.
3. Two bonds have the same term but different coupons. Which has the larger duration, and why?

ANSWERS

1. $\mathrm{v} = 1.08^{-1}$: $a_{\overline{4}|} = 3.312127$, $\mathrm{v}^4 = 0.735030$, $(Ia)_{\overline{4}|} = \dfrac{\ddot{a}_{\overline{4}|} - 4\mathrm{v}^4}{0.08} = 7.962219$. Then

$$\tau(0.08) = \frac{6(7.962219) + 4(100)(0.735030)}{6(3.312127) + 100(0.735030)} = \frac{341.78}{93.376} = 3.6603,$$

and $\nu(0.08) = \tau/1.08 = 3.3892$; check $1.08 \times 3.3892 = 3.6603$. Just under four years: nearly all the weight is the redemption, but the four coupons pull the balance point slightly forward.

2. $\mathrm{NPV} = a_{\overline{\infty}|} = 1/i$, $\mathrm{NPV}' = -1/i^2$, so $\nu = \frac{1/i^2}{1/i} = \frac{1}{i}$ and $\tau = (1+i)\nu = \frac{1+i}{i}$. As i falls, $\tau \to \infty$: at a low rate the distant payments are barely discounted and keep their weight, pushing the balance point of an infinite stream arbitrarily far out. A perpetuity is the most rate-sensitive instrument there is, and most so when rates are low.

3. The one with the *smaller* coupon. A large coupon puts more weight on the early dates and drags the balance point forward; a small coupon leaves almost all the weight on the redemption at time n. The extreme case is the zero-coupon bond, $\tau = n$: the largest duration for that term. (This is also why the low-coupon bond had the higher redemption yield on a rising curve in Chapter 22.)

CHAPTER 25

Convexity and Redington's conditions

LECTURE 29 · *The second derivative, twice used*

AFTER THIS CHAPTER YOU CAN

- ☐ Compute the convexity $\mathrm{c}(i)$ of a cashflow series.
- ☐ Use $\Delta\,\mathrm{NPV} / \mathrm{NPV} \approx -\varepsilon v + \frac{\varepsilon^2}{2}\,\mathrm{c}$.
- ☐ State Redington's three conditions and say what each does.
- ☐ Explain why overfunding is not immunisation, and why $\mathrm{c}_A < \mathrm{c}_L$ is the worst case.

The first-order estimate of the previous chapter got the five-year bond nearly right and the twenty-five-year bond wrong by about a tenth, *always in the same direction, always too pessimistic*. That is curvature, which a tangent ignores.

Warm-up. A level perpetuity in arrears has $\mathrm{NPV} = i^{-1}$, so $\mathrm{NPV}' = -i^{-2}$ and $\mathrm{NPV}'' = 2i^{-3}$, and $\mathrm{NPV}'' / \mathrm{NPV} = 2/i^2$: 200 at $i = 10\%$. The sign matters: NPV' is negative but NPV'' is *positive*. The value falls as i rises, but ever more slowly: the curve bends upwards.

25.1 Convexity

CONVEXITY

$\mathrm{c}(i) = \dfrac{\mathrm{NPV}''(i)}{\mathrm{NPV}(i)} = \dfrac{\sum_k t_k(t_k+1)c_k v^{t_k+2}}{\sum_k c_k v^{t_k}}$. It measures the change in duration when the interest rate changes. The numerator comes from differentiating $\mathrm{NPV}(i) = \sum c_k(1+i)^{-t_k}$ twice: $\sum c_k(-t_k)(-t_k-1)(1+i)^{-t_k-2}$. Note *no minus sign*: NPV is convex, so $\mathrm{c}(i) > 0$ when all $c_k > 0$.

ZERO-COUPON BOND

One cashflow: $\mathrm{c}(i) = \dfrac{n(n+1)cv^{n+2}}{cv^n} = n(n+1)v^2$, independent of the payment. At 8% ($v^2 = 0.857339$): $\mathrm{c}_4 = 17.15$, $\mathrm{c}_8 = 61.73$. Convexity grows roughly with the *square* of the term, duration only linearly: doubling the term roughly doubles the sensitivity but roughly *quadruples* the curvature.

EXAMPLE 25.1 · Three assets with the same duration

Using 10%, find the convexity of: A, an 11-year zero-coupon bond of face value €100; B, a lump sum of €9,663 in five years and €26,910 in twenty years; C, a level perpetuity paid annually in arrears.

SOLUTION

A: $c_A = 132v^2 = 109.1$ ($n = 11$: $11 \times 12 = 132$). **B:**

$$c_B(i) = \frac{9663(5)(6)v^7 + 26910(20)(21)v^{22}}{9663v^5 + 26910v^{20}}, \qquad c_B(0.1) = 153.7.$$

C: $c_C = 2/i^2 = 200$.
The ordering $109.1 < 153.7 < 200$ is controlled by how *spread out* the cashflows are: one date; two dates fifteen years apart; a stream running for ever. Yet at 10% all three have the *same* discounted mean term: $\tau = 11$ for A; for B the present values are 6000 and 4000, so $\tau = \frac{5(6000)+20(4000)}{10000} = 11$; and for C, $\tau = \frac{1+i}{i} = 11$. The same balance point, but a wider spread and hence larger convexity.

25.2 Approximate proportional change

TAYLOR TO SECOND ORDER

For small ε, $\text{NPV}(i+\varepsilon) \approx \text{NPV}(i) + \varepsilon\,\text{NPV}'(i) + \frac{\varepsilon^2}{2!}\text{NPV}''(i)$. Dividing by $\text{NPV}(i) \neq 0$, with $\text{NPV}' / \text{NPV} = -\nu$ and $\text{NPV}'' / \text{NPV} = c$,

$$\boxed{\frac{\text{NPV}(i+\varepsilon) - \text{NPV}(i)}{\text{NPV}(i)} \;\approx\; -\varepsilon\,\nu(i) + \frac{\varepsilon^2}{2}\,c(i)}$$

NPV is convex, so the tangent lies *below* the curve on both sides and the first-order estimate always understates the value. The second term $\frac{\varepsilon^2}{2}c$ is positive whatever the sign of ε, so it lifts the estimate back towards the curve.

EXAMPLE 25.2 · The 25-year bond, revisited

The 25-year, 3%-coupon bond of Chapter 24 at $i = 7\%$, $\varepsilon = 0.01$, with $\nu = 13.96$ and $c(0.07) = 280.52$:

	estimate	actual
First order, $-\varepsilon\nu$	-13.96%	-12.66%
Second order, $-\varepsilon\nu + \frac{\varepsilon^2}{2}c$	-12.56%	-12.66%

The first-order estimate was out by 1.3 percentage points; the second-order estimate by 0.1. For the five-year bond ($c = 24.03$) the second-order estimate is -4.26%, agreeing with the actual figure to two decimals: arguably better, but arguably less needed there, since the first-order error was only 0.12.

WHY A FIRST-ORDER ESTIMATE IS TOO PESSIMISTIC BOTH WAYS

Convex NPV lies above every tangent. The first-order estimate reads the tangent, so it is below the true value at $i+\varepsilon$ whatever the sign of ε: the error is the neglected $\frac{\varepsilon^2}{2}\,\mathrm{c}(i)$, and both factors are positive. For a *rise*, the estimate over-states the fall; for a *fall*, it under-states the gain. Either way the holder does better than the linear estimate suggests, which is why convexity is regarded as desirable, and why Redington asks for more of it in the assets than in the liabilities.

25.3 Immunisation

IMMUNISATION

Immunisation is the selection of an asset portfolio which protects the surplus against small changes in the interest rate. For a fund with assets A and liabilities L, write V_A, V_L for their present values, ν_A, ν_L for their volatilities and $\mathrm{c}_A, \mathrm{c}_L$ for their convexities. At rate i_0 the fund is **immunised** against small movements ε if

$$V_A(i_0) = V_L(i_0) \qquad \text{and} \qquad V_A(i_0+\varepsilon) > V_L(i_0+\varepsilon).$$

Take the **surplus** $S(i) = V_A(i) - V_L(i)$, with $S(i_0) = 0$, and require $S(i_0+\varepsilon) > 0$. By Taylor,

$$S(i_0+\varepsilon) \approx \underbrace{S(i_0)}_{=0} + \varepsilon S'(i_0) + \frac{\varepsilon^2}{2!} S''(i_0).$$

The first term is zero by assumption. The second changes sign with ε, so it must be killed. The third has a fixed sign, so it must be positive.

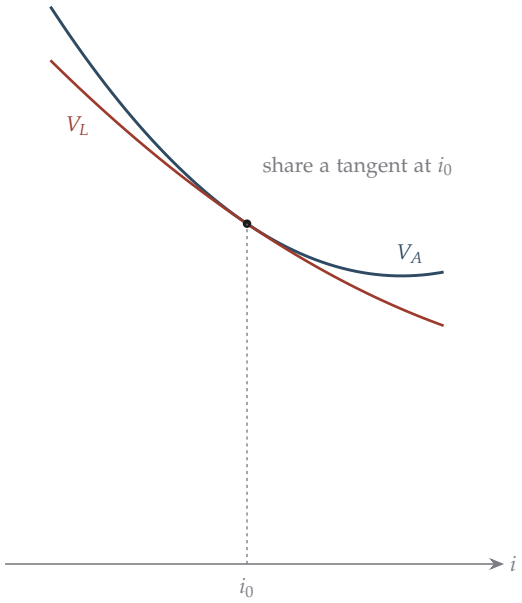
REDINGTON'S CONDITIONS FOR IMMUNISATION

Let i_0 be the initial rate of interest.

1. $V_A(i_0) = V_L(i_0)$: **values equal**.
2. $\nu_A(i_0) = \nu_L(i_0)$: **volatilities equal** (equivalently the discounted mean terms: $\tau = (1+i)\nu$ and the common factor cancels).
3. $\mathrm{c}_A(i_0) > \mathrm{c}_L(i_0)$: **convexity of the assets exceeds that of the liabilities.**

If these hold, the fund is protected against small movements in the interest rate.

Where each comes from. *Condition 2:* $\varepsilon S'(i_0)$ dominates the quadratic term for small ε, so a nonzero S' makes $S < 0$ in one direction. Hence $S'(i_0) = 0$, i.e. $V_A' = V_L'$; dividing by $-V_A = -V_L$ gives $\nu_A = \nu_L$. *Condition 3:* $S(i_0+\varepsilon) \approx \frac{\varepsilon^2}{2} S''(i_0)$ and $\varepsilon^2 > 0$, so we need $S'' > 0$, i.e. $V_A'' > V_L''$; dividing by the common positive value, $\mathrm{c}_A > \mathrm{c}_L$.



Conditions 1 and 2 make the curves *touch* at i_0 with the same slope. Condition 3 says the asset curve bends up more, so it lies above the liability curve on *both* sides: the surplus is positive whichever way rates move. Equivalently, the surplus has a local *minimum* of value zero at i_0.

OVERFUNDING IS NOT IMMUNISATION

Suppose $V_A(i_0) > V_L(i_0)$, so $S(i_0) > 0$, and even $v_A = v_L = v$. Then $S'(i_0) = -vV_A + vV_L = -vS(i_0) \neq 0$, and $S(i_0 + \varepsilon) \approx S(i_0)(1 - v\varepsilon)$: the surplus still has first-order sensitivity. The excess is a cushion, but the cushion itself is exposed. Equality $V_A = V_L$ is needed so that the first-order term can be killed by $v_A = v_L$.

"LOCAL" IS THE OPERATIVE WORD

Redington protects against *small* movements. For large changes the neglected Taylor terms grow with $|\varepsilon|$, and the surplus may not be protected.

EXAMPLE 25.3 · Two conditions but the wrong convexity

A fund satisfies conditions 1 and 2 but has $c_A < c_L$. What happens to the surplus if rates move either way?

SOLUTION

Conditions 1 and 2 give $S(i_0) = S'(i_0) = 0$, so $S(i_0 + \varepsilon) \approx \frac{\varepsilon^2}{2} S''(i_0)$, and $c_A < c_L$ makes $S'' < 0$. Since $\varepsilon^2 > 0$ for every $\varepsilon \neq 0$, the surplus becomes *negative in both directions*: i_0 is a local maximum of the surplus, at value zero. Matching values and durations alone is not enough: it removes the first-order exposure and can leave the fund exposed at second order in both directions at once. This is the worst of the three possible signs for S'' (with $S'' = 0$, higher terms decide).

CHECKING REDINGTON

1. Compute $V_A(i_0)$ and $V_L(i_0)$: equal?
2. Compute v (or τ) for both: equal?
3. Compute c for both: assets strictly larger?
4. If any fails, say which exposure remains and in which direction.

THE CHAPTER IN ONE BOX

$$\mathrm{c}(i) = \frac{\mathrm{NPV}''(i)}{\mathrm{NPV}(i)} = \frac{\sum_k t_k(t_k+1)c_k v^{t_k+2}}{\sum_k c_k v^{t_k}}, \qquad \frac{\Delta\,\mathrm{NPV}}{\mathrm{NPV}} \approx -\varepsilon v + \frac{\varepsilon^2}{2}\,\mathrm{c}\,.$$

$$\text{Redington at } i_0: \qquad V_A = V_L, \qquad v_A = v_L, \qquad \mathrm{c}_A > \mathrm{c}_L\,.$$

Convexity is positive, so the quadratic term lifts a first-order estimate whichever way rates move; the same one-sidedness is what makes the third condition protect the surplus in both directions. Zero-coupon bond: $\mathrm{c} = n(n+1)v^2$. Overfunding does not immunise; protection is local.

TRY IT YOURSELF

1. Find the convexity at $i = 8\%$ of a four-year and of an eight-year zero-coupon bond. What is the pattern?
2. A fund satisfies conditions 1 and 2 but has $\mathrm{c}_A < \mathrm{c}_L$. What happens to the surplus if rates move either way?
3. Explain why a first-order estimate of a price change is too pessimistic for a rise in i *and* for a fall.

ANSWERS

1. $\mathrm{c} = n(n+1)v^2$: $\mathrm{c}_4 = 20(0.857339) = 17.15$, $\mathrm{c}_8 = 72(0.857339) = 61.73$. Convexity grows roughly with the square of the term, while duration grows only linearly, which is why the second-order correction mattered so much more for the twenty-five-year bond.
2. The surplus is worse in both directions: $S(i_0+\varepsilon) \approx \frac{\varepsilon^2}{2}S''(i_0) < 0$. Local maximum of the surplus, at zero.
3. NPV is convex, so its graph lies above every tangent; the first-order estimate is the tangent's value. The error is $\frac{\varepsilon^2}{2}\,\mathrm{c}(i)$, positive whichever way rates move (and $\mathrm{c} > 0$ when all cashflows are positive).

CHAPTER 26

Building an immunised portfolio

LECTURES 30–31 · *Match, match, check, and what the match actually buys*

AFTER THIS CHAPTER YOU CAN

- ☐ Find the weights of a two-bond immunising portfolio with $w_a = \frac{b-\tau_L}{b-a}$.
- ☐ Turn weights into present values and redemption values, and check condition 3.
- ☐ Estimate the change in assets and liabilities after a move, to first and second order.
- ☐ Say whether an unmatched fund profits or loses when rates move.
- ☐ Explain what immunisation delivers, and what it gives up.

Chapter 25 gave three conditions but no portfolio satisfying them. Two of them are *equations* and the third is a *check*. Here they are in action.

26.1 Two zero-coupon bonds

Warm-up. A fund is split between two zero-coupon bonds of terms a and b, in proportions w_a and w_b by present value ($w_a + w_b = 1$). Each bond has a single cashflow, contributing aV_a and bV_b to the numerator of τ:

$$\tau = \frac{aV_a + bV_b}{V_a + V_b} = w_a\, a + w_b\, b.$$

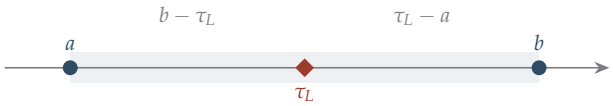
The discounted mean term of the fund is the weighted average of the two terms.

THE TWO-BOND RULE

Redington's second condition $\tau_A = \tau_L$ reads $w_a a + w_b b = \tau_L$, a linear equation in the weights. With $w_a + w_b = 1$:

$$\boxed{w_a = \frac{b - \tau_L}{b - a}, \qquad w_b = \frac{\tau_L - a}{b - a}}$$

It forces $a < \tau_L < b$: the assets must straddle the liabilities, because a weighted average lies between the two terms. The weights are a *lever*: each is proportional to the *distance of the other bond* from τ_L, so the bond sitting closer to the liability date carries more of the fund. The interest rate does not enter the proportions at all: it enters only when proportions are converted into redemption values.



EXAMPLE 26.1 · Why the assets must straddle

Could a fund owing €100,000 in ten years be immunised with five- and fifteen-year zeros? With eight- and five-year zeros instead?

SOLUTION

The liability is one payment at time 10, so $\tau_L = 10$. With $a = 5$, $b = 15$: $w_5 = \frac{15-10}{15-5} = \frac{1}{2}$, $w_{15} = \frac{1}{2}$: half and half, as symmetry demands, and independent of i. With five- and eight-year bonds, $\tau_L = 10$ lies outside $[5, 8]$: $w_5 = \frac{8-10}{8-5} = -\frac{2}{3}$. A negative weight would mean selling the five-year bond short to stretch the mean term to ten years; with a long-only portfolio the second condition *cannot be met*.

26.2 A worked immunisation

EXAMPLE 26.2 · Liabilities at 6 and 8, bonds at 5 and 10

A fund must pay €50,000 at the end of the sixth and eighth years. If rates are currently 7% at all terms, show that immunisation to small changes can be achieved by an appropriate combination of five- and ten-year zero-coupon bonds.

SOLUTION

Let R_5, R_{10} be the redemption values.
Condition 1 (values): $R_5 v^5 + R_{10} v^{10} = V_L = 50000(v^6 + v^8) = 62418$.
Condition 2 (volatilities, equivalently mean terms; below with τ): $V_A' = V_L'$ gives $5R_5 v^6 + 10R_{10} v^{11} = 50000(6v^7 + 8v^9) = 404398$. Multiply by 1.07 and subtract $5\times$ the first equation:

$$5R_{10} v^{10} = 404398(1.07) - 5(62418) = 120616, \quad R_{10} v^{10} = 24124, \quad R_{10} \approx €47{,}455,$$

$$R_5 v^5 = 62418 - 24124 = 38294, \quad R_5 \approx €53{,}709.$$

The amount invested in each is its present value: €38,294 (61.4%) in the five-year bond and €24,124 (38.6%) in the ten-year one, summing to V_L. Check with the lever: $\tau_L = \frac{6v^6 + 8v^8}{v^6 + v^8} = 6.93$ and $0.614(5) + 0.386(10) = 6.93$.
Condition 3 (convexity):

$$c_A(0.07) = \frac{30R_5 v^7 + 110R_{10} v^{12}}{62418} = 53.21, \qquad c_L(0.07) = \frac{50000(42v^8 + 72v^{10})}{62418} = 48.90.$$

Since $53.21 > 48.90$, the fund is immunised against small changes in the 7% rate.

WHY THE ASSETS ARE MORE CONVEX

Both portfolios have the same value and the same mean term 6.93 years, but they achieve it differently: the liabilities pay at 6 and 8, the assets at 5 and 10. **The assets are more spread out about the common mean term, and convexity measures exactly that spread.** The general principle: match the value and the mean term, then arrange the assets to engulf the liabilities more widely.

IMMUNISING WITH TWO ZERO-COUPON BONDS

1. Compute $V_L(i_0)$ and τ_L from the liability cashflows (a single payment at n has $\tau_L = n$).
2. Pick bond terms $a < \tau_L < b$.
3. Weights $w_a = \frac{b-\tau_L}{b-a}$, $w_b = 1 - w_a$. Present values $w_a V_L$, $w_b V_L$.
4. Redemption values $R = \text{PV} \times (1+i_0)^{\text{term}}$; **verify condition 1** as a check.
5. Check $\mathrm{c}_A > \mathrm{c}_L$: automatic when the assets straddle the liabilities more widely, but still worth the arithmetic.

Working with τ rather than ν avoids an extra factor of v in every term: $\sum t_k A_{t_k} \mathrm{v}^{t_k} = \sum t_j L_{t_j} \mathrm{v}^{t_j}$. A minor simplification: use whichever you prefer.

EXAMPLE 26.3 · CM1A April 2023, part (i)

An organisation must pay \$5 million in 18 years. It holds a cash fund equal to the present value of the liabilities and wishes to invest the whole of it in nine- and twenty-year zero-coupon bonds so as to immunise against small changes in the interest rate, currently 8% effective. Find the proportion of the fund in each bond.

SOLUTION

The assets straddle the liability: $9 < 18 < 20$. $V_L = 5{,}000{,}000\,\mathrm{v}^{18} = 1{,}251{,}245.1456$ and $\tau_L = 18$ (a single payment). With $x_9 = R_9\mathrm{v}^9$ and $x_{20} = R_{20}\mathrm{v}^{20}$:

$$x_9 + x_{20} = 1{,}251{,}245.1456, \qquad 9x_9 + 20x_{20} = 18 \times 1{,}251{,}245.1456 = 22{,}522{,}412.6208.$$

Subtract $9\times$ the first: $11x_{20} = 11{,}261{,}206.3104$, so $x_{20} = 1{,}023{,}746.0282$ and $x_9 = 227{,}499.1174$: 18.1818% in the nine-year bond and 81.8182% in the twenty-year one, i.e. $\frac{2}{11}$ and $\frac{9}{11}$. Exactly the lever: $w_9 = \frac{20-18}{20-9} = \frac{2}{11}$, so the shortcut finds it in one line, and the heavier bond is the one nearer the liability date.

26.3 What the match buys

Now move the rate. The single payment of \$5 million at $n = 18$ has, at 8%,

$$\tau_L = 18, \qquad \nu_L = \frac{18}{1.08} = 16.\overline{6}, \qquad \mathrm{c}_L = n(n+1)\mathrm{v}^2 = 342(1.08)^{-2} = 293.21.$$

Nothing about the *size* of the payment entered: all three measures are properties of the timing alone. (That is why part (i) never needed R_9 or R_{20}.) For the portfolio, $R_9 = 454771.79$, $R_{20} = 4771636.36$, and

$$\mathrm{c}_A(0.08) = \frac{9 \cdot 10\, R_9 \mathrm{v}^{11} + 20 \cdot 21\, R_{20} \mathrm{v}^{22}}{1251245.15} = 308.64 \;>\; 293.21 = \mathrm{c}_L,$$

so all three conditions hold.

EXAMPLE 26.4 · CM1A April 2023, part (ii): the rate rises to 8.5%

Estimate, using the volatility, the percentage change in the present value of the assets if the rate rises to 8.5% effective. Then see what second order adds.

SOLUTION

$\varepsilon = 0.005$. Condition 2 forced $\tau_A = \tau_L = 18$, so $\nu_A = 18/1.08 = 16.\overline{6}$ and, to first order,

$$\frac{\Delta V_A}{V_A} \approx -0.005 \times 16.\overline{6} = -8.33\%.$$

The *liabilities* give exactly the same: $\nu_L = \nu_A$ by construction. To first order the two sides move identically, so the surplus is unchanged; that is what condition 2 is designed to achieve. But it also means the first-order calculation *cannot tell us whether the fund has come out ahead*. For that we need convexity:

$$\frac{\Delta V_A}{V_A} \approx -0.005(16.\overline{6}) + \frac{0.005^2}{2}(308.64) = -7.95\%,$$
$$\frac{\Delta V_L}{V_L} \approx -0.005(16.\overline{6}) + \frac{0.005^2}{2}(293.21) = -7.97\%.$$

The assets fall by *less*; the whole difference is the convexity gap $308.64 - 293.21$, scaled by $\varepsilon^2/2$. Exact values: -7.96% and -7.98%. The second-order estimate is good to a hundredth of a point; the first-order one was out by nearly four tenths.

THE LAST WORD ON THE THREE CONDITIONS

$$\frac{\Delta V_A}{V_A} - \frac{\Delta V_L}{V_L} \;\approx\; \underbrace{-\varepsilon\,(\nu_A - \nu_L)}_{\text{zero under condition 2}} \;+\; \underbrace{\frac{\varepsilon^2}{2}\,(\mathrm{c}_A - \mathrm{c}_L)}_{\text{positive under condition 3}}$$

If the durations match, the first-order terms cancel and the surplus is protected against the *direction* of a move; the convexity gap then decides its size, and is positive whichever way rates go. If the durations do *not* match, the first term dominates and the fund is a directional bet. The surplus curve sits at zero at i_0, is flat there (condition 2) and curves upward (condition 3): a move in either direction leaves a small surplus. This is all immunisation delivers.

A FALL INSTEAD

If rates *fall* to 7.5%, $\varepsilon = -0.005$: the first-order term changes sign but the second-order term does not.

$$\frac{\Delta V_A}{V_A} \approx +0.005(16.\bar{6}) + \frac{0.005^2}{2}(308.64) = +8.72\%, \qquad \frac{\Delta V_L}{V_L} \approx +8.70\%.$$

The assets rise by *more*, so again $V_A > V_L$ afterwards (exact: +8.73% and +8.71%). ε^2 is positive whichever way rates go, so the convexity gap always works in the fund's favour.

26.4 A fund that is not immunised

EXAMPLE 26.5 · CM1A April 2019

An individual buys an annuity for a single premium, paying £10000 annually in arrears for 15 years. The insurer invests the premium in a bond paying 6% coupons annually in arrears, redeemable at par in exactly nine years. (i) Calculate the duration at 5% of (a) the annuity and (b) the bond. (ii) Explain whether the insurer makes a profit or a loss if rates fall slightly at all terms.

SOLUTION

The annuity is what the company *must pay*: the liability; the bond is the asset.

(i)(a) Level payments, so $\tau_L = \dfrac{10000\,(Ia)_{\overline{15}|}}{10000\,a_{\overline{15}|}}$. At 5%: $a_{\overline{15}|} = 10.3797$, $(Ia)_{\overline{15}|} = \frac{1.05a_{\overline{15}|} - 15(1.05)^{-15}}{0.05} = 73.6677$, so

$$\tau_L = \frac{73.6677}{10.3797} = 7.0973 \text{ years.}$$

(b) $\tau_A = \dfrac{6\,(Ia)_{\overline{9}|} + 900(1.05)^{-9}}{6\,a_{\overline{9}|} + 100(1.05)^{-9}}$ with $a_{\overline{9}|} = 7.1078$, $(Ia)_{\overline{9}|} = 33.2347$:

$$\tau_A = \frac{779.5559}{107.1078} = 7.2782 \text{ years.}$$

(ii) A small fall raises both present values; which rises more? $\tau_A = 7.2782 > 7.0973 = \tau_L$, so $\nu_A > \nu_L$ (the factor $1+i$ is common). A larger volatility is a steeper response: the assets rise by more than the liabilities, and the company makes a **profit**. To first order the surplus changes by $-\varepsilon(\nu_A - \nu_L)V$, positive for $\varepsilon < 0$.

PROFIT FROM A FALL IS THE SAME EXPOSURE AS A LOSS FROM A RISE

This fund is **not** immunised: condition 2 fails, $\tau_A \neq \tau_L$. The company profits from a fall in rates *precisely because it is exposed to them*, and would make a loss of the same size if rates rose: both assets and liabilities fall, but the assets by more, so the fund is in deficit. Unmatched means the fund has taken a position on the direction of rates, and a position pays only when the guess is right.

IMMUNISATION IS INSURANCE

Redington is a defensive, local, first-order hedge that leaves a nonnegative second-order cushion. It is not a profit strategy: profit needs an intentional mismatch and a view on direction, which immunisation deliberately avoids. To immunise, the company would shorten its assets slightly (say by selling part of the nine-year bond and holding the proceeds in a shorter instrument) until $\nu_A = \nu_L$, and it would then need only $c_A > c_L$. What is given up is the profit of part (ii). Like all insurance it is paid for by surrendering the upside; the choice between the two funds is a commercial question, not a mathematical one.

THE CHAPTER IN ONE BOX

$$\text{Two bonds of terms } a < b: \quad w_a + w_b = 1, \quad w_a a + w_b b = \tau_L, \quad \text{so} \quad w_a = \frac{b - \tau_L}{b - a}.$$

$$\frac{\Delta V_A}{V_A} - \frac{\Delta V_L}{V_L} \approx -\varepsilon(\nu_A - \nu_L) + \frac{\varepsilon^2}{2}(c_A - c_L).$$

Match the value, match the mean term, then *check* the convexity. The first two are linear equations in the holdings; the third holds when the asset dates engulf the liability dates. A single payment at n has $\tau = n$, $\nu = \frac{n}{1+i}$, $c = n(n+1)v^2$. Immunisation is local: it protects against *small* movements only. A mismatched fund gains if rates go one way and loses if they go the other.

TRY IT YOURSELF

1. A fund owes €100,000 in ten years. At $i = 6\%$ it holds five-year and fifteen-year zero-coupon bonds. Find the proportion of the fund in each, using the mean-term argument.
2. For the same fund, find the redemption value of each bond and verify condition 1.
3. Could that fund be immunised with five-year and eight-year zero-coupon bonds instead? Answer without calculating.
4. In the April 2023 fund, estimate the percentage change in V_A and V_L if rates *fell* to 7.5%, and say which is larger.
5. For the April 2019 company, would a slight *rise* in rates give a profit or a loss? Justify using the durations.
6. An insurer holds assets of duration 7.28 against liabilities of duration 7.10, with equal values. What single change would immunise the fund, and what would be given up?

ANSWERS

1. $\tau_L = 10$: $w_5 = \frac{15-10}{15-5} = \frac{1}{2}$, $w_{15} = \frac{1}{2}$; independent of i.
2. $V_L = 100000(1.06)^{-10} = €55{,}839.48$, so each bond has present value €27,919.74: $R_5 = 27919.74(1.06)^5 = €37{,}362.91$, $R_{15} = 27919.74(1.06)^{15} = €66{,}911.28$. Check: $27919.74 + 27919.74 = 55839.48 = V_L$.
3. No. A weighted average of 5 and 8 lies between them, but $\tau_L = 10$ does not;

algebraically $w_5 = -\frac{2}{3}$ (a short sale). The assets must straddle the liabilities.

4. $\varepsilon = -0.005$: V_A +8.72%, V_L +8.70% (exact +8.73%, +8.71%): the assets rise by more.
5. A loss: a rise lowers both present values, and the assets have the larger volatility ($7.2782 > 7.0973$), so they fall by more and the fund is in deficit.
6. Shorten the assets until their duration falls to 7.10 (condition 1 already holds); then only $c_A > c_L$ remains to check. What is given up is the profit that the mismatch earns when rates fall; with matched durations a fall gives no first-order gain and a rise no first-order loss.