LECTURE 31

What immunisation is worth

Two past papers, and the last word on the third condition

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The three measures

LECTURES 27–29

$$\nu(i) \;=\; -\frac{\mathrm{NPV}'(i)}{\mathrm{NPV}(i)}, \qquad \tau(i) \;=\; (1+i)\nu(i), \qquad \mathsf{c}(i) \;=\; \frac{\mathrm{NPV}''(i)}{\mathrm{NPV}(i)},$$

$$\frac{\mathrm{NPV}(i+\varepsilon) - \mathrm{NPV}(i)}{\mathrm{NPV}(i)} \;\approx\; -\varepsilon\,\nu(i) + \frac{\varepsilon^2}{2}\,\mathsf{c}(i).$$

REDINGTON, AT i_0

1. $V_A(i_0) = V_L(i_0)$
2. $\nu_A(i_0) = \nu_L(i_0)$, equivalently $\tau_A = \tau_L$
3. $\mathsf{c}_A(i_0) > \mathsf{c}_L(i_0)$

Last lecture, we built a portfolio meeting all three. Today, we will observe what happens when the rate does move. We will also encounter a counterexample, a portfolio meeint only two of the 3 conditoins.

Today

1. CM1A April 2023, continued

2. CM1A April 2019

Warm-up

EXERCISE 31.1

Consider a liability with a single payment at time 18*, and assume the rate is* 8%*. Write down* τ_L*, and hence* ν_L *and* c_L*.*

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A single cashflow at time n has all its weight at that date, so

$$\tau_L \;=\; 18, \qquad \nu_L \;=\; \frac{\tau_L}{1+i} \;=\; \frac{18}{1.08} \;=\; 16.\dot{6},$$

$$\mathsf{c}_L \;=\; n(n+1)\mathsf{v}^2 \;=\; 342\,(1.08)^{-2} \;=\; 293.2099.$$

ASIDE

Nothing about the *size* of the payment entered: all three measures are properties of the timing alone. That is why the April 2023 question can be answered without ever computing R_9 or R_{20}.

SECTION 1

CM1A April 2023, continued

The question

EXAMPLE 31.2

An organisation must pay \$5 million in 18 years' time. It holds a cash fund equal to the present value of the liabilities and invests the whole of it in nine-year and twenty-year zero-coupon bonds so as to immunise against small changes in the interest rate. The current rate is 8% per annum effective.

(i) Calculate the proportion of the fund invested in each bond.

(ii) Estimate, using the volatility, the percentage change in the present value of the assets if the interest rate rose to 8.5% per annum effective.

We did part (i) in Lecture 30 and came to the solution: invest 18.1818% *in the nine-year bond and* 81.8182% *in the twenty-year one, that is* $\frac{2}{11}$ *and* $\frac{9}{11}$ *of the fund.*

Checking the third condition

Not asked, but worth doing. Converting the two present values into redemption amounts,

$$R_9 \;=\; 227{,}499.1174\,\mathrm{v}^{-9} \;=\; 454{,}771.79,$$

$$R_{20} \;=\; 1{,}023{,}746.0282\,\mathrm{v}^{-20} \;=\; 4{,}771{,}636.36.$$

Then, using $\mathrm{c} = \sum_k t_k(t_k+1)c_k\mathrm{v}^{t_k+2}/\,\mathrm{NPV}$,

$$\mathrm{c}_A(0.08) \;=\; \frac{9\cdot 10\cdot R_9\mathrm{v}^{11} + 20\cdot 21\cdot R_{20}\mathrm{v}^{22}}{1{,}251{,}245.15} \;=\; 308.6420,$$

against $\mathrm{c}_L(0.08) = 293.2099$ from the warm-up. Since $308.6420 > 293.2099$, the fund is immunised.

(ii) Using the volatility

To first order, from the expansion on the recall page,

$$\frac{V_A(0.085) - V_A(0.08)}{V_A(0.08)} \;\approx\; -0.005\,\nu_A(0.08).$$

Part (i) forced $\tau_A(0.08) = \tau_L(0.08) = 18$, which *was* the second condition, so

$$\nu_A(0.08) \;=\; \frac{\tau_A(0.08)}{1.08} \;=\; \frac{18}{1.08} \;=\; 16.\dot{6},$$

and the estimated change is

$$-0.005 \times 16.\dot{6} \;=\; -0.08\dot{3} \;=\; -8.33\%.$$

So the present value of the assets falls by about 8.33%.

First order sees no difference

Note what the same calculation gives for the liabilities. Since $\nu_L(0.08) = \nu_A(0.08)$ by construction,

$$\frac{V_L(0.085) - V_L(0.08)}{V_L(0.08)} \;\approx\; -0.005 \times 16.\dot{6} \;=\; -8.33\%,$$

exactly as for the assets.

> KEY IDEA
>
> To first order the two sides move *identically*, so the surplus is unchanged. This is precisely what Condition 2 was designed to achieve. But it also means the first-order calculation cannot tell us whether the fund has come out ahead or behind.

For that we need the term that distinguishes them: the convexity.

Second order separates them

Including the quadratic term, with $\varepsilon = 0.005$:

$$\frac{\Delta V_A}{V_A} \approx -0.005(16.\dot{6}) + \frac{(0.005)^2}{2}(308.6420) = -7.95\%,$$

$$\frac{\Delta V_L}{V_L} \approx -0.005(16.\dot{6}) + \frac{(0.005)^2}{2}(293.2099) = -7.97\%.$$

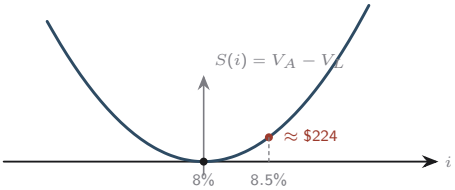
KEY IDEA

The assets fall by *less* than the liabilities, by two hundredths of a percentage point. The entire difference is the convexity gap $308.6420 - 293.2099$, scaled by $\varepsilon^2/2$.

Exactly: -7.96% and -7.98%. The second-order estimate is good to a hundredth of a point; the first-order one was out by nearly four tenths.

The surplus that results

Since the assets fall by less, $V_A(0.085) > V_L(0.085)$, as expected.



KEY IDEA

The surplus curve sits at zero at 8%, is flat there (condition 2), and curves upward (condition 3). A move in *either* direction leaves the fund with a small surplus. This, in effect, is all what immunisation delivers.

SECTION 2

CM1A April 2019

The question

EXAMPLE 31.3

An individual buys an annuity for a single lump-sum premium. The annuity pays 10,000 annually in arrears for 15 years. The insurance company invests the premium in a fixed-interest bond paying 6% coupons annually in arrears, redeemable at par in exactly nine years.

(i) Calculate the duration at 5% per annum effective of (a) the annuity and (b) the bond.

(ii) Explain whether the insurance company will make a profit or a loss if interest rates decrease slightly at all terms.

Which is which? The *annuity* is what the company must pay, so it is the liability; the *bond* is what it holds, so it is the asset.

(i)(a) The annuity

The present value of the annuity is $10{,}000\, a_{\overline{15}|}$, and its numerator is the increasing annuity of Lecture 28:

$$\tau_{\text{annuity}}(0.05) \;=\; \frac{10{,}000\left(\mathrm{v} + 2\mathrm{v}^2 + \cdots + 15\mathrm{v}^{15}\right)}{10{,}000\, a_{\overline{15}|}} \;=\; \frac{(Ia)_{\overline{15}|}}{a_{\overline{15}|}}.$$

With $\mathrm{v} = (1.05)^{-1}$, $a_{\overline{15}|} = \frac{1-(1.05)^{-15}}{0.05} = 10.3797$ and $(Ia)_{\overline{15}|} = \frac{(1.05)a_{\overline{15}|} - 15(1.05)^{-15}}{0.05} = 73.6677$, so

$$\tau_{\text{annuity}}(0.05) \;=\; \frac{73.6677}{10.3797} \;=\; 7.0973 \text{ years}.$$

(i)(b) The bond

From Lecture 28, an n-year bond paying D annually in arrears and redeemable at R has

$$\tau_{\text{bond}}(i) \;=\; \frac{D\,(Ia)_{\overline{n}|} + nR\mathrm{v}^{n}}{D\,a_{\overline{n}|} + R\mathrm{v}^{n}}.$$

Here $D = 6$, $R = 100$, $n = 9$, with $a_{\overline{9}|} = 7.1078$ and $(Ia)_{\overline{9}|} = \frac{(1.05)a_{\overline{9}|} - 9(1.05)^{-9}}{0.05} = 33.2347$, so

$$\tau_{\text{bond}}(0.05) \;=\; \frac{6(33.2347) + 900(1.05)^{-9}}{6(7.1078) + 100(1.05)^{-9}} \;=\; \frac{779.5559}{107.1078} \;=\; 7.2782 \text{ years}.$$

(ii) Profit or loss?

A small *decrease* in interest rates raises the present value of both the assets (the bond) and the liabilities (the annuity). The question is which rises more.

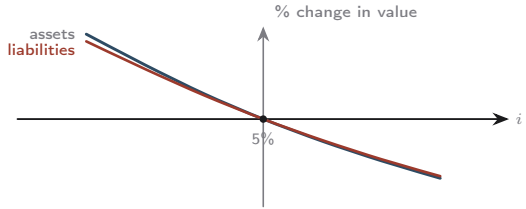
$$\tau_A \;=\; 7.2782 \;>\; 7.0973 \;=\; \tau_L \qquad \Longrightarrow \qquad \nu_A \;>\; \nu_L,$$

since $\tau = (1+i)\nu$ and the factor is the same for both.

> KEY IDEA
>
> A larger volatility means a steeper response, so the assets rise by more than the liabilities. The insurance company therefore makes a **profit**.

To first order the surplus changes by $-\varepsilon(\nu_A - \nu_L)V$, which for $\varepsilon < 0$ is positive.

The picture behind the answer



Both curves pass through zero at 5%, where the values match. The asset curve is the *steeper*, so to the left of 5% it lies above; and to the right, below.

What this fund is not

CAUTION

This fund is **not** immunised. Condition 2 fails as $\tau_A = 7.2782 \neq 7.0973 = \tau_L$. The company profits from a fall in rates precisely because it is exposed to them. It is same size of *loss* it would make if rates rose.

KEY IDEA

Redington is a defensive, local, first-order hedge that leaves a nonnegative second-order cushion; it is not a profit strategy, because profit requires an intentional mismatch and a directional view, which immunisation deliberately avoids.

To immunise, the company would have to shorten its assets slightly, and would then give up the profit.

Lecture 31 on one slide

THE LAST WORD ON THE THREE CONDITIONS

$$\frac{\Delta V_A}{V_A} - \frac{\Delta V_L}{V_L} \approx \underbrace{-\varepsilon\,(\nu_A - \nu_L)}_{\text{zero under cond. 2}} + \underbrace{\frac{\varepsilon^2}{2}\,(\mathsf{c}_A - \mathsf{c}_L)}_{\text{positive under cond. 3}}$$

If the durations match, the first-order terms cancel and the surplus is protected against the *direction* of a move; the convexity gap then decides its size, and is positive whichever way rates go. If the durations do *not* match, the first term dominates and the fund is a directional bet.

Exercises

EXERCISE 31.4

In the April 2023 fund, estimate the percentage change in V_A and V_L if rates instead fell to 7.5%, and say which is larger.

EXERCISE 31.5

For the April 2019 company, would a slight rise in rates give a profit or a loss? Justify your answer using the durations.

EXERCISE 31.6

An insurer holds assets of duration 7.28 against liabilities of duration 7.10, with equal values. What single change would immunise the fund, and what would be given up?

Solutions are in the backup slides.

END OF LECTURE 31

Value, duration, convexity: match the first, match the second, and keep more of the third. That is the what immunisation does, and it cloes the. My thanks for your attention; best of luck in your exams; and your future endeavours. Next: recap and excel.

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Solution 1

Now $\varepsilon = -0.005$, so the first-order term changes sign but the second-order term does not:

$$\frac{\Delta V_A}{V_A} \approx +0.005(16.\dot{6}) + \frac{(0.005)^2}{2}(308.6420) = +8.72\%,$$

$$\frac{\Delta V_L}{V_L} \approx +0.005(16.\dot{6}) + \frac{(0.005)^2}{2}(293.2099) = +8.70\%.$$

The assets rise by *more*, so again $V_A > V_L$ after the move. This is the point of the third condition: ε^2 is positive whichever way rates go, so the convexity gap always works in the fund's favour.

Solution 2

A loss.

A rise in rates lowers both present values, and the assets have the larger volatility, so they fall by more:

$$\Delta V_A \;\approx\; -\varepsilon \nu_A V, \qquad \Delta V_L \;\approx\; -\varepsilon \nu_L V, \qquad \nu_A > \nu_L, \;\; \varepsilon > 0.$$

Hence V_A falls below V_L and the company is in deficit.

The same exposure that produced a profit in part (ii) produces a loss here. That is what being unmatched means: the fund has taken a position on the direction of rates, and a position pays only when the guess is right.

Solution 3

Shorten the assets until their duration falls from 7.28 to 7.10. For instance, by selling part of the nine-year bond and holding the proceeds in a shorter instrument. Condition 1 already holds, so matching the duration establishes condition 2, and the fund would then need only $c_A > c_L$ to be immunised.

What is given up is the profit of part (ii). With $\nu_A = \nu_L$, the first-order terms cancel, so a fall in rates no longer produces a gain, but neither does a rise produce a loss.

> KEY IDEA
>
> Immunisation is insurance, and like all insurance it is paid for by surrendering the upside. The choice between the two funds is not a mathematical question but a commercial one.