LECTURE 30

Immunisation in practice

Choosing a portfolio that meets the three conditions

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
16 November 2026 · Winter 2026

Three conditions, no portfolio

$$V_A(i_0) = V_L(i_0), \qquad \nu_A(i_0) = \nu_L(i_0),$$
$$\mathsf{c}_A(i_0) > \mathsf{c}_L(i_0)$$

Redington, from Lecture 29

We saw three conditions but no portfolio satisfying them, yet. We will now see these three rules — two of which are equations, and the third a check — in action.

Today

1. Two zero-coupon bonds

2. CM1A April 2023

Warm-up

EXERCISE 30.1

A fund's whole value is split between two zero-coupon bonds of terms a and b, in proportions w_a and w_b by present value. Show that the discounted mean term of the fund is $w_a a + w_b b$.

Note, saying *in proportions* means $w_b + w_a = 1$.

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Write V_a and V_b for the two present values, so $w_a = V_a/(V_a + V_b)$. A zero-coupon bond of term a has a single cashflow at time a, contributing aV_a to the numerator of τ. Hence

$$\tau \;=\; \frac{aV_a + bV_b}{V_a + V_b} \;=\; a\,\frac{V_a}{V_a + V_b} + b\,\frac{V_b}{V_a + V_b} \;=\; w_a a + w_b b.$$

Warm-up

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KEY IDEA

So Redington's second condition, $\tau_A = \tau_L$, reads $w_a a + w_b b = \tau_L$. This is a linear equation in the weights. With $w_a + w_b = 1$ that is two equations in two unknowns, and can be solved with basic linear algebra.

And it forces $a < \tau_L < b$: the assets must *straddle* the liabilities, as a weighted average lies between the two terms.

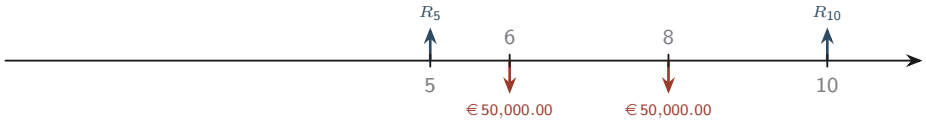
SECTION 1

Two zero-coupon bonds

The question

EXAMPLE 30.2

A fund must make payments of €50,000.00 at the end of the sixth and eighth years. Show that, if interest rates are currently 7% p.a. at all durations, immunisation to small changes in interest rates can be achieved by holding an appropriately chosen combination of five- and ten-year zero-coupon bonds.



The liabilities (red) are paid at 6 and 8; the assets (blue) pay out at 5 and 10, so they surround the liabilities, as pointed out in the warm-up. As we shall see, this is also what makes the third condition hold.

Condition 1

€50,000.00 *outlays at* $t = 6, 8$*;* $i = 0.07$ *p.a. effective.* *To hold: at* $t = 5$ *and* $t = 10$.

Let R_5, R_{10} be the redemption values of the five- and ten-year zero-coupon bonds. The present value of the assets is

$$V_A(0.07) \;=\; R_5 \mathrm{v}^5 + R_{10} \mathrm{v}^{10}.$$

By Redington's first condition this must equal the present value of the liabilities,

$$V_L(0.07) \;=\; 50{,}000\left(\mathrm{v}^6 + \mathrm{v}^8\right) \;=\; 62{,}418.$$

Hence

$$R_5 \mathrm{v}^5 + R_{10} \mathrm{v}^{10} \;=\; 62{,}418.$$

Condition 2

€50,000.00 *outlays at* $t = 6, 8$*;* $i = 0.07$ *p.a. effective.* *To hold: at* $t = 5$ *and* $t = 10$.

Differentiating, $V_A'(0.07) = -5R_5\mathrm{v}^6 - 10R_{10}\mathrm{v}^{11}$, so the volatility of the assets is

$$\nu_A(0.07) \;=\; -\frac{V_A'(0.07)}{V_A(0.07)} \;=\; \frac{5R_5\mathrm{v}^6 + 10R_{10}\mathrm{v}^{11}}{62{,}418}.$$

By the second condition this must equal the volatility of the liabilities,

$$\nu_L(0.07) \;=\; \frac{50{,}000\left(6\mathrm{v}^7 + 8\mathrm{v}^9\right)}{62{,}418} \;=\; \frac{404{,}398}{62{,}418},$$

and hence $5R_5\mathrm{v}^6 + 10R_{10}\mathrm{v}^{11} = 404{,}398$.

Solving for the cashflows

We must solve

$$R_5 v^5 + R_{10} v^{10} = 62{,}418 \qquad \text{and} \qquad 5R_5 v^6 + 10R_{10} v^{11} = 404{,}398.$$

Multiplying the second by 1.07 (i.e. by v^{-1}) and the first by 5 to match the R_5 terms:

$$\begin{aligned} 5R_5 v^5 + \ 5R_{10} v^{10} &= 312{,}090, \\ 5R_5 v^5 + 10R_{10} v^{10} &= 404{,}398 \times 1.07 = 432{,}706. \end{aligned}$$

Subtracting kills R_5, so

$$R_{10} = \frac{432{,}706 - 312{,}090}{5v^{10}} = 47{,}454, \qquad R_5 = \frac{62{,}418 - R_{10} v^{10}}{v^5} = 53{,}710.$$

KEY IDEA

The second equation is nothing but the *DMT* condition $\tau_A = \tau_L$: equating volatilities directly gives it in one line, which is why the τ form is the quicker route (Section 3).

The portfolio

So $R_5 = 53{,}710$ and $R_{10} = 47{,}454$, which determines the portfolio. The amount invested in each bond is its present value:

$$R_5 v^5 \;=\; 53{,}710 \times (1.07)^{-5} \;=\; 38{,}294, \qquad R_{10} v^{10} \;=\; 47{,}454 \times (1.07)^{-10} \;=\; 24{,}124.$$

The systematic way of solving systems of linear equations is through Gauss-Jordan. For two equations with two unknowns, this is overkill to introduce in the module.

> KEY IDEA
>
> These sum to €62,418.00, the present value of the liabilities, being Condition 1. About 61% of the fund goes into the short bond and 39% into the long one.

Check against the warm-up: $\tau_L = 6.93$, and $0.614(5) + 0.386(10) = 6.93$.

Checking condition 3

Conditions 1 and 2 hold by construction. It remains to check the third. We have

$$c_A(0.07) = \frac{30R_5v^7 + 110R_{10}v^{12}}{62{,}418} = 53.21,$$

$$c_L(0.07) = \frac{50{,}000\left(42v^8 + 72v^{10}\right)}{62{,}418} = 48.90.$$

Since $c_A(0.07) > c_L(0.07)$, the fund is immunised against small changes in the 7% p.a. interest rate.

Why the assets are more convex

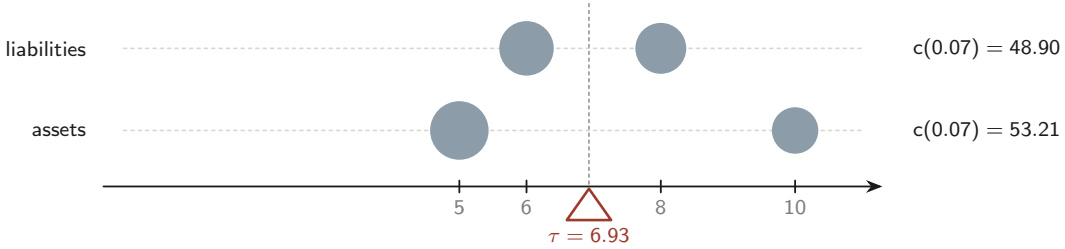
Both portfolios have the same value and the same mean term 6.93 years, but they achieve it differently:

	payment dates	convexity
Liabilities	6 and 8	48.90
Assets	5 and 10	53.21

KEY IDEA

The assets are *more spread out* about the common mean term. That is the general principle. To immunise, match the value and the mean term, then arrange the assets to engulf the liabilities more widely. Convexity measures exactly that spread.

Spread about the mean term



Same value, same balance point, but the assets sit further out on both sides, and that is what the larger convexity records.

Disc area is the share of the present value paid at that date.

SECTION 2

CM1A April 2023

The question

EXAMPLE 30.3

An organisation is required to pay \$5 million in 18 years' time. It holds a cash fund equal to the present value of the liabilities and wishes to use the whole fund to purchase a combination of nine-year and twenty-year zero-coupon bonds in order to immunise against small changes in the interest rate. The current interest rate is 8% per annum effective. Calculate, showing all working, the proportion of the cash fund that should be invested in each bond.

Again the assets straddle the liability: $9 < 18 < 20$.

Condition 1

Let R_9 and R_{20} be the redemption values of the nine- and twenty-year zero-coupon bonds. The present value of the assets is

$$V_A(0.08) \;=\; R_9 \mathrm{v}^9 + R_{20} \mathrm{v}^{20},$$

and the present value of the liabilities is

$$V_L(0.08) \;=\; 5{,}000{,}000\,\mathrm{v}^{18} \;=\; 1{,}251{,}245.1456.$$

Redington's first condition gives

$$R_9 \mathrm{v}^9 + R_{20} \mathrm{v}^{20} \;=\; 1{,}251{,}245.1456.$$

Condition 2, via the DMT

By the second condition the DMT of the assets equals the DMT of the liabilities. Now

$$\tau_A(0.08) \;=\; \frac{9R_9\mathrm{v}^9 + 20R_{20}\mathrm{v}^{20}}{1{,}251{,}245.1456}, \qquad \tau_L(0.08) \;=\; \frac{18 \times 5{,}000{,}000\,\mathrm{v}^{18}}{5{,}000{,}000\,\mathrm{v}^{18}} \;=\; 18,$$

the liability being a single payment. Hence

$$9R_9\mathrm{v}^9 + 20R_{20}\mathrm{v}^{20} \;=\; 18 \times 1{,}251{,}245.1456 \;=\; 22{,}522{,}412.6208.$$

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WORKING WITH τ RATHER THAN ν AVOIDS THE EXTRA FACTOR OF v

Both equations are already in the same unknowns $R_9\mathsf{v}^9$ and $R_{20}\mathsf{v}^{20}$ and in general

$$\underbrace{\sum_k t_k A_{t_k} \frac{v^{t_k}}{1+i} = \sum_j t_j L_{t_j} \frac{v^{t_j}}{1+i}}_{\nu \text{ form}} \quad\Longleftrightarrow\quad \underbrace{\sum_k t_k A_{t_k} v^{t_k} = \sum_j t_j L_{t_j} v^{t_j}}_{\tau \text{ form}}.$$

At the end of the day, it's a minor simplification, so use whatever you prefer.

Solving

We need to solve

$$R_9 v^9 + R_{20} v^{20} = 1{,}251{,}245.1456, \qquad 9R_9 v^9 + 20R_{20} v^{20} = 22{,}522{,}412.6208.$$

Multiplying the first by 9 and subtracting,

$$9R_9 v^9 + 9R_{20} v^{20} = 11{,}261{,}206.3104,$$

$$R_{20} v^{20} = \frac{22{,}522{,}412.6208 - 11{,}261{,}206.3104}{11} = 1{,}023{,}746.0282,$$

and hence $R_9 v^9 = 1{,}251{,}245.1456 - R_{20} v^{20} = 227{,}499.1174$.

The proportions

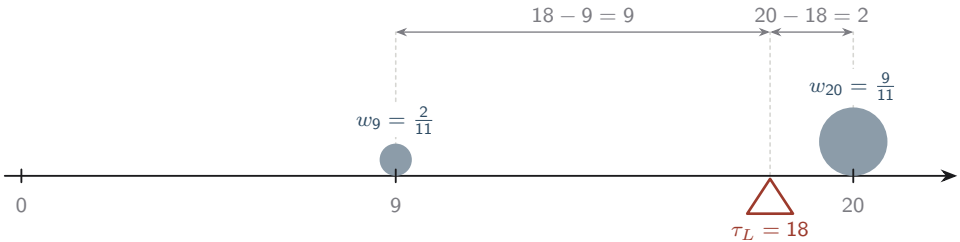
The organisation should therefore invest

$$\frac{R_9 \mathrm{v}^9}{R_9 \mathrm{v}^9 + R_{20} \mathrm{v}^{20}} \times 100 \;=\; \frac{227{,}499.1174}{1{,}251{,}245.1456} \times 100 \;=\; 18.1818\%$$

of the cash into the nine-year zero-coupon bond, and 81.8182% into the twenty-year one.

> KEY IDEA
>
> Those proportions are exactly $\frac{2}{11}$ and $\frac{9}{11}$. This is of course no accident and ties back to the warm-up, $9w_9 + 20w_{20} = 18$ with $w_9 + w_{20} = 1$, so $w_9 = \frac{20-18}{20-9} = \frac{2}{11}$. We could have used this as a bit of a shortcut.

The proportions as a lever



The weights are inversely proportional to the distances from the balance point, $w_9 : w_{20} = 2 : 9$, so the heavier bond is the one sitting closer to the liability date.

Lecture 30 on one slide

MATCH, MATCH, CHECK

Two bonds of terms $a < b$:

$$w_a + w_b = 1, \qquad w_a a + w_b b = \tau_L,$$

so

$$w_a = \frac{b - \tau_L}{b - a}.$$

Match the value, match the mean term, then *check* the convexity. The first two are linear equations in the holdings; the third is satisfied whenever the asset dates engulf the liability dates, which is what convexity measures. Immunisation remains local: it protects against small movements only.

Exercises

EXERCISE 30.4

A fund owes €100,000.00 in ten years. At $i = 6\%$ it holds five-year and fifteen-year zero-coupon bonds. Find the proportion of the fund in each, using the mean-term argument.

EXERCISE 30.5

For the same fund, find the redemption value of each bond, and verify condition 1.

EXERCISE 30.6

Could that fund be immunised using five-year and eight-year zero-coupon bonds instead? Answer without calculating.

Solutions are in the backup slides.

END OF LECTURE 30

We saw how the conditions can be met. Next: what is the actual benefit of them when the rate actually moves.

ilia.pirashvili@universityofgalway.ac.ie

Solution 1

The liability is a single payment at time 10, so $\tau_L = 10$. With bonds of terms $a = 5$ and $b = 15$, the warm-up gives

$$5w_5 + 15w_{15} \;=\; 10, \qquad w_5 + w_{15} \;=\; 1,$$

so $w_5 = \dfrac{15-10}{15-5} = \dfrac{1}{2}$ and $w_{15} = \dfrac{1}{2}$.

Half the fund in each, as symmetry demands, the two terms being equally far from the liability date. Note that the answer does not depend on i at all: the interest rate enters only when converting these proportions into redemption values.

Solution 2

The present value of the liability at 6% is

$$V_L \;=\; 100{,}000\,(1.06)^{-10} \;=\; €\,55{,}839.48,$$

so by Solution 1 each bond must have present value €27,919.74. Converting to redemption values,

$$R_5 \;=\; 27{,}919.74\,(1.06)^5 \;=\; €\,37{,}363.11, \qquad R_{15} \;=\; 27{,}919.74\,(1.06)^{15} \;=\; €\,66{,}913.44.$$

Condition 1 holds by construction: $R_5 v^5 + R_{15} v^{15} = 27{,}919.74 + 27{,}919.74 = €\,55{,}839.48 = V_L$.

Solution 3

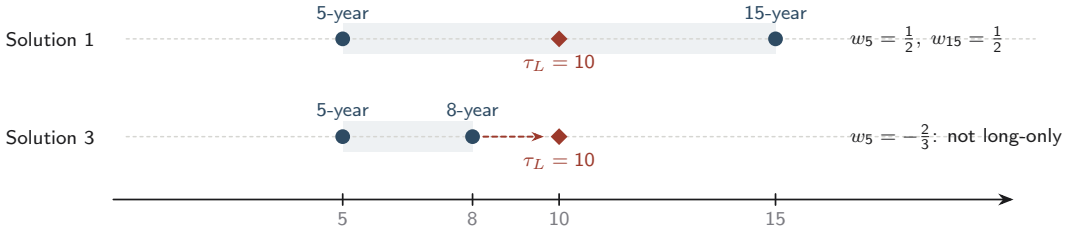
No.

By the warm-up, the mean term of a two-bond fund is the weighted average $w_a a + w_b b$, and a weighted average of 5 and 8 lies between 5 and 8. The liability has $\tau_L = 10$, which is outside that range, so no choice of non-negative weights can satisfy condition 2.

Formally, $w_5 = \frac{8-10}{8-5} = -\frac{2}{3}$: the algebra returns a negative weight, meaning the fund would have to *sell* the five-year bond short to stretch its mean term out to ten years.

> KEY IDEA
>
> Hence the straddling requirement: the assets must bracket the liabilities in time, or the second condition cannot be met with a long-only portfolio.

Solutions 1 and 3, drawn



The mean term of a long-only mix lies in the shaded range between the two terms. Ten is inside $[5, 15]$ but outside $[5, 8]$.