LECTURE 29

Convexity and immunisation

The second derivative, and what it protects

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A first-order estimate, and its error

	predicted	actual
5-year bond	-4.38%	-4.26%
25-year bond	-13.96%	-12.66%

$-\varepsilon\nu$ at $\varepsilon = 0.01$, from Lecture 28

Duration got the short bond nearly right, off by 2.81%, and the long one wrong by 10.27%. And always in the same direction, always too pessimistic. This stems from not taking curvature into account.

Today

1. Convexity
2. Approximate proportional change
3. Immunisation

Warm-up

EXERCISE 29.1

A level perpetuity is paid annually in arrears, so $\mathrm{NPV}(i) = 1/i$. *Find* $\mathrm{NPV}'(i)$ *and* $\mathrm{NPV}''(i)$, *and hence* $\mathrm{NPV}''(i)/\mathrm{NPV}(i)$.

Two derivatives of i^{-1}. The third quantity is today's new definition, computed before we give it a name.

Warm-up

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Differentiating $\mathrm{NPV}(i) = i^{-1}$ twice, we get

$$\mathrm{NPV}'(i) \;=\; -i^{-2}, \qquad \mathrm{NPV}''(i) \;=\; 2i^{-3}, \qquad \frac{\mathrm{NPV}''(i)}{\mathrm{NPV}(i)} \;=\; \frac{2/i^3}{1/i} \;=\; \frac{2}{i^2}.$$

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> ASIDE
>
> Note the sign: NPV′ is negative but NPV″ is *positive*. The value falls as i rises, but ever more slowly — the curve bends upwards. At $i = 10\%$ the ratio is 200.

That last quantity is the convexity, and 200 is the largest of the three numbers computed in Section 1.

SECTION 1

Convexity

Convexity

DEFINITION 29.2 · Convexity (in finance)

Convexity is a measure of the change in duration when the interest rate changes:

$$\mathrm{c}(i) \;=\; \frac{1}{\mathrm{NPV}(i)}\frac{\mathrm{d}^2}{\mathrm{d}i^2}\,\mathrm{NPV}(i) \;=\; \boxed{\frac{\mathrm{NPV}''(i)}{\mathrm{NPV}(i)}} \;=\; \frac{\sum_{k=1}^{n} t_k(t_k+1)c_k \mathrm{v}^{\,t_k+2}}{\sum_{k=1}^{n} c_k \mathrm{v}^{\,t_k}}.$$

The third expression follows by differentiating $\mathrm{NPV}(i) = \sum_k c_k(1+i)^{-t_k}$ twice:

$$\mathrm{NPV}''(i) \;=\; \sum_k c_k(-t_k)(-t_k-1)(1+i)^{-t_k-2} \;=\; \sum_k t_k(t_k+1)c_k \mathrm{v}^{\,t_k+2}.$$

Note there is no minus sign here: NPV is convex, so $\mathrm{c}(i) > 0$ when all $c_k > 0$.

Example: three assets

EXAMPLE 29.3

Using 10% p.a. interest, calculate the convexity of the following assets.

A. An 11-year zero-coupon bond of face value €100.00.

B. A lump sum of €9,663.00 in five years' time and a lump sum of €26,910.00 in twenty years' time.

C. A level perpetuity paid annually in arrears.

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A. The price of 100 nominal is $P_A(i) = 100v^{11}$, so

$$c_A(i) = \frac{100(11)(12)v^{13}}{100v^{11}} = 132v^2, \qquad c_A(0.1) = 109.1.$$

Example: three assets

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Using 10% p.a. interest, calculate the convexity of the following assets.

A. An 11-year zero-coupon bond of face value €100.00. **109.1**

B. A lump sum of €9,663.00 in five years' time and a lump sum of €26,910.00 in twenty years' time.

C. A level perpetuity paid annually in arrears.

B. Here $P_B(i) = 9663v^5 + 26910v^{20}$, so

$$c_B(i) = \frac{9663(5)(6)v^7 + 26910(20)(21)v^{22}}{9663v^5 + 26910v^{20}}, \qquad c_B(0.1) = 153.7.$$

Example: three assets

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Using 10% p.a. interest, calculate the convexity of the following assets.

A. An 11-year zero-coupon bond of face value €100.00. **109.1**

B. A lump sum of €9,663.00 in five years' time and a lump sum of €26,910.00 in twenty years' time. **153.7**

C. A level perpetuity paid annually in arrears.

C. Here $P_C(i) = a_{\overline{\infty}|} = 1/i$, and by the warm-up

$$\mathsf{c}_C(i) \;=\; \frac{2}{i^2}, \qquad \mathsf{c}_C(0.1) \;=\; 200.$$

Example: three assets

EXAMPLE 29.3

Using 10% p.a. interest, calculate the convexity of the following assets.

A. An 11-year zero-coupon bond of face value €100.00. **109.1**

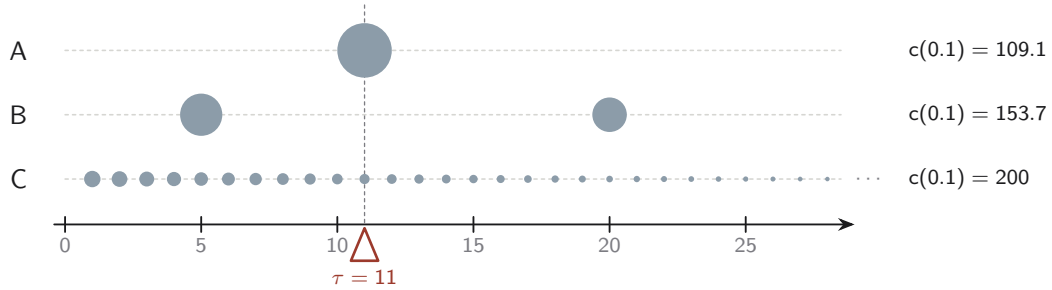
B. A lump sum of €9,663.00 in five years' time and a lump sum of €26,910.00 in twenty years' time. **153.7**

C. A level perpetuity paid annually in arrears. **200**

KEY IDEA

The ordering $109.1 < 153.7 < 200$ is controlled by how *spread out* the cashflows are. One date, then two dates fifteen years apart, then a stream running for ever.

Same duration, different spread



Disc radius represents the share of the present value paid at that date. The share is measured within the cashflow, not the combined portfolio. That's why the disc at 11 is largest, representing 100%, even though it is only €100.00.

At 10% all three have the same discounted mean term: $\tau = 11$ for A; $\tau = \frac{5(6000)+20(4000)}{10000} = 11$ for B, whose present values are 6000 and 4000; and $\tau = \mathrm{v}^{-1}\frac{\mathrm{NPV}'(i)}{\mathrm{NPV}(i)} = (1+i)\frac{1}{i} = \frac{1+i}{i} = 11$ for C. They have the same balance point, but a wider spread and thus larger convexity.

SECTION 2

Approximate proportional change

Taylor to second order

Recall the Taylor expansion for a smooth function f:

$$f(x+h) \;=\; f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

Applying this to $\mathrm{NPV}(i)$, for small $\varepsilon \in \mathbb{R}$, we get

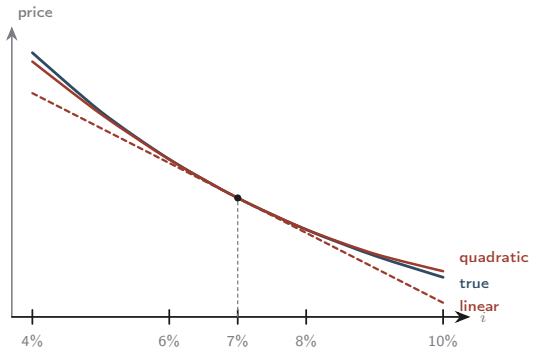
$$\mathrm{NPV}(i+\varepsilon) \;\approx\; \mathrm{NPV}(i) + \varepsilon\,\mathrm{NPV}'(i) + \frac{\varepsilon^2}{2!}\,\mathrm{NPV}''(i).$$

So, for $\mathrm{NPV}(i) \neq 0$, subtracting and dividing through by $\mathrm{NPV}(i)$,

$$\boxed{\frac{\mathrm{NPV}(i+\varepsilon) - \mathrm{NPV}(i)}{\mathrm{NPV}(i)} \;\approx\; -\varepsilon\,\nu(i) + \frac{\varepsilon^2}{2}\,\mathrm{c}(i)}$$

since $\mathrm{NPV}'(i)/\,\mathrm{NPV}(i) = -\nu(i)$ and $\mathrm{NPV}''(i)/\,\mathrm{NPV}(i) = \mathrm{c}(i)$.

What the second term gives us



The 25-year bond of Lecture 28, expanded around $i = 7\%$.

NPV is convex, so the tangent lies *below* the curve on both sides, and the first-order estimate always understates the value.

Adding $\frac{\varepsilon^2}{2}\,\mathrm{c}(i)$, which is positive whatever the sign of ε, lifts the estimate back towards the curve, improving it rather drastically.

The long bond, revisited

For the 25-year bond with $\varepsilon = 0.01$, $\nu(0.07) = 13.96$ and $\mathrm{c}(0.07) = 280.52$:

	estimate	actual
First order, $-\varepsilon\nu$	-13.96%	-12.66%
Second order, $-\varepsilon\nu + \frac{\varepsilon^2}{2}\,\mathrm{c}$	-12.56%	-12.66%

KEY IDEA

The first-order was off by 10.27%; the second-order error is 0.79%! It is arguably better for the five-year bond, where the second-order apporximation is agree with the actual -4.26% up to two decimal points. However, it is also arguably less needed, here. In terms of absolute errors, the 25-year bond improved and error of 1.3% to 0.1%. But in the 5-year bod, it only improved by about 0.12%.

SECTION 3

Immunisation

Set-up

DEFINITION 29.4 · Immunisation

Immunisation is a selection of an asset portfolio which protects the surplus against small changes in the interest rate.

Consider a fund with asset cashflows $\{A_{t_k}\}$ and liabilities $\{L_{t_k}\}$, and write V_A, V_L for their present values, ν_A, ν_L for their volatilities and $\mathsf{c}_A, \mathsf{c}_L$ for their convexities. At rate i_0 the fund is **immunised** against small movements ε provided

$$V_A(i_0) \;=\; V_L(i_0) \qquad \text{and} \qquad V_A(i_0+\varepsilon) \;>\; V_L(i_0+\varepsilon).$$

The surplus function

Assume $V_A(i_0) = V_L(i_0)$ and consider the **surplus**

$$S(i) \;=\; V_A(i) - V_L(i).$$

We require $S(i_0 + \varepsilon) > 0$. The Taylor expansion for small $\varepsilon \in \mathbb{R}$ gives us

$$S(i_0 + \varepsilon) \;\approx\; S(i_0) + \varepsilon S'(i_0) + \frac{\varepsilon^2}{2!} S''(i_0).$$

REDINGTON'S THREE CONDITIONS

Take the three terms in turn. The first is zero by assumption. The second changes sign with ε, so it must be killed (see below). The third has a fixed sign, so it must positive. Those are Redington's three conditions.

The first two terms

$$S(i_0+\varepsilon) \;\approx\; \underbrace{S(i_0)}_{=0} + \varepsilon S'(i_0) + \frac{\varepsilon^2}{2!} S''(i_0).$$

The first term vanishes since $V_A(i_0) = V_L(i_0)$. The second, $\varepsilon S'(i_0)$, must be zero: for sufficiently small ε, it dominates the quadratic term, and for positive or negative ε, $\varepsilon S'(i_0)$ would be negative in one direction, making $S(i_0+\varepsilon) < 0$. Hence we need

$$S'(i_0) \;=\; 0, \qquad \text{that is} \qquad V_A'(i_0) \;=\; V_L'(i_0).$$

Dividing by $-V_A(i_0) = -V_L(i_0)$, this is equivalent to

$$\nu_A(i_0) \;=\; \nu_L(i_0), \qquad \text{or equivalently} \qquad \tau_A(i_0) \;=\; \tau_L(i_0),$$

since $\tau = (1+i)\nu$ and the common factor $(1+i_0)$ cancels.

The third term

$$S(i_0 + \varepsilon) \;\approx\; \frac{\varepsilon^2}{2!} S''(i_0).$$

Here $\varepsilon^2/2!$ is always *positive*, whatever the sign of ε. So if $S''(i_0) > 0$ the whole term is positive, which is equivalent to requiring

$$V_A''(i_0) \;>\; V_L''(i_0), \qquad \text{that is} \qquad \mathrm{c}_A(i_0) \;>\; \mathrm{c}_L(i_0),$$

dividing again by the common positive value $V_A(i_0) = V_L(i_0)$.

> KEY IDEA
>
> Look at the picture on slide 15: with equal durations the two curves share a tangent at i_0, so the one that bends up more is higher on *both* sides.

Overfunding is not immunisation

WHY $V_A(i_0) > V_L(i_0)$ IS NOT IMMUNISATION

If $V_A(i_0) > V_L(i_0)$, then $S(i_0) > 0$. Even for equal volatilities, say $\nu_A = \nu_L = \nu$,

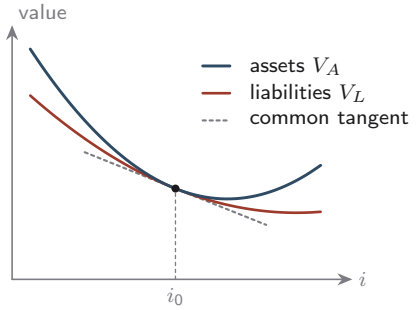
$$S'(i_0) \;=\; -\nu_A V_A(i_0) + \nu_L V_L(i_0) \;=\; -\nu\big[V_A(i_0) - V_L(i_0)\big] \;=\; -\nu S(i_0) \neq 0.$$

Thus

$$S(i_0 + \varepsilon) \approx S(i_0) - \nu S(i_0)\varepsilon \;=\; S(i_0)(1 - \nu\varepsilon),$$

so the surplus still has first-order interest-rate sensitivity. The excess assets are a cushion, but the cushion itself is exposed: the overfunded itself is not immunised. Equality $V_A(i_0) = V_L(i_0)$ is needed so that the first-order term can be killed by $\nu_A(i_0) = \nu_L(i_0)$.

Two curves, one tangent



Conditions 1 and 2 make the two curves *touch* at i_0, with the same slope there.

Condition 3 says the asset curve bends up more. It then lies above the liability curve on *both* sides of i_0: the surplus is positive whichever way rates move.

Redington's conditions

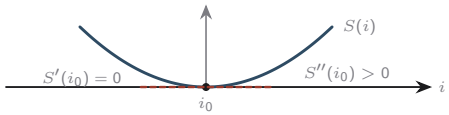
REDINGTON'S CONDITIONS FOR IMMUNISATION

Let i_0 be the initial rate of interest.

1. $V_A(i_0) = V_L(i_0)$: the value of the assets equals the value of the liabilities.
2. $\nu_A(i_0) = \nu_L(i_0)$: the volatilities are equal (equivalently, the discounted mean terms are equal).
3. $\mathsf{c}_A(i_0) > \mathsf{c}_L(i_0)$: the convexity of the assets exceeds that of the liabilities.

If these three hold, the fund is protected against small movements in interest rates.

What the three conditions look like



Condition 1 puts the curve *on* the axis at i_0; condition 2 makes it *flat* there; condition 3 makes it curve *upwards*, forcing i_0 to be a local minimum with value zero.

> CAUTION
>
> "Local" is the operative word: the surplus may **not** be immunised against *large* rate changes, since the neglected Taylor terms grow with $|\varepsilon|$.

Lecture 29 on one slide

THE SECOND DERIVATIVE, TWICE USED

$$\mathsf{c}(i) \;=\; \frac{\mathsf{NPV}''(i)}{\mathsf{NPV}(i)} \;=\; \frac{\sum_k t_k(t_k+1)c_k \mathsf{v}^{t_k+2}}{\sum_k c_k \mathsf{v}^{t_k}}, \qquad \frac{\Delta\,\mathsf{NPV}}{\mathsf{NPV}} \approx -\varepsilon\nu + \frac{\varepsilon^2}{2}\,\mathsf{c}\,.$$

Redington: $V_A = V_L$, $\nu_A = \nu_L$, $\mathsf{c}_A > \mathsf{c}_L$ at i_0.

Convexity is positive, so the quadratic term lifts a first-order estimate whichever way rates move. That same one-sidedness is what makes the third condition protect the surplus in both directions.

Exercises

EXERCISE 29.5

Find the convexity at $i = 8\%$ of a four-year zero-coupon bond, and of an eight-year one. What is the pattern?

EXERCISE 29.6

A fund satisfies conditions 1 and 2 but has $c_A < c_L$. What happens to the surplus if rates move either way?

EXERCISE 29.7

Explain why a first-order estimate of a price change is always too pessimistic for a rise in i and too pessimistic for a fall.

Solutions are in the backup slides.

END OF LECTURE 29

Three conditions, and no examples. Next: choosing a portfolio that meets them.

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Solution 1

A zero-coupon bond of term n has one cashflow, so

$$\mathsf{c}(i) \;=\; \frac{n(n+1)c\mathsf{v}^{\,n+2}}{c\mathsf{v}^{\,n}} \;=\; n(n+1)\mathsf{v}^2,$$

independent of the payment. At 8%, $\mathsf{v}^2 = 0.857339$, so

$$\mathsf{c}_4 \;=\; 20(0.857339) \;=\; 17.15, \qquad \mathsf{c}_8 \;=\; 72(0.857339) \;=\; 61.73.$$

The pattern is that convexity grows roughly with the *square* of the term, while duration grows only linearly. So doubling the term roughly doubles the sensitivity but roughly quadruples the curvature, which is why the second-order correction mattered so much more for the twenty-five-year bond.

Solution 2

The surplus is *worse* whichever way rates move.

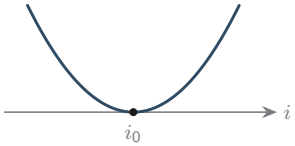
Conditions 1 and 2 give $S(i_0) = 0$ and $S'(i_0) = 0$, so

$$S(i_0 + \varepsilon) \approx \frac{\varepsilon^2}{2} S''(i_0),$$

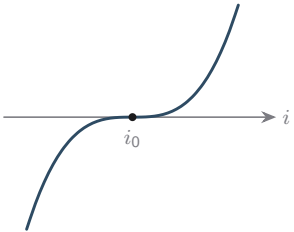
and $c_A < c_L$ makes $S''(i_0) < 0$. Since $\varepsilon^2 > 0$ for every $\varepsilon \neq 0$, the surplus becomes negative in both directions: i_0 is a local *maximum* of the surplus, at value zero.

So matching values and durations alone is not enough. It removes the first-order exposure but can leave the fund exposed at second order in both directions at once. This is the worst of the three possible signs for S''.

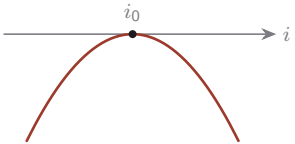
Solution 2, drawn



$S''(i_0) > 0$
surplus on both sides



$S''(i_0) = 0$
higher terms decide



$S''(i_0) < 0$
deficit on both sides

With $S(i_0) = S'(i_0) = 0$, the sign of $S''(i_0)$ is all that is left. Redington asks for the left-hand picture; the fund in the exercise is the right-hand one.

Solution 3

Because NPV is convex, its graph lies *above* every tangent. The first-order estimate is the value read off the tangent at i, so it is below the true value at $i + \varepsilon$ whatever the sign of ε.

Algebraically, the error is the neglected term $\frac{\varepsilon^2}{2}\,\mathrm{c}(i)$, and both factors are positive: $\varepsilon^2 > 0$ whichever way rates move, and $\mathrm{c}(i) > 0$ whenever all the cashflows are positive.

So for a rise the estimate over-states the fall, and for a fall it under-states the gain. In both cases the investor does better than the linear estimate suggests, which is why convexity is regarded as a desirable property. This is why Redington asks for more of it in the assets than in the liabilities.