LECTURE 28

Duration

The average date at which a portfolio is paid

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Where Lecture 27 got to

$$\nu(i) = -\frac{\mathrm{NPV}'(i)}{\mathrm{NPV}(i)} = \frac{\sum_k t_k c_k \mathsf{v}^{t_k+1}}{\sum_k c_k \mathsf{v}^{t_k}}$$

Effective duration, or volatility

A five-year zero had $\nu = 4.72$ and a five-year coupon bond $\nu = 4.21$. Both numbers are suspiciously close to a number of *years*. Today, we make this precise.

Today

1. The instantaneous forward rate
2. Discounted mean term
3. Basic durations

Warm-up

EXERCISE 28.1

At $i = 6\%$, find the effective duration $\nu(i)$ of a zero-coupon bond of term n, redeemable at R. What does it depend on?

Warm-up

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At $i = 6\%$, find the effective duration $\nu(i)$ of a zero-coupon bond of term n, redeemable at R. What does it depend on?

A single cashflow R at time n gives $\mathrm{NPV}(i) = R\mathrm{v}^n$, so

$$\nu(i) \;=\; \frac{nR\mathrm{v}^{n+1}}{R\mathrm{v}^n} \;=\; n\mathrm{v} \qquad (\,=\; 5(0.943396) \;=\; 4.7170 \text{ at } n = 5,\; i \;=\; 6\%)\,.$$

We are differentiating with respect to i, not v. As $\mathrm{v} = 1/1+i$, the power of v increases by one instead of decreasing.

> R CANCELS
>
> Duration does not depend on the size of the payment, only on *when* it falls. And the answer is the term, discounted once.

So ν is nearly the number of years, and $(1+i)\nu = n$ exactly. That stray factor of v is the subject of Section 2.

SECTION 1

The instantaneous forward rate

Forces, from rates

Lecture 25 gave the spot rate y_t and forward rate $f_{t,r}$. Each has a continuous counterpart, obtained as in Lecture 2: Take the logarithm of the growth factor.

DEFINITION 28.2 · Spot force, forward force

$$Y_t \;=\; \ln(1+y_t), \qquad F_{t,r} \;=\; \ln(1+f_{t,r}).$$

Since $(1+y_t)^t = 1/P_t$ and $(1+f_{t,r})^r = P_t/P_{t+r}$, taking logarithms gives $e^{tY_t} = 1/P_t$ and

$$Y_t \;=\; -\frac{1}{t}\ln P_t, \qquad F_{t,r} \;=\; \frac{1}{r}\ln\left(\frac{P_t}{P_{t+r}}\right).$$

Note that t nor r can be any real number here.

The instantaneous forward rate

DEFINITION 28.3 · Instantaneous forward rate

The **instantaneous forward rate** is defined as $F_t = \lim_{r \to 0} F_{t,r}$.

It is the forward force applying in the instant $t \to t + \Delta t$ as $\Delta \to 0$. From $F_{t,r} = \frac{1}{r} \ln(P_t / P_{t+r})$, we get

$$F_t = \lim_{r \to 0} \frac{1}{r} \ln \left(\frac{P_t}{P_{t+r}} \right) = - \lim_{r \to 0} \frac{\ln P_{t+r} - \ln P_t}{r} = - \frac{\mathrm{d}}{\mathrm{d}t} \ln P_t = - \frac{1}{P_t} \frac{\mathrm{d}}{\mathrm{d}t} P_t = - \frac{P_t'}{P_t}.$$

The third equation is just the definition of differentiation. We use the more informal $t + r$ instead of $t + \Delta t$.

F_t IS A LOGARITHMIC DERIVATIVE

F_t is a rate of change divided by a level, just like $\delta(t) = V_t'/V_t$ from Lecture 2 and $\nu(i)$ from Lecture 27. We will shortly see another.

Recovering the prices

Integrating $F_t = -\frac{\mathrm{d}}{\mathrm{d}t} \ln P_t$ from 0 to t, and using $P_0 = 1$ gives us the following:

$$\int_0^t F_s \, \mathrm{d}s \;=\; -\Big[\ln P_s\Big]_0^t \;=\; -\ln P_t, \qquad \text{so} \qquad \boxed{P_t \;=\; \mathrm{e}^{-\int_0^t F_s \, \mathrm{d}s}.}$$

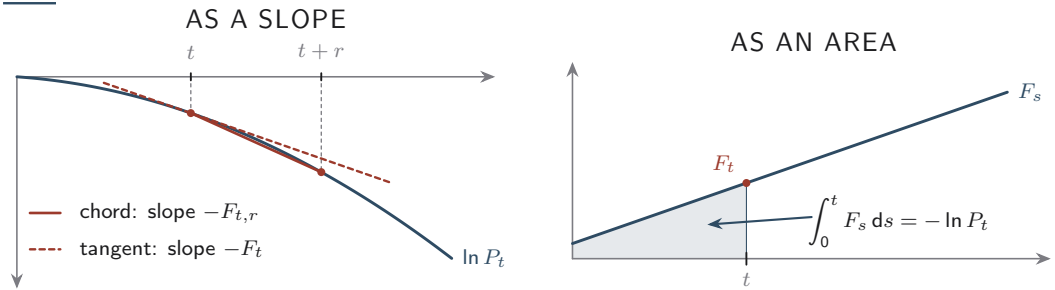
The reformulation has nothing to do with the integral, but only with e and ln, and the fact that $x = e^{\ln(x)}$.

Note what it says: A bond maturing immediately is worth its face value.

F_s RECOVERS THE TERM STRUCTURE

Compare Lecture 2, where a varying force of interest gave $\mathrm{v}(t) = \mathrm{e}^{-\int_0^t \delta(s)\,\mathrm{d}s}$. The same formula, with the instantaneous forward rate in place of the force of interest.

F_t as a slope, and as an area



As $r \to 0$, $F_{t,r} \to F_t$ and the chord turns to the tangnt. Read the other way, the area under F_s up to t is $-\ln P_t$.

Both panels draw the same $F_s = as + b$.

NOTE: $F_s = -\ln P_t$ DOES **NOT** MEAN F_s IS A LOGARITHMIC CURVE

F_s can be any curve in general, and is linear in the illustration. In that case, $P_t = e^{-(at^2/2+bt)}$, so the log-price plotted on the left, $\ln P_t$, is quadratic.

Example

EXAMPLE 28.4

Suppose the price at time 0 of a zero-coupon bond of term $2 < t < 4$ is given by $P_t = 100 - 2t^2$. Calculate the instantaneous forward rate F_3.

Since

$$F_t = -\frac{1}{P_t}\frac{\mathrm{d}}{\mathrm{d}t}P_t = -\frac{-4t}{100-2t^2} = \frac{4t}{100-2t^2},$$

we have

$$F_3 = \frac{12}{100-18} = \frac{12}{82} = 0.146,$$

so the instantaneous forward rate at time 3 is $F_3 = 14.6\%$.

SECTION 2

Discounted mean term

Discounted mean term

DEFINITION 28.5 · Macaulay duration

The **(Macaulay) duration**, or **discounted mean term** (DMT), is the weighted average of the future times at which the cashflows occur:

$$\tau(i) \;=\; \frac{\sum_{k=1}^{n} t_k c_k \mathrm{v}^{t_k}}{\mathrm{NPV}(i)} \;=\; \frac{\sum_{k=1}^{n} t_k \boxed{c_k \mathrm{v}^{t_k}}}{\sum_{k=1}^{n} \boxed{c_k \mathrm{v}^{t_k}}} \qquad \frac{\sum_k x_k w_k}{\sum_k w_k}$$

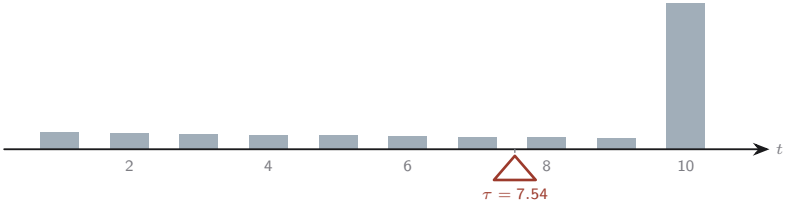
The weight attached to the date t_k is the present value $c_k \mathrm{v}^{t_k}$ at rate i of the cashflow due at that time. Note that

$$\boxed{\tau(i) \;=\; (1+i)\nu(i)}$$

which is immediate on comparing the two formulas: every term of ν carries one extra factor of v.

τ is a centre of mass

Place a weight $c_k \mathrm{v}^{t_k}$ at each payment date. Then τ is the point at which the whole arrangement balances.



Weights for a ten-year 8% bond at $i = 5\%$; the redemption at $t = 10$ dominates and pulls τ far to the right.

A second derivation, via the force of interest

Another way of deriving the Macaulay duration is in terms of the force of interest δ. Since $i = \mathrm{e}^{\delta} - 1$, we have $\frac{\mathrm{d}i}{\mathrm{d}\delta} = \mathrm{e}^{\delta} = 1 + i$, and so

$$
\begin{aligned}
-\frac{1}{\mathrm{NPV}(i)} \frac{\mathrm{d}}{\mathrm{d}\delta} \mathrm{NPV}(i) \;&=\; -\frac{1}{\mathrm{NPV}(i)} \frac{\mathrm{d}}{\mathrm{d}i} \mathrm{NPV}(i) \frac{\mathrm{d}i}{\mathrm{d}\delta} \;=\; \nu(i) \frac{\mathrm{d}i}{\mathrm{d}\delta} \\
&= \nu(i) \frac{d}{d\delta} \left(\mathrm{e}^{\delta} - 1 \right) \;=\; \mathrm{e}^{\delta} \nu(i) \;=\; (1+i)\nu(i) \;=\; \tau(i).
\end{aligned}
$$

THUS, WE GET: $\tau(i) = -\frac{\mathrm{d}}{\mathrm{d}\delta} \ln \mathrm{NPV}(i)$

The discounted mean term is the volatility measured against the *force* of interest rather than the effective rate. This is the fourth logarithmic derivative.

We use the general rule $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$ and the chain rule for $f(\delta) = \mathrm{NPV}(i(\delta))$.

SECTION 3

Basic durations

A zero-coupon bond

EXAMPLE 28.6

Find the duration of an n-year zero-coupon bond redeemable at R.

The price is $\text{NPV}(i) = R\mathsf{v}^n$, a single cashflow, so

$$\tau(i) \;=\; \frac{nR\mathsf{v}^n}{R\mathsf{v}^n} \;=\; n.$$

> INTUITIVELY OBVIOUS
>
> The average term of a series of cashflows with only one payment must be the time of that payment. So the warm-up's stray v is gone, and τ is exactly a measure of time.

This is the extreme case: all the weight sits at one date, so the balance point is that date.

A coupon bond: the numerator

EXAMPLE 28.7

Find the duration of an n-year bond paying coupons of D annually in arrears and redeemable at R.

As the price of the bond is $\text{NPV}(i) = Da_{\overline{n}|} + R\mathrm{v}^n$, the numerator of τ becomes

$$\sum_k t_k c_k \mathrm{v}^{t_k} = \underbrace{D\sum_{t=1}^{n} t\mathrm{v}^t}_{\text{coupons}} + \underbrace{nR\mathrm{v}^n}_{\text{redemption}}.$$

The times $t = 1, 2, \ldots$ form an increasing annuity and thus $D\sum_t t\mathrm{v}^t = D\,(Ia)_{\overline{n}|}$. The redemption contributes $nR\mathrm{v}^n$. Its date n weighted by the present value $R\mathrm{v}^n$.

$(Ia)_{\overline{n}|}$ is the time-weighted companion of $a_{\overline{n}|}$: each v^t is multiplied by its date t.

A coupon bond: the duration

Collecting the two pieces,

$$\sum_k t_k c_k \mathsf{v}^{t_k} \;=\; D\,(Ia)_{\overline{n}|} + nR\mathsf{v}^n, \qquad \mathsf{NPV}(i) \;=\; D\,a_{\overline{n}|} + R\mathsf{v}^n,$$

we get

$$\boxed{\tau(i) \;=\; \frac{D\,(Ia)_{\overline{n}|} + nR\mathsf{v}^n}{D\,a_{\overline{n}|} + R\mathsf{v}^n}}$$

In pure interest terms we can "simplify" it by recalling

$$a_{\overline{n}|} \;=\; \frac{1-v^n}{i}, \qquad \ddot{a}_{\overline{n}|} \;=\; \frac{1-v^n}{d}, \qquad (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\mathsf{v}^n}{i}$$

$$\tau(i) \;=\; \frac{D\left(\frac{\frac{1-v^n}{d} - nv^n}{i}\right) + nRv^n}{D\frac{1-v^n}{i} + Rv^n} \;=\; \boxed{\frac{D\left(\frac{1-v^n}{d} - nv^n\right) + nR\,i\,v^n}{D(1-v^n) + R\,i\,v^n}}$$

Example 1

EXAMPLE 28.8

Find the discounted mean term of a ten-year fixed-interest bond, paying annual coupons of 8% and redeemable at par, at interest rates 5%, 10% and 15%.

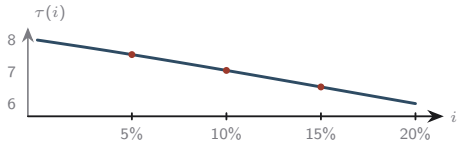
From the formula above with $D = 8$, $R = 100$, $n = 10$, we get

$$\tau(0.05) = \frac{928.90}{123.17} = 7.54, \qquad \tau(0.10) = \frac{617.83}{87.71} = 7.04, \qquad \tau(0.15) = \frac{423.17}{64.87} = 6.52.$$

A larger τ means more sensitive to changes in the interest rate.

The first of these is the balance point drawn two slides ago.

Why τ falls as i rises



A higher i discounts the *late* cashflows more heavily than the early ones, so weight shifts towards the front and the balance point moves left.

At $i = 0$ nothing is discounted and $\tau = \frac{8(55)+10(100)}{180} = 8.0$. As i grows the redemption payment loses weight and τ falls back towards the coupon dates.

Example 2

EXAMPLE 28.9

How would the price of a bond paying annual coupons of 3% and redeemable at par be affected if future rates of interest over all terms increased from 7% to 8%, if the term of the bond is (a) five years and (b) twenty-five years?

The price per 100 nominal is $P = 3a_{\overline{n}|} + 100\mathrm{v}^n$.

	at 7%	at 8%	fall
5-year bond	83.60	80.04	4.26%
25-year bond	53.39	46.63	12.66%

The change in interest rate has a greater effect on the longer bond.

Example 2: and the durations agree

Based on the 7% interest rate:

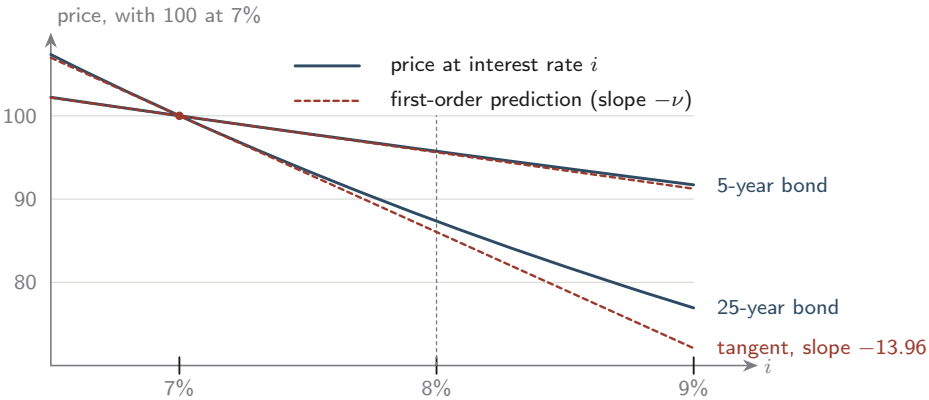
- 5-year bond: $\tau(0.07) = 4.69$ years, volatility $\nu(0.07) = 4.38$;
- 25-year bond: $\tau(0.07) = 14.94$ years, volatility $\nu(0.07) = 13.96$.

GOOD FOR THE SHORT BONDS, BUT TOO PESSIMISTIC FOR THE LONG ONES.

The volatilities predict the falls. A rise of $\varepsilon = 0.01$ should cost roughly $\varepsilon\nu$, that is 4.38% and 13.96%. In actuallity, the falls are 4.26% and 12.66%. So, the predictions overshot by 2.81% and 10.27% respectively.

That gap is what the *second* derivative corrects, and it is larger for the long bond because the long bond is more curved.

Duration is the tangent



Duration is the slope of the price curve at 7%. The curve bends away above its tangent, and the 25-year bond bends far more than the 5-year bond. This leads to the gap of the previous slide.

Lecture 28 on one slide

TWO MORE LOGARITHMIC DERIVATIVES

$$F_t \;=\; -\frac{\mathrm{d}}{\mathrm{d}t}\ln P_t, \qquad P_t \;=\; \mathrm{e}^{-\int_0^t F_s\,\mathrm{d}s};$$

$$\tau(i) \;=\; \frac{\sum_k t_k c_k \mathsf{v}^{t_k}}{\sum_k c_k \mathsf{v}^{t_k}} \;=\; (1+i)\nu(i) \;=\; -\frac{\mathrm{d}}{\mathrm{d}\delta}\ln \mathsf{NPV}(i).$$

The discounted mean term is the *centre of mass* of the discounted cashflows. It is n for a zero-coupon bond, and $\frac{D(Ia)_{\overline{n}|}+nR\mathsf{v}^n}{Da_{\overline{n}|}+R\mathsf{v}^n}$ for a coupon bond. It falls as i rises, and predicts price movements — to first order.

Exercises

EXERCISE 28.10

Find τ and ν at $i = 8\%$ for a four-year bond redeemable at par with 6% annual coupons, and check $\tau = (1 + i)\nu$.

EXERCISE 28.11

Show that the duration of a level perpetuity in arrears is $\frac{1+i}{i}$, and say what happens to it as i falls.

EXERCISE 28.12

Two bonds have the same term but different coupons. Which has the larger duration, and why?

Solutions are in the backup slides.

END OF LECTURE 28

Duration is the first derivative, and it was too pessimistic fo long bonds. Next, we take the second derivative into account as well.

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Solution 1

With $D = 6$, $R = 100$, $n = 4$ and $\mathsf{v} = (1.08)^{-1}$,

$$a_{\overline{4}|} = 3.312127, \qquad \mathsf{v}^4 = 0.735030, \qquad (Ia)_{\overline{4}|} = \frac{\ddot{a}_{\overline{4}|} - 4\mathsf{v}^4}{0.08} = 7.586278.$$

Hence

$$\tau(0.08) = \frac{6(7.586278) + 4(100)(0.735030)}{6(3.312127) + 100(0.735030)} = \frac{339.529}{93.376} = 3.6362,$$

and $\nu(0.08) = \tau/(1.08) = 3.3669$. Checking, $1.08 \times 3.3669 = 3.6362$.

Just under four years: nearly all the weight is the redemption payment, but the four coupons pull the balance point slightly forward.

Solution 2

A perpetuity of 1 in arrears has $\mathrm{NPV}(i) = a_{\overline{\infty}|} = 1/i$, so $\mathrm{NPV}'(i) = -1/i^2$ and

$$\nu(i) \;=\; -\frac{\mathrm{NPV}'(i)}{\mathrm{NPV}(i)} \;=\; \frac{1/i^2}{1/i} \;=\; \frac{1}{i}, \qquad \tau(i) \;=\; (1+i)\nu(i) \;=\; \frac{1+i}{i}.$$

As i falls, $\tau \to \infty$. The reason is the centre-of-mass reading: at a low rate the distant payments are barely discounted, so they retain weight, and the balance point of an infinite stream is pushed arbitrarily far out. A perpetuity is the most rate-sensitive instrument there is, and most so when rates are low.

Solution 3

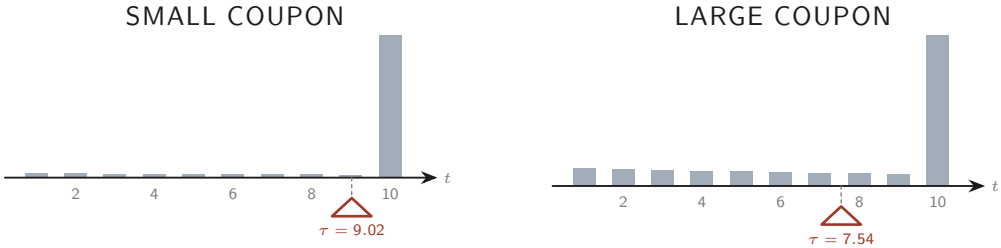
The one with the *smaller* coupon.

Duration is the balance point of the discounted cashflows. A large coupon puts more weight on the early dates $1, \ldots, n-1$ and so drags the balance point forward; a small coupon leaves almost all the weight on the redemption payment at time n, where the balance point can only be close to n itself.

The extreme case is Section 3's zero-coupon bond, with no coupon at all and $\tau = n$ exactly. This is the largest duration any bond of that term can have.

> KEY IDEA
>
> So a low-coupon bond is the more rate-sensitive, which is also why it was the one with the higher redemption yield on a rising curve in Lecture 26.

Solution 3, drawn



Ten-year bonds at 5%, drawn to the same scale: coupons of 2% on the left and 8% on the right. The small coupon leaves almost all the weight on the last date, so the balance point slides right towards $t = 10$.