LECTURE 27

Yield curves and interest rate risk

Understanding the curve and what influences it

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A curve, and what moves it

$$y_1,\ y_2,\ y_3,\ \ldots$$

a different rate at every term

Two lectures have built the term structure and summarised it. Neither asked what *shape* it has, why, or what happens to a portfolio when the whole thing shifts.

Today

1. The shape of the curve
2. Interest rate risk
3. Effective duration

Warm-up

EXERCISE 27.1

A three-year bond redeemable at par pays annual coupons of 5%. *Find its price per* €100.00 *nominal at a yield of* 6%, *and again at* 6.1%. *By what percentage does the price fall?*

Warm-up

EXERCISE 27.1

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With $v = (1+i)^{-1}$ and $P(i) = 5v + 5v^2 + 105v^3$,

$$P(0.06) = 97.3270, \qquad P(0.061) = 97.0651, \qquad \frac{97.3270 - 97.0651}{97.3270} = 0.2691\%.$$

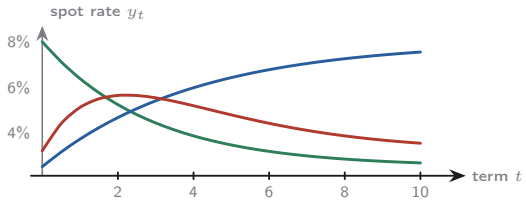
> ASIDE
>
> A tenth of a percentage point on the yield costs a quarter of a percentage point of the price. The ratio of the two, about 2.7, will come back in Section 3.

SECTION 1

The shape of the curve

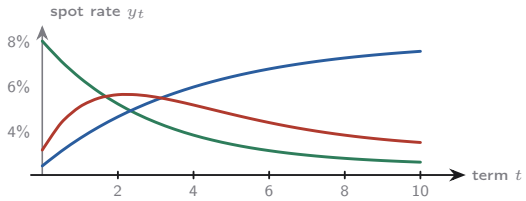
Three shapes

The curve of spot rates y_t, $t \geq 0$, is an example of a **yield curve**. Typical examples:



A yield curve may be **downward sloping** (green), **upward sloping** (blue), or **humped** (red).

What the shapes mean



GREEN long-term bonds more expensive than short-term bonds.

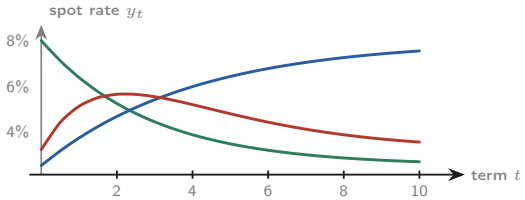
BLUE long-term bonds cheaper than short-term bonds.

RED short-term bonds of term > 1 year cheaper than both very short-term and long-term bonds.

Price and yield move in *opposite* directions. A low yield at some term means a high price at that term.

To avoid confusion, observe that *cheaper* means more profitable for the buyer. Higher yield means higher profit.

Supply and demand

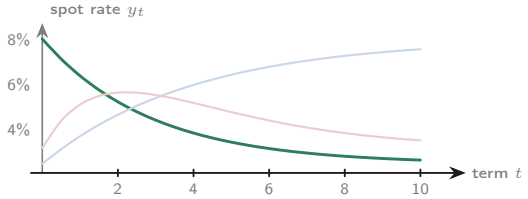


A dip in the curve marks a term investors want; a bulge marks one they do not.

KEY IDEA

If investors believe there will be a higher overall return from bonds of a certain duration, higher demand pushes up the price and hence pushes *down* the yield compared with other terms.

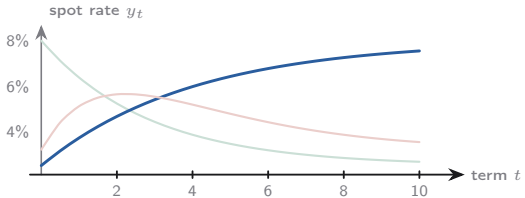
1. Expectations theory



The green curve suggests demand for long-term bonds is greater, and hence that a fall in rates is expected.

Expectations theory: the relative attraction of short- and long-term investments varies with expectations of future rate movements. If rates are expected to fall, short-term investments become less attractive and long-term more attractive, so short-term yields rise and long-term yields fall.

2. Liquidity preference



Liquidity preference: long-term bonds are more sensitive to interest-rate movements (and other risks), so risk-averse investors require higher yields to compensate. This is reflected in the blue curve.

> KEY IDEA
>
> "More sensitive" is made precise in Section 3: a longer bond has a larger effective duration.

3. Market segmentation

Market segmentation: bonds of different terms attract different investors, who choose assets similar in term to their liabilities.

For example, banks have short-term liabilities, because of potential withdrawals at short notice, while pension funds have long-term liabilities. So banks invest in short-term bonds and pension funds in long-term bonds, and demand differs across terms.

Moreover, government and company issuance strategies may differ from investors' requirements. The market-segmentation hypothesis suggests that the term structure emerges from these forces of supply and demand.

The term structure moves

Note that we have described the *initial* term structure, where everything is fixed at time 0. In practice the term structure varies rapidly over time, and the five-year spot rate tomorrow may be quite different from the five-year spot rate today.

More sophisticated treatments model the change over time. Then every variable we have used — P_t, y_t, $f_{t,r}$ — needs another argument s giving the starting point: $y_{s,t}$ is the t-year spot rate applying at time s, and $f_{s,t,r}$ the rate agreed at time s for an amount invested at $s+t$ for r years.

That is beyond this module. But we will still ask what the effect of this is o the price as part of this module.

SECTION 2

Interest rate risk

The set-up

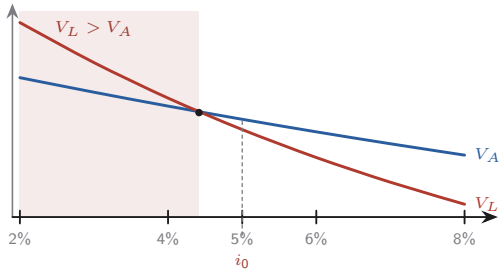
We now consider simple measures of vulnerability to interest rate movements. For simplicity, from now on assume the term structure is *flat*:

$$y_t \;=\; f_{t,r} \;=\; i \quad \text{for all } t, r.$$

Suppose an institution holds assets of value V_A to meet liabilities of value V_L. Both are discounted values of future cashflows, so both are sensitive to the rate of interest, and we assume the institution is healthy at time 0: $V_A > V_L$.

> KEY IDEA
>
> If rates *fall*, both V_A and V_L increase; if rates *rise*, both decrease. The danger is not the movement itself but the two values moving by different amounts.

Where the risk lies



V_L is the *steeper* curve. At $i_0 = 5\%$ the institution is healthy, but a fall below about 4.4% reverses that: the liabilities gain more than the assets.

Stating the risk

THE RISK

That following a *downward* movement in interest rates the value of the assets increases by *less* than the value of the liabilities; or that following an *upward* movement the value of the assets decreases by *more* than the value of the liabilities.

To examine the impact of interest rate movements on different cashflow sequences, we use changes in the yield to maturity to represent changes in the underlying term structure.

What we need, then, is a number measuring *how steep* each of those two curves is. Of course, it must not depend on teh size of the fund.

SECTION 3

Effective duration

Effective duration

Consider cashflows $c_1, \dots, c_n$ at times $t_1 \le \cdots \le t_n$. At effective rate (yield to maturity) i, the net present value is $\mathrm{NPV}(i) = \sum_k c_k \mathsf{v}^{t_k}$.

DEFINITION 27.2 · Effective duration

If $\mathrm{NPV}(i) \neq 0$, the **effective duration**, or **volatility**, is

$$\nu(i) \;=\; -\frac{1}{\mathrm{NPV}(i)}\frac{\mathrm{d}}{\mathrm{d}i}\mathrm{NPV}(i) \;=\; -\frac{\mathrm{NPV}'(i)}{\mathrm{NPV}(i)} \;=\; \frac{\sum_{k=1}^{n} t_k c_k \mathsf{v}^{t_k+1}}{\sum_{k=1}^{n} c_k \mathsf{v}^{t_k}}.$$

Note, we are dividing by $\mathrm{NPV}(i)$ as otherwise, bigger funds would give higher volatility: We are interested in how much a unit changes, not much total value changes.

Where the third expression comes from

Differentiating $\mathrm{NPV}(i) = \sum_k c_k(1+i)^{-t_k}$ term by term,

$$\mathrm{NPV}'(i) \;=\; \sum_{k=1}^{n} c_k(-t_k)(1+i)^{-t_k-1} \;=\; -\sum_{k=1}^{n} t_k c_k \mathrm{v}^{\,t_k+1},$$

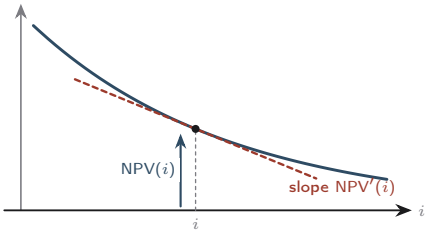
and dividing by $-\mathrm{NPV}(i)$ gives the formula on the previous slide.

KEY IDEA

Note that $\nu(i) = -\frac{\mathrm{d}}{\mathrm{d}i}\ln \mathrm{NPV}(i)$ is the second logarithmic derivative of the module, after $\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln V_t$ in Lecture 2. The reasoning is the same: We are dividing the slope by the level, and that removes the size of the fund and leaves a rate.

By *logarithmic* we mean tha general shape $\frac{d}{di}\ln f(i) = \frac{f'(i)}{f(i)}$.

The picture



KEY IDEA

$\nu(i)$ is "*minus the slope*" divided by the "*height*". If we took the slope alone, it would double as the cashflow doubled. But dividing it the raw value cancels that, so ν measures *sensitivity*, not size.

Properties

Effective duration measures the sensitivity of a series of cashflows to movements in i, and is independent of the size of NPV(i).

- If all $c_k > 0$ then NPV(i) decreases as i increases, so NPV$'(i) < 0$ and $\nu(i) > 0$ for all i.
- $\nu(i)$ gives a quantitative measure of the rate of change of the price of a bond with respect to i. For a small increase in i, it creates a drop in price of approximately "$\nu(i)$%".

> CAUTION
>
> That last statement is a *first-order* approximation: NPV$(i+h) \approx$ NPV$(i)\,(1 - h\,\nu(i))$ for small h. Since NPV is convex, it under-states the value for both signs of h.

Two examples

EXAMPLE 27.3

The warm-up bond: three years, 5% coupons, at $i = 6\%$. Here $\mathrm{NPV}(0.06) = 97.3270$ and

$$\nu(0.06) \;=\; \frac{1(5)\mathrm{v}^2 + 2(5)\mathrm{v}^3 + 3(105)\mathrm{v}^4}{97.3270} \;=\; 2.6956,$$

so a rise of 0.1% in i costs about $0.001 \times 2.6956 = 0.2696\%$ of the price. The exact value in the warm-up was 0.2691%.

EXAMPLE 27.4

A ten-year zero-coupon bond at $i = 6\%$ has a single cashflow, so $\nu(0.06) = 10\mathrm{v} = 9.434$. This is far more sensitive.

The second is liquidity preference made quantitative: the longer bond really is the riskier one.

Lecture 27 on one slide

HOW STEEP, NOT HOW BIG

$$\nu(i) \;=\; -\frac{\mathrm{NPV}'(i)}{\mathrm{NPV}(i)} \;=\; -\frac{\mathrm{d}}{\mathrm{d}i}\ln \mathrm{NPV}(i) \;=\; \frac{\sum_k t_k c_k \mathrm{v}^{\,t_k+1}}{\sum_k c_k \mathrm{v}^{\,t_k}}$$

$$\mathrm{NPV}(i+h) \;\approx\; \mathrm{NPV}(i)\,(1 - h\,\nu(i)) \qquad \text{for small } h.$$

A yield curve may slope down, slope up, or hump, and expectations, liquidity preference and market segmentation each offer a reason. The risk is not that rates move but that assets and liabilities move by different amounts. *Effective duration* is the number that says by how much.

Exercises

EXERCISE 27.5

Find the effective duration at $i = 6\%$ of a five-year zero-coupon bond, and of a three-year zero-coupon bond. What is the pattern?

EXERCISE 27.6

A five-year bond redeemable at par pays 6% annual coupons, at $i = 6\%$. Find NPV *and ν, and compare with the five-year zero.*

EXERCISE 27.7

An institution's assets have duration 6 and its liabilities duration 9, with $V_A = V_L$ at present. Which way must rates move to make it insolvent, and why?

Solutions are in the backup slides.

END OF LECTURE 27

Duration answers “how much” to first order. Next: what the second order adds, and the older mean-term measure that sits behind both.

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Solution 1

A zero-coupon bond has a single cashflow c at time n, so

$$\nu(i) \;=\; \frac{nc\mathsf{v}^{\,n+1}}{c\mathsf{v}^{\,n}} \;=\; n\mathsf{v},$$

independent of c. At 6%, $\mathsf{v} = 0.943396$ and

$$\nu_5 \;=\; 5(0.943396) \;=\; 4.7170, \qquad \nu_3 \;=\; 3(0.943396) \;=\; 2.8302.$$

The pattern is that duration is proportional to term: for a zero-coupon bond it is just $n\mathsf{v}$, a little under the term itself. This is the precise form of liquidity preference — the longer bond is more rate-sensitive, in exact proportion.

Solution 2

A bond whose coupon equals the yield prices at par, so NPV$(0.06) = 100$ without calculation (Lecture 22). For the duration, with $\mathrm{v} = 0.943396$,

$$\nu(0.06) \;=\; \frac{\sum_{k=1}^{4} k(6)\mathrm{v}^{\,k+1} + 5(106)\mathrm{v}^{6}}{100} \;=\; 4.2124.$$

Against 4.7170 for the five-year zero: the coupon bond is *less* sensitive, even though both run five years. The reason is that the coupons return money early, so the average date at which the investor is paid is earlier than five years — and duration is essentially that average date.

> KEY IDEA
>
> Term and duration are not the same thing. The next lecture makes the "average date" reading exact.

Solution 3

Rates must *fall*.

To first order, a change of h in the yield changes each value by a factor $1 - h\nu$. With $V_A = V_L$ at present,

$$V_A(i+h) - V_L(i+h) \;\approx\; V_A\,(1-6h) - V_L\,(1-9h) \;=\; V_A\,(3h),$$

which is negative exactly when $h < 0$.

In words: the liabilities are the steeper curve, so they gain more when rates fall. That is the situation drawn in Section 2, and it is why matching durations — not merely matching values — is what an institution actually needs.