LECTURE 26

One number for a whole curve

Redemption yields, par yields, and two past papers

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A curve, and the wish to avoid it

$$P = D\left(\mathsf{v}_1 + \mathsf{v}_2^2 + \cdots + \mathsf{v}_n^n\right) + R\mathsf{v}_n^n$$

n rates, one for each date

Correct, and inconvenient. To compare, a trader wants *one* number for a bond. Today, we will see two ways of producing one, and see what each keeps and what each throws away.

Today

1. Yields to maturity
2. CM1A April 2022
3. Par yields
4. CM1A September 2021

Warm-up

EXERCISE 26.1

In a market with $y_2 = 4\%$ *and* $y_5 = 5\%$*, find the forward rate* $f_{2,3}$.

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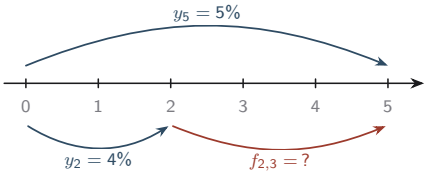
In a market with $y_2 = 4\%$ and $y_5 = 5\%$, find the forward rate $f_{2,3}$. With $t = 2$ and $r = 3$, so that $t + r = 5$,



$$(1 + f_{2,3})^3 = \frac{(1 + y_5)^5}{(1 + y_2)^2} = \frac{1.276282}{1.081600} = 1.180000, \qquad f_{2,3} = 5.6720\%.$$

HERE, $f_{2,3}$ IS HIGHER THAN BOTH $y_2 = 4\%$ AND $y_5 = 5\%$

Since $1 + y_5$ is the geometric mean of five one-year factors, and the first two average only 4%, the last three must exceed 5% to pull the average up.

A single rate standing for a stretch of the curve is an *average* (mean). And an average tells you nothing about the spread (variance). Both of today's summaries have that property.

SECTION 1

Yields to maturity

The redemption yield

DEFINITION 26.2 · Yield to maturity

The **yield to maturity** for a coupon-paying bond, also called the **redemption yield**, is the effective rate of interest at which the discounted proceeds of the bond equal its price.

It is widely used, but depends on the *coupon rate* of the bond, and so gives no simple model of the relationship between term and yield. A **gross** redemption yield ignores tax; a **net** one allows for it.

KEY IDEA

Two bonds of the same term but different coupons have the same spot rates behind them, yet different redemption yields. The curve is defined by the market, the redemption yield by the bond.

SECTION 2

CM1A April 2022

The question

EXAMPLE 26.3

The annual effective forward rate applicable over the period t to $t+r$ is $f_{t,r}$, with t and r in years. You are given

$$f_{0,1} = 4.2\%, \quad f_{1,1} = 4.8\%, \quad f_{2,1} = 5.3\%, \quad f_{2,2} = 5.9\%.$$

(i) Calculate $f_{3,1}$, as a percentage to four decimal places.

(ii) Calculate all possible spot rates of interest that the above information allows you to calculate.

(iii) Calculate, using linear interpolation, the annual effective gross redemption yield of a four-year bond, redeemable at 105%, with a 2.5% coupon payable annually in arrears.

(i) Calculate $f_{3,1}$: Two routes from 2 to 4

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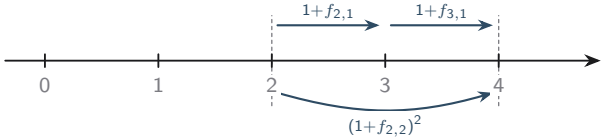

Note that $1 + f_{2,2}$ is not the rate that takes €1.00 from $t = 2$ to $t = 2 + 2 = 4$. It is the average rate one-year rate that takes €1.00 from $t = 2$ to $t = 2 + 2 + 4$. Hence, we must squar it (chaing it with itself).

Both routes carry money from time 2 to time 4, so they must agree by consistency: $(1 + f_{2,2})^2 = (1 + f_{2,1})(1 + f_{3,1})$. Hence

$$f_{3,1} = \frac{(1 + f_{2,2})^2}{1 + f_{2,1}} - 1 = \frac{(1.059)^2}{1.053} - 1 = 0.065034 = 6.5034\%.$$

(ii) Calculate all* spot rates: Chaining the one-year rates

that is possible to derive

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With the information given, and using $(1+y_t)^t = (1+f_0)\cdots(1+f_{t-1})$:

$$\begin{aligned}
(1+y_1) &= (1+f_{0,1}) \\
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(1+y_3)^3 &= (1+f_{0,1})(1+f_{1,1})(1+f_{2,1}) \\
(1+y_4)^4 &= (1+f_{0,1})(1+f_{1,1})(1+f_{2,1})(1+f_{3,1})
\end{aligned}$$

FOUR ONE-YEAR RATES GIVE FOUR SPOT RATES AND NO MORE

We could not, for instance, produce y_5: nothing in the data reaches beyond year four. Part (i) was needed precisely to supply the fourth link in the chain.

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We could not, for instance, produce y_5: nothing in the data reaches beyond year four. Part (i) was needed precisely to supply the fourth link in the chain.

(ii) Calculate all* spot rates: Chaining the one-year rates

that is possible to derive

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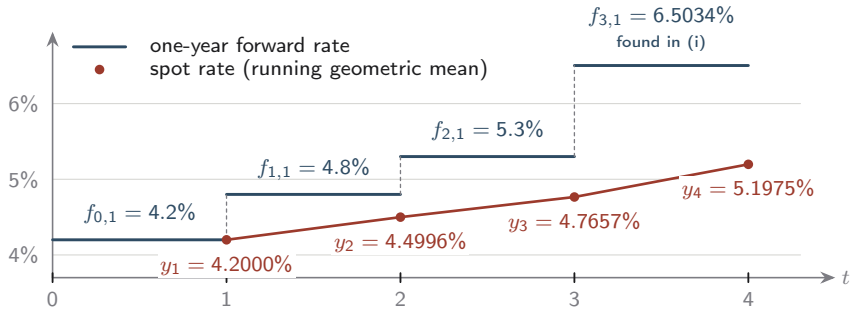
(ii) Forwards in steps, spots as averages



SPOT RATES ARE ROLLING AVERAGES WITH EXPANDING WINDOWS

A forward rate covers one year; a spot rate averages all the years so far. With the forwards rising, each y_t lags behind the forward just reached.

(iii) Redemption yield: The price of the bond (four-year, redeemable at 105%, annual arrear coupon 2.5%)

$$y_1 = 4.2000\%, \quad y_2 = 4.4996\%, \quad y_3 = 4.7657\%, \quad y_4 = 5.1975\%.$$

A gross (without tax) redemption yield equates price and proceeds, so we need the price first, and that comes from the spot rates. Per €100.00 nominal, with coupons of 2.5 and redemption 105 at $t = 4$,

$$P \;=\; 2.5\left(\mathrm{v}_1 + \mathrm{v}_2^2 + \mathrm{v}_3^3 + \mathrm{v}_4^4\right) + 105\,\mathrm{v}_4^4, \qquad \mathrm{v}_t^t \;=\; (1+y_t)^{-t}:$$

t	1	2	3	4
v_t^t	0.959693	0.915738	0.869646	0.816543

$$P \;=\; 2.5(3.561620) + 105(0.816543) \;=\; 8.904 + 85.737 \;=\; 94.6411.$$

(iii) The redemption yield (effective, gross, four-year, redeemable at 105%, annual arrear coupon 2.5%)

Now find the single rate i with

$$2.5\, a_{\overline{4}|} + 105\, v^4 \;=\; 94.6411.$$

Writing $g(i) = 2.5\, a_{\overline{4}|} + 105 v^4 - 94.6411$, we have

$$\begin{aligned} g(0.05) &= 95.2486 - 94.6411 = +0.6076 \\ g(0.052) &= 94.5526 - 94.6411 = -0.0885, \end{aligned}$$

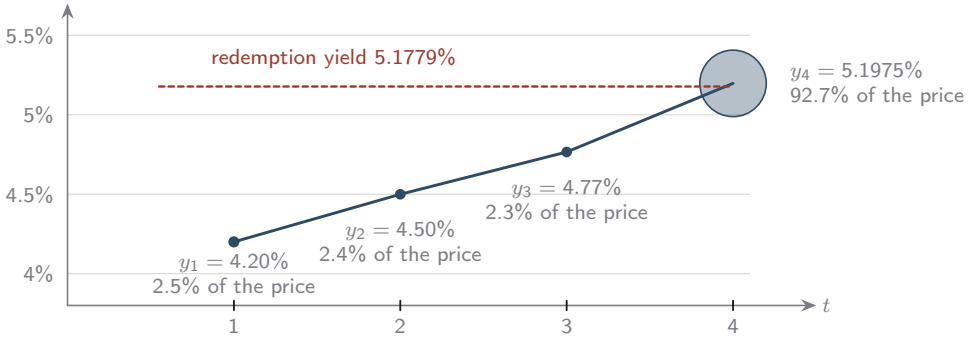
so linear interpolation gives

$$i \;\approx\; 0.05 + \frac{0.6076}{0.6076 + 0.0885}(0.002) \;=\; 0.051746 \;=\; 5.1746\%.$$

HOW TO GUESS A SMALL BRACKET? HERE (0.05, 0.052)

The bracket is only 0.2% points wide, so the estimate is close. The true value is 5.1745%. Clearly $i < \max(y_i) = 0.051975$. The next frame shows why we could guess 0.05 as the lower bound.

Where the redemption yield sits



Size of the dot signifies its share of the total price. The last is so big since it contains the redemption payment. that is why the yield so close to its discount capacity. Note also that the vertical axis does not start at zero.

What the answer tells us

The redemption yield 5.1779% sits just below the four-year spot rate $y_4 = 5.1975\%$.

REMARK 26.4

The redemption yield is a single rate standing in for the whole term structure. It is a weighted average of $y_1, \ldots, y_4$ and as 90.6% of the price is the redemption payment at $t = 4$, the answer is pulled close to y_4. The small early coupons, discounted at lower spot rates, drag it slightly down, but can not move it far.

> USEFUL WHEN INTERPOLATING
>
> Knowing where the main payments fall lets you guess where the redemption yield will roughly be. This can be very useful when interpolating.

SECTION 3

Par yields

Par yields

DEFINITION 26.5 · Par yield

The n-**year par yield** is the coupon per €1.00 nominal payable on a bond of term n years which would give the bond the current price; under the current term structure, of €1.00 per €1.00 nominal, assuming redemption at par. Mathematically express, yc_n is defined as

$$1 \;=\; (yc_n)\left(\mathsf{v}_1 + \mathsf{v}_2^2 + \cdots + \mathsf{v}_n^n\right) + \mathsf{v}_n^n, \qquad \mathsf{v}_t \;=\; (1+y_t)^{-1}.$$

KEY IDEA

Informally, it asks *what coupon would make this bond sell at par?* It is a fourth way of quoting the same term structure, alongside P_n, y_n and $f_{t,r}$.

Coupon bias

Rearranging the definition,

$$yc_n = \frac{1 - \mathsf{v}_n^n}{\mathsf{v}_1 + \mathsf{v}_2^2 + \cdots + \mathsf{v}_n^n}.$$

The par yields give an alternative measure of the relationship between the yield and the term of investments.

DEFINITION 26.6 · Coupon bias

The difference between the n-year par yield rate and the n-year spot rate is called the **coupon bias**.

It is not zero because a par bond pays coupons at $1, 2, \ldots, n$, each discounted at its own spot rate, whereas y_n discounts a single payment at time n.

Example

EXAMPLE 26.7

Calculate the three-year par yield if the current annual term structure is $y_1 = 5\%$, $y_2 = 4\%$, $Y_3 = 3\%$ — **where the capital letter indicates it's a force**.

The third rate is quoted as a *force*, so by Lecture 2 the corresponding discount factor is $v_3^3 = e^{-3(0.03)}$, and

$$yc_3 \;=\; \frac{1 - e^{-3(0.03)}}{(1.05)^{-1} + (1.04)^{-2} + e^{-3(0.03)}} \;=\; 0.03084,$$

so the three-year par yield is $yc_3 = 3.084\%$. The discrete three-year spot rate is $y_3 = e^{0.03} - 1 = 3.046\%$, so the coupon bias is only 0.04 percentage points.

SECTION 4

CM1A September 2021

The question

EXAMPLE 26.8

A bond is issued at time $t = 0$ at a price of €107.60 per €100.00 nominal. The bond pays coupons of 6% p.a., annually in arrears, and will be redeemed at par in three years' time. The two-year par yield at time $t = 0$ is 6.5% p.a. The one-year forward rate of interest at time $t = 1$ year is 4.5% p.a. effective. Calculate, assuming no arbitrage, the implied one-, two- and three-year annual effective spot rates.

The question

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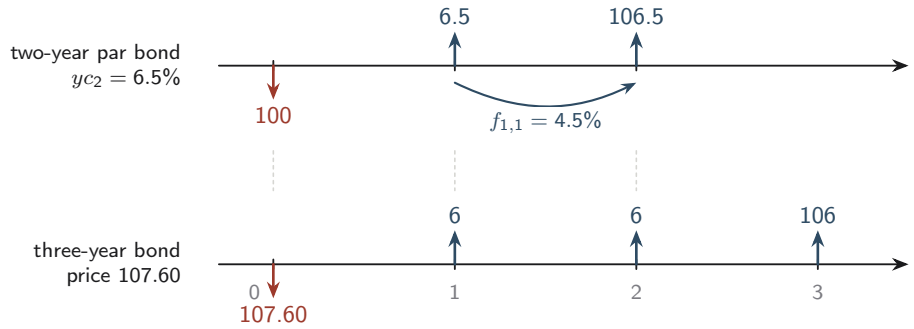
KEY IDEA

We are given $P = 107.60,\ c = 6\%,\ yc_2 = 6.5\%,\ f_{1,1} = 4.5\%$ and we seek y_1, y_2, y_3.

The three pieces of information

Given: $yc_2 = 6.5\%$, $f_{1,1} = 4.5\%$, $P = 107.60$*; seek* y_1, y_2, y_3



The par yield and the forward rate give y_1 and y_2. The price of the three-year bond then fixes y_3.

Using the par yield for the y_1 equation

Given: $yc_2 = 6.5\%$, $f_{1,1} = 4.5\%$*; seek* y_1

By definition, $1 = (yc_2)(\mathsf{v}_1 + \mathsf{v}_2^2) + \mathsf{v}_2^2$, which, scaled to €100.00 nominal, gives us

$$100 \;=\; 6.5\left(\mathsf{v}_1 + \mathsf{v}_2^2\right) + 100\mathsf{v}_2^2 \;=\; 6.5\mathsf{v}_1 + 106.5\mathsf{v}_2^2.$$

Recall $f_{0,1} = y_1$, so $(1+y_2)^2 = (1+y_1)(1+f_{1,1})$ and

$$\mathsf{v}_2^2 \;=\; (1+y_2)^{-2} \;=\; \frac{1}{(1+y_1)(1+f_{1,1})} \;=\; (1+y_1)^{-1}(1.045)^{-1}.$$

Every term now carries the single unknown factor $\mathsf{v}_1 = (1+y_1)^{-1}$ which gives us

$$100 \;=\; (1+y_1)^{-1}\left(6.5 + 106.5(1.045)^{-1}\right).$$

y_1 and y_2

Given: $yc_2 = 6.5\%,\ f_{1,1} = 4.5\%$*; seek* y_1, y_2

Hence

$$1 + y_1 \;=\; \frac{6.5 + 106.5(1.045)^{-1}}{100} \;=\; \frac{108.413876}{100}, \qquad y_1 \;=\; 8.41388\%.$$

To compute y_2 we use $(1 + y_2)^2 = (1 + y_1)(1 + f_{1,1})$ again and write

$$y_2 \;=\; \left((1 + y_1)(1 + f_{1,1})\right)^{1/2} - 1 \;=\; \sqrt{(1.0841388)(1.045)} - 1 \;=\; 6.43895\%.$$

> KEY IDEA
>
> $1 + y_2$ is the geometric mean of the two one-year factors (recall that $y_1 = f_0 = f_{0,1}$), here of 1.0841388 and 1.045, giving something between the two.

y_3 from the bond price

Given: $P = 107.60,\ c = 6\%,\ y_1 = 0.0841388, y_2 = 0.0643895$*; seek* y_3

Since there is no arbitrage, the bond is worth the sum of its zero-coupon parts:

$$\begin{aligned} 107.60 &= 6\left(\mathrm{v}_1 + \mathrm{v}_2^2 + \mathrm{v}_3^3\right) + 100\mathrm{v}_3^3 \\ &= 6\left((1.0841388)^{-1} + (1.0643895)^{-2}\right) + 106(1+y_3)^{-3}. \end{aligned}$$

The bracket is $6(0.922391 + 0.882671) = 10.83037$, so

$$(1+y_3)^{-3} = \frac{107.60 - 10.83037}{106} = 0.912921,$$

$$y_3 = (0.912921)^{-1/3} - 1 = 3.08345\%.$$

The curve we found

	y_1	y_2	y_3
spot rate	8.41388%	6.43895%	3.08345%

A steeply *falling* curve

TWO CHECKS WORTH DOING

The two-year par yield should come back, $6.5v_1 + 106.5v_2^2 = 100.00$; and the bond price should be, $6(v_1 + v_2^2 + v_3^3) + 100v_3^3 = 107.60$. Both do.

The bond sells above par since its 6% coupon beats the later spot rates by a wide margin.

Lecture 26 on one slide

TWO SUMMARIES, OPPOSITE IN DIRECTION

REDEMPTION YIELD: $P = D\,a_{\overline{n}|} + R\mathsf{v}^n$ solved for one i,

PAR YIELD: $yc_n = \dfrac{1-\mathsf{v}_n^n}{\mathsf{v}_1+\mathsf{v}_2^2+\cdots+\mathsf{v}_n^n}$ read off the curve.

The redemption yield is defined by a bond and moves with its coupon. The par yield is intrinsic to the market and is a fourth language for the term structure, along with P_n, y_n and $f_{t,r}$. Their gap from y_n is the coupon bias.

Exercises

EXERCISE 26.9

With $y_1 = 5\%$, $y_2 = 6\%$, find the two-year par yield and the coupon bias.

EXERCISE 26.10

A two-year bond redeemable at par pays annual coupons at the par yield you found. Verify that it prices at 100 *per* €100.00 *nominal, and find its redemption yield.*

EXERCISE 26.11

Two four-year bonds have coupons of 2% *and* 9%*. Under the spot curve of CM1A April 2022, which has the higher redemption yield? Answer without calculating.*

Solutions are in the backup slides.

END OF LECTURE 26

Two numbers, both averages of a curve. Next, we see to unndersatnd the curve, what moves it, and how much a portfolio depends on it.

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Solution 1

The discount factors are $v_1 = (1.05)^{-1} = 0.952381$ and $v_2^2 = (1.06)^{-2} = 0.889996$, so

$$yc_2 \;=\; \frac{1 - v_2^2}{v_1 + v_2^2} \;=\; \frac{0.110004}{1.842377} \;=\; 0.059708 \;=\; 5.9708\%.$$

The coupon bias is $yc_2 - y_2 = 5.9708\% - 6\% = -0.0292$ percentage points. It is slightly negative, as it must be. The par bond pays part of its money at time 1, where the rate is only 5%, so it needs a coupon a little below y_2 to sell at par.

Solution 2

Pricing the bond strand by strand, with a coupon of 5.9708 per 100 nominal,

$$5.9708(0.952381) + 105.9708(0.889996) \;=\; 5.6865 + 94.3135 \;=\; 100.00,$$

as the definition of the par yield requires.

For the redemption yield, solve $5.9708\, a_{\overline{2}|} + 100\mathrm{v}^2 = 100$ for a single i. But a bond priced at par with coupon c has redemption yield exactly c, by the argument of Lecture 22: the coupon precisely services the capital, which is then returned intact. So the redemption yield is 5.9708%, equal to the par yield.

> KEY IDEA
>
> That is the point of the par yield: it is the one coupon for which the two summaries agree.

Solution 3

The 2% bond.

The April 2022 curve rises: $y_1 = 4.20\%$ up to $y_4 = 5.20\%$. The redemption yield is an average of the spot rates weighted by where the money is. A low coupon puts almost all of the value in the redemption payment at $t = 4$, so the average sits close to the highest rate y_4. A high coupon shifts weight onto the early dates, where the rates are lower, and drags the average down.

So on a rising curve the lower-coupon bond has the higher redemption yield — and on a falling curve, such as the September 2021 one, it would be the other way round. This is exactly the warning made twice today: the redemption yield belongs to the bond, not to the market.