LECTURE 25

Spot rates and forward rates

When the rate depends on the term

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One rate for every date

$$\mathsf{PV} = \sum_t c_t \, \mathsf{v}^t, \qquad \mathsf{v} = \frac{1}{1+i}$$

One i for all t.

Thus far, we have mainly considered constant rates i. Today, each date gets its own rate.

Today

1. Zero-coupon bonds and spot rates

2. Forward rates

Warm-up

EXERCISE 25.1

A contract paying €1.00 *in one year's time currently costs* €0.95*. Another is paying* €1.00 *in two years costs* €0.89*, and yet a third pays* €1.00 *in three years and costs* €0.82*. What annual effective rate of interest does each imply?*

EXERCISE 25.1

A contract paying €1.00 in one year's time currently costs €0.95. Another is paying €1.00 in two years costs €0.89, and yet a third pays €1.00 in three years and costs €0.82. What annual effective rate of interest does each imply?

Each contract is a single payment, so $P = (1+y)^{-n}$ and $y = P^{-1/n} - 1$:

$$\frac{1}{0.950} - 1 \;=\; 5.2632\%, \qquad (0.890)^{-1/2} - 1 \;=\; 6.0000\%, \qquad (0.820)^{-1/3} - 1 \;=\; 6.8387\%.$$

> ASIDE
>
> Three rates, all from the same market on the same day, and no two of them equal.

There is no single i here, and IRL, there never is. The rest of this lecture considers a market that quotes a different rate at every term.

SECTION 1

Zero-coupon bonds and spot rates

The assumption we are dropping

So far we have assumed the interest rate i, or force of interest δ, earned on an investment is independent of the *term* of that investment. In practice the rate offered does usually vary with the term, and it is often important to take that into consideration.

DEFINITION 25.2 · Zero-coupon bond

A **unit zero-coupon bond** of term n is an agreement to pay €1.00 at the end of n years, with no coupon payments. It is also called a **pure discount bond**. Its price at issue is denoted P_n.

> KEY IDEA
>
> These are the atoms: a zero-coupon bond isolates *one* date, so its price says what the market charges for money at that **horizon** and nothing else.

Spot rates

DEFINITION 25.3 · Spot rate

The yield on a unit zero-coupon bond of term n years, y_n, is the **n-year spot rate of interest**. It measures the average interest rate over the n-year period.

The equation of value for the zero-coupon bond is simply

$$P_n = \frac{1}{(1+y_n)^n}, \qquad \text{so} \qquad y_n = P_n^{-1/n} - 1,$$

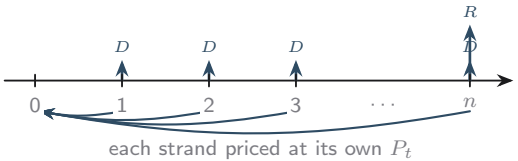
and since rates differ by term, in general $y_t \neq y_s$ for $s \neq t$.

KEY IDEA

The variation by term is the **term structure of interest rates**; P_n and y_n carry identical information.

Every bond is a portfolio of zero-coupon bonds

A bond paying coupons of D for n years, with a redemption payment of R at time n, is a combined investment of n zero-coupon bonds of maturity value D at terms $1, \dots, n$, plus one of nominal value R at term n.



Let P be the price of the bond, P_i the price of the i-th coupon, and $\mathsf{v}_t = (1+y_t)^{-1}$,

$$P \;=\; D\left(P_1 + \cdots + P_n\right) + RP_n \;=\; D\left(\mathsf{v}_1 + \mathsf{v}_2^2 + \cdots + \mathsf{v}_n^n\right) + R\mathsf{v}_n^n.$$

Why the prices must agree

NO ARBITRAGE

This is a consequence of **no arbitrage**. The portfolio of zero-coupon bonds has the same payouts as the fixed-interest bond, so the prices must be the same.

Otherwise, an investor could buy the cheaper side, sell the dearer, and keep the difference with no risk. The combined arguments of Lecture 1 and of the bond strips of Lecture 15.

CAUTION

Note what has changed in the notation. Previously, all discounting used a single v, but now each date carries its *own* factor $\mathrm{v}_t^t = (1 + y_t)^{-t}$.

Example: pricing from spot rates

EXAMPLE 25.4

Calculate the price of a five-year fixed-interest security, redeemable at par, with 6% annual coupons, assuming the annual term structure of interest rates is

$$y_1 = 7\%, \quad y_2 = 7\tfrac{1}{4}\%, \quad y_3 = 7\tfrac{1}{2}\%, \quad y_4 = 7\tfrac{3}{4}\%, \quad y_5 = 8\%.$$

Per 100 nominal,

$$\begin{aligned} P &= 6\left(\mathrm{v}_1 + \mathrm{v}_2^2 + \cdots + \mathrm{v}_5^5\right) + 100\,\mathrm{v}_5^5 \\ &= 6\left(\frac{1}{1.07} + \frac{1}{(1.0725)^2} + \frac{1}{(1.075)^3} + \frac{1}{(1.0775)^4} + \frac{1}{(1.08)^5}\right) + \frac{100}{(1.08)^5} = 92.25. \end{aligned}$$

Where the value sits

t	1	2	3	4	5	total
P_t	0.934579	0.869372	0.804961	0.741875	0.680583	
coupon $6P_t$	5.607	5.216	4.830	4.451	4.083	24.19
redemption					68.06	68.06
						92.25

NO YIELD, EACH ATOM IS DISCOUNTED AT *ITS OWN* SPOT RATE

We could immediately say that the price is below par as the 6% coupon is below every spot rate in the structure. Note also that we did not use a single "yield", but discounted each cashflow at the rate appropriate to *its own* date.

SECTION 2

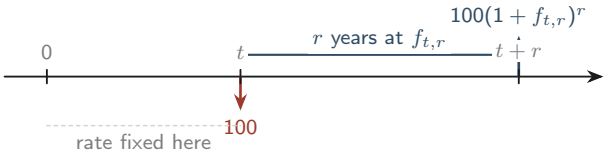
Forward rates

Discrete time forward rates

DEFINITION 25.5 · Forward rate

The **discrete time forward rate** $f_{t,r}$ is the annual interest rate *agreed at time* 0 for an investment made at time $t > 0$ for a period of r years.

So if an investor agrees at time 0 to invest €100.00 at time t for r years, the accumulated investment at time $t + r$ is $100(1 + f_{t,r})^r$.



The rate is settled *now*, but the money moves *later*, necessitating the definition.

Linking the three languages

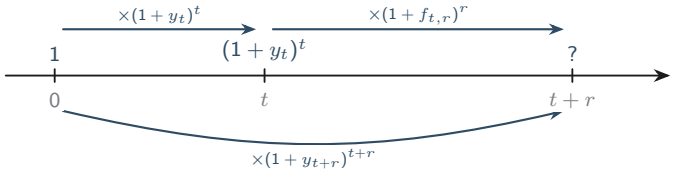
Forward rates, spot rates and zero-coupon bond prices are all connected. The accumulation at time t of an investment of 1 at time 0 is $(1 + y_t)^t$. If we agree at time 0 to invest that amount at time t for r years, we earn an annual rate of $f_{t,r}$, so 1 invested for $t + r$ years accumulates to

$$(1 + y_t)^t (1 + f_{t,r})^r.$$

But from the $(t + r)$-year spot rate, 1 invested for $t + r$ years accumulates to $(1 + y_{t+r})^{t+r}$. Calculated from the zero-coupon bond price, 1 goes to P_{t+r}^{-1}.

Three descriptions of the same growth, so they all must agree.

Two routes, one answer



KEY IDEA

This is the **principle of consistency** from Lecture 1: the proceeds cannot depend on the route taken. Hence

$$(1+y_t)^t(1+f_{t,r})^r \;=\; (1+y_{t+r})^{t+r} \;=\; \frac{1}{P_{t+r}}.$$

The conversion formula

Rearranging,

$$\boxed{(1+f_{t,r})^r = \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} = \frac{P_t}{P_{t+r}}}$$

KEY IDEA

This means the full term structure may be determined given *any one* of the spot rates, the forward rates, or the zero-coupon bond prices. They are three descriptions of the same object.

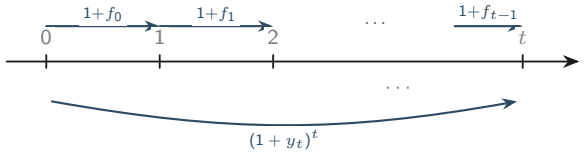
Note how natural the "P" form is: P_t/P_{t+r} compares the price of money at two horizons, and the forward rate is exactly the ratio between them.

One-period forward rates

One-period forward rates are of particular importance. The one-period forward rate at time t, agreed at time 0, is denoted

$$f_t \;=\; f_{t,1}, \qquad \text{and we define} \qquad f_0 \;=\; y_1.$$

Compare 1 invested for t years at the spot rate y_t with the same amount invested one year at a time, the proceeds reinvested at the appropriate one-year forward rate:



$$(1+y_t)^t \;=\; (1+f_0)(1+f_1)(1+f_2)\cdots(1+f_{t-1}).$$

The spot rate is a geometric mean

Taking t-th roots of that identity gives us

$$1 + y_t \;=\; \sqrt[t]{(1+f_0)(1+f_1)\cdots(1+f_{t-1})},$$

so $1 + y_t$ is the **geometric mean** of the t one-year growth factors.

> REMINDER
>
> The geometric mean of $a_1, \ldots, a_n$ is $\sqrt[n]{a_1 a_2 \cdots a_n}$: the right average whenever quantities *compound* rather than add.

This makes precise the earlier remark that the n-year spot rate "measures the average interest rate over the n-year period". It is an average; but the geometric one, because growth factors multiply.

Lecture 25 on one slide

THREE LANGUAGES, ONE TERM STRUCTURE

$$P_n \;=\; \frac{1}{(1+y_n)^n}, \qquad (1+f_{t,r})^r \;=\; \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} \;=\; \frac{P_t}{P_{t+r}},$$

$$(1+y_t)^t \;=\; \prod_{k=0}^{t-1}(1+f_k), \qquad P \;=\; D\,(\mathsf{v}_1 + \cdots + \mathsf{v}_n^n) + R\mathsf{v}_n^n.$$

Any one of P_n, y_n, $f_{t,r}$ determines the other two. Every coupon bond is a portfolio of zero-coupon bonds, priced strand by strand.

Exercises

EXERCISE 25.6

For the market of the warm-up, find the one-year forward rates f_1 and f_2, and verify that $(1 + y_3)^3 = (1 + f_0)(1 + f_1)(1 + f_2)$.

EXERCISE 25.7

A three-year bond redeemable at par pays annual coupons of 5%. Using the same three prices, find its price per €100.00 nominal.

EXERCISE 25.8

With $y_2 = 4\%$ and $y_5 = 5\%$, find the forward rate $f_{2,3}$.

Solutions are in the backup slides.

END OF LECTURE 25

Three languages for one growth. Next: Two ways of collapsing that whole curve/growth into a single number

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Solution 1

The cleanest route is through the prices: $(1 + f_{t,r})^r = P_t/P_{t+r}$, so with $r = 1$,

$$1 + f_1 \;=\; \frac{P_1}{P_2} \;=\; \frac{0.950}{0.890} \;=\; 1.067416, \qquad 1 + f_2 \;=\; \frac{P_2}{P_3} \;=\; \frac{0.890}{0.820} \;=\; 1.085366,$$

that is $f_1 = 6.7416\%$ and $f_2 = 8.5366\%$. Also $f_0 = y_1 = 5.2632\%$.

For the check, the product telescopes:

$$(1 + f_0)(1 + f_1)(1 + f_2) \;=\; \frac{1}{P_1} \cdot \frac{P_1}{P_2} \cdot \frac{P_2}{P_3} \;=\; \frac{1}{P_3} \;=\; \frac{1}{0.820} \;=\; 1.219512,$$

and $(1 + y_3)^3 = (1.068387)^3 = 1.219512$ too.

Solution 2

The bond is a portfolio of three zero-coupon bonds: coupons of 5 at times 1, 2, 3, and the redemption of 100 at time 3. So, per €100.00 nominal,

$$P \;=\; 5\,(P_1 + P_2 + P_3) + 100P_3 \;=\; 5(0.950 + 0.890 + 0.820) + 100(0.820),$$

that is $5(2.660) + 82.00 = 13.30 + 82.00 = €95.30$.

Below par, as it must be: the 5% coupon is below every spot rate in the market (5.26%, 6.00%, 6.84%).

> **CAUTION**
>
> No annuity symbol appears. $a_{\overline{3}|}$ would require one rate for all three dates, and there is no such rate here.

Solution 3

Here $t = 2$ and $r = 3$, so $t + r = 5$ and the conversion formula gives

$$(1 + f_{2,3})^3 \;=\; \frac{(1 + y_5)^5}{(1 + y_2)^2} \;=\; \frac{(1.05)^5}{(1.04)^2} \;=\; \frac{1.276282}{1.081600} \;=\; 1.180000,$$

whence

$$f_{2,3} \;=\; (1.180000)^{1/3} - 1 \;=\; 5.6720\%.$$

Above both y_2 and y_5, and it must be: the five-year rate is a geometric average of the first two years at 4% and the last three at $f_{2,3}$, so with the average at 5% the later stretch has to exceed it.