LECTURE 24

Finding the payback date

Four problems, one approach

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
2 November 2026 · Winter 2026

The definition, and the difficulty

$$A(t_1) = \sum_{s \leq t_1} (1 + j_1)^{t_1 - s} c_s$$

$$+ \int_0^{t_1} (1 + j_1)^{t_1 - s} \rho(s)\, \mathrm{d}s \geq 0$$

The discounted payback period is the smallest t_1 for which the formula on the left is true. In general, the unknown sits in three places at once: the exponent, the bounds of the integral, and the range of the sum. This makes it a bit for work to determine.

Today

1. Continuous income
2. A cashflow with a blip
3. Two rates: Payback and profit
4. A level annuity: The integer case

Warm-up

EXERCISE 24.1

A single outlay C is met by income at rate R per annum, payable continuously, with borrowing at rate j_1. Show that

$$A(t) \;=\; -C(1+j_1)^t + R\,\bar{s}_{\overline{t}|} \;\geq\; 0 \qquad \Longleftrightarrow \qquad \bar{a}_{\overline{t}|} \;\geq\; \frac{C}{R}.$$

Our first abstract warm-up. It is not too hard, only needs one division, and remembering annuity formulas. We will do this and little more throughout the whole lecture, so it is worth doing it once in general, before doing it four times as special cases.

Warm-up: solution

Divide throughout by $(1+j_1)^t$, which is positive, so no inequality reverses. We get

$$-C + R\,(1+j_1)^{-t}\,\bar{s}_{\overline{t}|} \;\geq\; 0.$$

But $(1+j_1)^{-t}\bar{s}_{\overline{t}|} = \mathrm{v}^t\bar{s}_{\overline{t}|} = \bar{a}_{\overline{t}|}$ by the transfer principle of Lecture 4 (accumulating to t and discounting back to 0 returns the present value). Hence $R\,\bar{a}_{\overline{t}|} \geq C$, which is the claim.

> KEY IDEA
>
> The point is where t now appears. Before, it was in an exponent *and* inside an accumulated annuity. Now it sits only inside $\bar{a}_{\overline{t}|}$, which increases with t.

SECTION 1

Continuous income

Example 1: Question

EXAMPLE 24.2

An investor borrows money at an effective rate of 10% p. a. to invest in a six-year project with an internal rate of return of approximately 18.9%. Calculate the discounted payback period if the cashflows are

- initial outlay 25,000;
- regular income 10,000 p.a. payable continuously during the first five years;
- regular expenditure 2,000 p.a. payable continuously during the first five years;
- decommissioning expense of 5,000 at the end of the project.

Income and expenditure run over the same interval, so we may net them and get $\rho(t) = 10 - 2 = 8$ per annum for $0 \leq t \leq 5$ (in thousands).

Example 1: Setup

We expect to get out of the red within the first five years, i.e., we expect DPP $\in [0, 5]$, since the income is received only during that period. So find the first t with $A(t) \geq 0$:

$$A(t) \;=\; -25(1.1)^t + \int_0^t 8\,(1.1)^{t-s}\,\mathrm{d}s \;=\; -25(1.1)^t + 8\,\bar{s}_{\overline{t}|}.$$

> KEY IDEA
>
> The decommissioning cost plays no part here. It falls after the last income and cannot affect *when* the sign changes to positive, which, if it happens, must happen before it. The decomissoining cost *does*, of course, affect the final profit, and the quoted internal rate of return (IRR).

So $A(t) = 0 \iff 8\,\bar{s}_{\overline{t}|} = 25(1.1)^t$, which is the warm-up with $C = 25$ and $R = 8$.

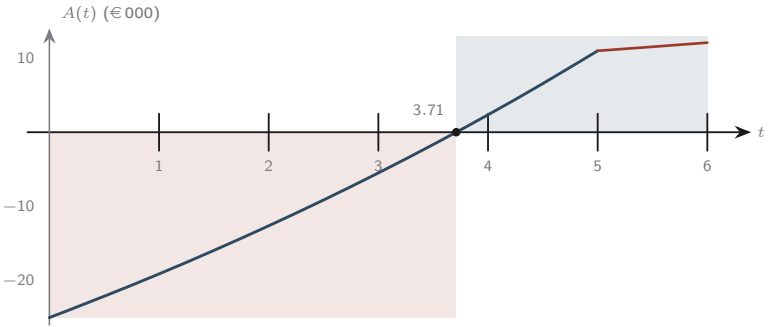
Example 1: Solution

By the warm-up, dividing by $(1.1)^t$,

$$\begin{aligned} 8\,\bar{s}_{\overline{t}|} \;=\; 25(1.1)^t \;&\iff\; \bar{a}_{\overline{t}|} \;=\; \frac{25}{8} \;=\; 3.125 \\ &\iff\; \frac{1-(1.1)^{-t}}{\ln(1.1)} \;=\; 3.125 \\ &\iff\; (1.1)^{-t} \;=\; 1 - 3.125\ln(1.1) \;=\; 0.70215569 \\ &\iff\; t \;=\; \frac{\ln(0.70215569)}{-\ln(1.1)} \;=\; 3.71. \end{aligned}$$

It follows that DPP $= 3.71$ years, or about three years and eight and a half months.

Example 1, drawn



The balance climbs steadily while the income is coming in, crosses zero at $t = 3.71$, and then, income having stopped at $t = 5$, merely accrues interest until the project ends.

SECTION 2

A cashflow with a blip

Example 2: The question

EXAMPLE 24.3

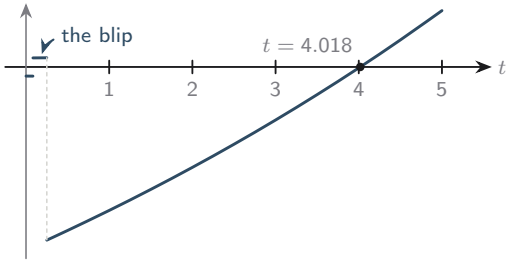
A business plan for a new company that has obtained a five-year lease for operating a local bus service (Y = year, M = month):

Cashflow	Timing	Amount (€000s)
Initial set up	Start Y1	−100
Advertising income	End M1	+200
Vehicle purchases	End M3	−2,000
Passenger fares	Continuous from M3	+1,000 p.a.
Staff costs	Continuous from M3	−400 p.a.
Asset resale value	End Y5	+500

Determine the DPP, assuming the project is financed by a flexible loan facility at an effective rate of interest of 10% per annum.

Example 2: The blip

The net cashflow changes sign more than once here. After the advertising income at the end of month 1 the balance is $-100(1.1)^{1/12} + 200 = +99.20$, *positive*, and stays so until the vehicles are bought at the end of month 3.



We treat that as a **blip**, as an outlier and look for the first time *after the vehicle purchases* at which $A(t) \geq 0$.

Example 2: Setting up

For $t \geq \frac{3}{12}$,

$$\begin{aligned} A(t) &= -100(1.1)^t + 200(1.1)^{t-1/12} - 2000(1.1)^{t-3/12} + \int_{3/12}^{t} 600\,(1.1)^{t-s}\,\mathrm{d}s \\ &= \left(-100(1.1)^{3/12} + 200(1.1)^{2/12} - 2000\right)(1.1)^{t-3/12} + 600\,\overline{s}_{\overline{t-3/12|}} \\ &= -1899.21\,(1.1)^{t-3/12} + 600\,\overline{s}_{\overline{t-3/12|}}. \end{aligned}$$

KEY IDEA

The three discrete flows are first accumulated to the common date $t = \frac{3}{12}$, where they total -1899.21. From there, the problem is exactly the same as the warm-up.

Example 2: Solving

Therefore, writing $u = t - \frac{3}{12}$,

$$\begin{aligned} A(t) = 0 &\iff 600\,\bar{s}_{\overline{u}|} = 1899.21\,(1.1)^u \\ &\iff \bar{a}_{\overline{u}|} = \frac{1899.21}{600} = 3.16535 \\ &\iff (1.1)^{-u} = 1 - 3.16535\ln(1.1) = 0.6983099 \\ &\iff u = \frac{\ln(0.6983099)}{-\ln(1.1)} = 3.768. \end{aligned}$$

So $t = 3.768 + 0.25 = 4.018$ years, just over four years, and comfortably inside the five-year lease.

SECTION 3

Two rates: Payback and profit

Example 3: The question

EXAMPLE 24.4

A company is considering investing in one of a number of projects. The chosen project is to be financed by a €500,000.00 bank loan at an effective rate of 9% per annum, and surplus funds may be invested at 6% per annum effective. One project has an initial outlay of €500,000.00 and income of €70,000.00 p.a. payable continuously for twenty years. Determine the discounted payback period, and the accumulated profit after twenty years.

Both rates will (eventually) appear here: $j_1 = 0.09$ up to the payback date, and $j_2 = 0.06$ afterwards.

Example 3: The payback period

The accumulated value at time t, in €0.00s, is

$$A(t) \;=\; -500(1.09)^t + \int_0^t 70(1.09)^{t-s}\,\mathrm{d}s \;=\; -500(1.09)^t + 70\,\bar{s}_{\overline{t}|},$$

so we solve $70\,\bar{s}_{\overline{t}|} = 500(1.09)^t$. Dividing by $(1.09)^t$ as before, we get

$$\bar{a}_{\overline{t}|} \;=\; \frac{1-(1.09)^{-t}}{\ln(1.09)} \;=\; \frac{500}{70} \qquad \Longrightarrow \qquad t \;=\; 11.093 \;\text{(DPP)}.$$

OBSERVE THE 9% IN THE INTEGRAL

While the balance is negative, the project is in debt, so the €70,000 income is used to repay the loan and must be accumulated at 9% (since $9\% > 6\%$ it is better to pay back as fast as possible). This applies only up to the DPP; after payback, surplus would earn 6%, but that does not affect the DPP calculation from below (from debt).

Example 3: The profit

Thus far, we only used $j_1 = 9\%$, but the 6% appears now: Since $A(\text{DPP}) = 0$, the balance restarts from zero at $t = 11.093$, and the income of 70 per annum then accumulates at $j_2 = 6\%$. Hence

$$A(20) \;=\; 70\,\bar{s}_{\overline{20-\text{DPP}|}} \text{ at } 0.06 \;=\; 70\,\frac{(1.06)^{8.907}-1}{\ln(1.06)} \;=\; 817.34,$$

that is, €817,340.00.

KEY IDEA

The first 11.093 years contribute *nothing* beyond telling us that **that** is where balace hits zero. In general, however, even at DPP, balence need **not** be zero. For example a a large lump sum can cause balance to turn to profit, and immediately leave a positive credit. This is why the general formula of Lecture 23 carries an $A(t_1)$ term.

SECTION 4

A level annuity: The integer case

A special case

A case of interest is a single investment of C in return for payments of R, annually in arrear for n years. Here t_1 is the smallest *integer* t with $A(t) \geq 0$, where at rate j_1

$$A(t) \;=\; -C(1+j_1)^t + R\, s_{\overline{t}|},$$

that is, the smallest integer t with $R\, s_{\overline{t}|} \geq C(1+j_1)^t$. Dividing by $(1+j_1)^t$ once more,

$$\boxed{a_{\overline{t}|} \;\geq\; \frac{C}{R} \quad \text{at rate } j_1}$$

THE WARM-UP ALL OVER AGAIN

The annuity is now discrete, not continuous, but the identity $\mathrm{v}^t s_{\overline{t}|} = a_{\overline{t}|}$ is the same.

The profit in the special case

The project is viable if $t_1 \leq n$, in which case the accumulated profit after n years is

$$A(t_1)(1+j_2)^{n-t_1} + R\, s_{\overline{n-t_1|}} \quad \text{at rate } j_2.$$

> CAUTION
>
> Because t_1 must be a whole number, $A(t_1)$ is generally *strictly* positive rather than zero. The balance overshoots during the year in which it crosses. Unlike Example 3, the first term does not vanish, and dropping it would be a common error.

This is another example to our remark on page 15 as to why the general formula in Lecture 23 carries an $A(t_1)$ term.

Example 4: The question

EXAMPLE 24.5

An investment of €100,000.00 will produce an annuity of €10,500.00 annually in arrear for twenty-five years. Find the discounted payback period when the interest rate on borrowed money is 9% per annum. Find also the accumulated profit after twenty-five years if money can be invested at 7% per annum.

By the special case, the DPP is the smallest integer t such that

$$a_{\overline{t}|} \geq \frac{100{,}000}{10{,}500} = 9.5238 \qquad \text{at } 9\%.$$

Example 4: The payback period

$$a_{\overline{t}|} \geq \frac{100{,}000}{10{,}500} \iff (1.09)^{-t} \leq 1 - \frac{100{,}000}{10{,}500} \times 0.09 = 0.142857$$

$$\iff -t \ln(1.09) \leq \ln(0.142857) \iff t \geq \frac{\ln(0.142857)}{-\ln(1.09)} = 22.58,$$

so $t_1 = 23$ years.

> CAUTION
>
> The inequality reverses on dividing by $-\ln(1.09) < 0$. The answer is then rounded *up* as payments only arrive at integer times, so 23 and not 22.

Example 4: The profit

With $t_1 = 23$ and $n = 25$, two years of annuity remain:

$$\begin{aligned}\text{Profit} &= A(23)(1.07)^2 + 10{,}500\, s_{\overline{2}|}\ \text{(at } 0.07\text{)} \\ &= \left(-100{,}000(1.09)^{23} + 10{,}500\,\frac{(1.09)^{23}-1}{0.09}\right)(1.07)^2 + 10{,}500\,\frac{(1.07)^2-1}{0.07} \\ &= 4{,}920.67 + 21{,}735 \;=\; €\,26{,}655.67.\end{aligned}$$

KEY IDEA

Here $A(23) = €\,4{,}297.91 > 0$. The balance passed zero partway through year 22 to 23 and kept climbing to the year end. Ignoring that surplus would understate the profit by nearly €5,000.00.

Lecture 24 on one slide

DIVIDE BY $(1+j_1)^t$, EVERY TIME

$$R\,\bar{s}_{\overline{t}|} \;\geq\; C(1+j_1)^t \iff \bar{a}_{\overline{t}|} \;\geq\; \frac{C}{R}, \qquad R\,s_{\overline{t}|} \;\geq\; C(1+j_1)^t \iff a_{\overline{t}|} \;\geq\; \frac{C}{R}.$$

Accumulate any discrete flows to a common date first, then restart the clock there. Ignore a blip, and ignore anything falling after the crossing. And when t_1 must be an integer, the balance overshoots, so $A(t_1) > 0$ and its accumulation is part of the profit.

Exercises

EXERCISE 24.6

An outlay of €200,000.00 *produces* €30,000.00 *per annum in arrear for twelve years, borrowing at* 8%*. Find the DPP, and the profit after twelve years if surplus funds earn* 5%*.*

EXERCISE 24.7

A project has an outlay of €40,000.00 *and income of* €6,000.00 *p.a. continuously, borrowing at* 7%*. Find the DPP, and say what happens if the income were only* €2,500.00*.*

EXERCISE 24.8

In Example 1 the decommissioning cost was ignored. Justify this, and say when such a cost would change the answer.

Solutions are in the backup slides.

END OF LECTURE 24

The payback date is set by what you pay to borrow; the profit by what you earn once you have stopped.

ilia.pirashvili@universityofgalway.ac.ie

Solution 1

Annual payments, so this is the integer case: find the least integer t with $a_{\overline{t}|} \geq \frac{200{,}000}{30{,}000} = 6.6667$ at 8%. Then

$$(1.08)^{-t} \;\leq\; 1 - 6.6667 \times 0.08 \;=\; 0.466667, \qquad t \;\geq\; \frac{\ln(0.466667)}{-\ln(1.08)} \;=\; 9.903,$$

so $t_1 = 10$ years.

The balance at that date is $A(10) = -200{,}000(1.08)^{10} + 30{,}000\, s_{\overline{10}|} = €2{,}811.87$, strictly positive as the caution warned, and two years remain:

$$\text{Profit} \;=\; 2{,}811.87(1.05)^2 + 30{,}000\, s_{\overline{2}|} \text{ at } 0.05 \;=\; 3{,}100.09 + 61{,}500 \;=\; €64{,}600.09.$$

Solution 2

Continuous income, so $A(t) = -40(1.07)^t + 6\,\bar{s}_{\overline{t}|}$ in €0.00s, and the warm-up gives $\bar{a}_{\overline{t}|} = \frac{40}{6} = 6.6667$:

$$(1.07)^{-t} \;=\; 1 - 6.6667\ln(1.07) \;=\; 0.548942, \qquad t \;=\; \frac{\ln(0.548942)}{-\ln(1.07)} \;=\; 8.86 \text{ years}.$$

If the income were only €2,500.00 then $\bar{a}_{\overline{t}|} = 16$ would be required, giving

$$(1.07)^{-t} \;=\; 1 - 16\ln(1.07) \;=\; 1 - 1.0825 \;<\; 0,$$

which is impossible. The income never covers the interest on the outstanding debt, so the balance never turns and there is no discounted payback period at all, and no profit either.

Solution 3

The decommissioning cost falls at $t = 6$, after the balance turned positive at $t = 3.71$. The definition of the DPP asks for the first t with $A(t) \geq 0$, and $A(t)$ for $t < 6$ does not involve a payment made at $t = 6$. So, the cost cannot affect when the crossing happened.

It would change the answer in either of two cases:

- if it fell *before* the crossing, since it would then be one of the terms in $A(t)$ and would delay the crossing;
- if it were large enough to push the balance back below zero afterwards, in which case the first crossing is no longer the last, and one must ask which crossing the question means.

The second is the blip of Example 2, with the signs reversed.