LECTURE 23

Borrowing dearer than lending

Tracking the balance, and the date it turns

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One rate, all the way through

$$\mathrm{NPV}(i) \;=\; \sum_t \mathsf{v}^t c_t + \int_0^T \mathsf{v}^t \rho(t)\,\mathrm{d}t$$

The same i borrowed and lent alike thus far

Every appraisal so far has rested on a *single* i. IRL, an investor faces two, and today, we drop it that assumption.

Today

1. Two rates
2. Three projects
3. The discounted payback period

Warm-up

EXERCISE 23.1

A company can borrow at 8% *and lend at* 5%*. It borrows* €50,000.00 *to fund a project returning* €20,000.00 *at the end of each of three years. Repayments and lendings may be made at years end's. What is the company's balance after three years?*

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	interest	receipt	balance owed
End Y1	4.000	20	$50 + 4 - 20 = 34.000$
End Y2	2.720	20	$34 + 2.72 - 20 = 16.720$
End Y3	1.338	20	$16.72 + 1.338 - 20 = -1.942$

The balance, and thus the project, turns positive during the third year, and subsequenlty, it can not lend at all: Nothing but the *borrowing* rate was used as the balance was never in credit. The debt changed sign somewhere between years 2 and 3, which we will talk a bit more of today.

SECTION 1

Two rates

Dropping the last assumption

We have assumed so far that the investor may borrow or lend money at the same rate of interest i. In practice the investor will probably have to pay a higher rate j_1 on borrowings than the rate j_2 received on investments.

The difference $j_1 - j_2$ depends on various factors, including the credit-worthiness of the investor and the expense of raising a loan.

> **NO SINGLE $i \iff$ NO NPV(i) OR YIELD**
>
> The concepts of net present value and yield are in general no longer meaningful in these circumstances. As there is no single i, we cannot simply consider NPV(i), and hence nothing whose root could be a yield.

What to do instead

We must calculate the accumulation of net cashflows *from first principle*, the rate depending on whether the investor's account is in credit.

> **CHARGE j_1 WHEN IN LOSS, CREDIT j_2 WHEN IN PROFIT**
>
> In many practical problems the balance, meaning the accumulation of net cashflows, is negative until a certain time t_1 and positive afterwards. So, use the rate j_1 while the balance is negative, and the rate j_2 once it is positive.

In some cases the investor must finance the project by a *fixed-term loan with no early repayment option*. A positive cashflow then cannot be used to repay the loan gradually; it must instead be accumulated at the lending rate j_2.

That distinction changes the answer, as Projects C and E below will show.

SECTION 2

Three projects

The question

EXAMPLE 23.2

Consider Projects C, D and E over five years, financed by loans. Loan repayments for C may be made at the end of each year; for D and E only annual interest repayments may be made until the projects end. Rate for money borrowed is 6.25% but for money deposited it is only 4%.

All outlays are at years start. All proceedings are at years end.

	Project C		Project D		Project E	
Timing	Outlay	Proceeds	Outlay	Proceeds	Outlay	Proceeds
Y1	70	10	100		80	10
Y2	30	20			20	30
Y3	20	40			5	87
Y4	10	80				
Y5				140		

All amounts in €0.00s.

Project C: The running balance

Borrow to cover the negative net cashflows. Since $j_1 = 0.0625 > 0.04 = j_2$, it pays to use positive net cashflows to make loan repayments *immediately* rather than to invest them.

Note that "start of year $r{+}1$" and "end of year r" are the same date, so the two cashflows there combine.

$$\textbf{End Y1:}\quad -70(0.0625) - 30 + 10 \;=\; -24.375 \quad\Longrightarrow\quad \text{borrow}\,,$$

giving a loan total of $70 + 24.375 = 94.375$.

$$\textbf{End Y2:}\quad -94.375(0.0625) - 20 + 20 \;=\; -5.898 \quad\Longrightarrow\quad \text{borrow}\,,$$

giving a loan total of $94.375 + 5.898 = 100.273$.

Project C: the last two years

End Y3: $-100.273(0.0625) - 10 + 40 = 23.733 \implies$ make a repayment,

leaving $100.273 - 23.733 = 76.540$ outstanding.

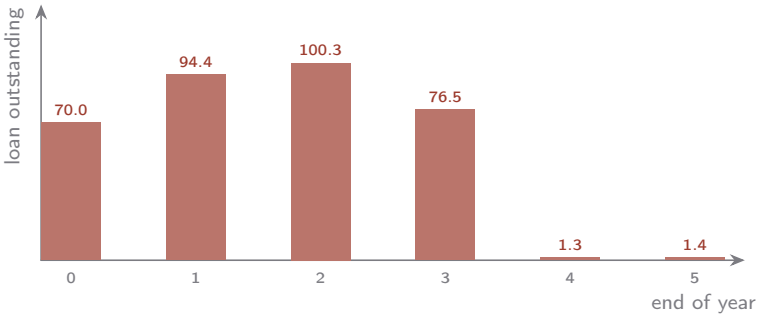
End Y4: $-76.540(0.0625) + 80 = 75.216 \implies$ make a repayment,

leaving $76.540 - 75.216 = 1.324$ outstanding!

PROJECT C IS NOT PROFITABLE.

The company is **unable to repay the loan**, with a balance of 1.324 remaining, which by the end of year 5 has grown to $1.324(1.0625) = 1.407$.

Project C, drawn



The debt rises for two years, falls sharply in years 2-3 and 3-4, but never quite reaches zero. With no further proceeds, the debt will keep growing.

Project D

Project D has a single outlay of 100 at the start of year 1 and a single receipt of 140 at the end of year 5. With no income before the end, the company must borrow to cover the interest repayments too, so the loan simply accumulates at j_1 for five years. After repaying it, the accumulated profit at the end of year 5 is

$$-100(1.0625)^5 + 140 \;=\; -135.408 + 140 \;=\; 4.592.$$

Project D is profitable.

> KEY IDEA
>
> Note that j_2 never appears, as the company holds no positive balance at any point before $t = 5$, so the lending rate has nothing to act on.

Project E: The first two years

Borrow to cover the negative net cashflows; but since no capital repayments are permitted, positive net cashflows must be *invested* at $j_2 = 4\%$.

$$\textbf{End Y1:} \quad -80(0.0625) - 20 + 10 \;=\; -15 \quad \Longrightarrow \quad \text{borrow}\,,$$

giving a loan total of $80 + 15 = 95$.

$$\textbf{End Y2:} \quad -95(0.0625) - 5 + 30 \;=\; 19.063 \quad \Longrightarrow \quad \text{invest}\,.$$

The loan total stays at 95, but we can not pay it back yet. We can, however, cover the interest with the gains, and will, of course, do that, as borrowing is dearer to us than investing. Thus, the investments stand at 19.063.

Project E: The finish

End Y3: $-95(0.0625)+87 = 81.063 \implies$ repay the loan, invest the balance.

The balance at the end of Y3 is

$$\underbrace{19.063(1.04)}_{\text{gain Y2 accumulated to Y3}} + \underbrace{81.063}_{\text{gain Y3}} - \underbrace{95}_{\text{unrepaid capital}} = 5.888.$$

This is a positive number and earns at rate j_2.

End Y5: accumulated profit $= 5.888(1.04)^2 = 6.368.$

EASY MISTAKE TO MAKE:

We note that Project E finishes in Y3, not Y5, so we can pay back at the end of Y3. As the other projects ended in Y5, and we evaluate the projects in Y5, this is an easy detail to miss in an exam situation.

The three projects compared

	Accumulated profit at end Y5	Verdict
Project C	—	loan not repaid; **not profitable**
Project D	4.592	profitable
Project E	6.368	**most profitable**

PROJECT E IS THE MOST PROFITABLE.

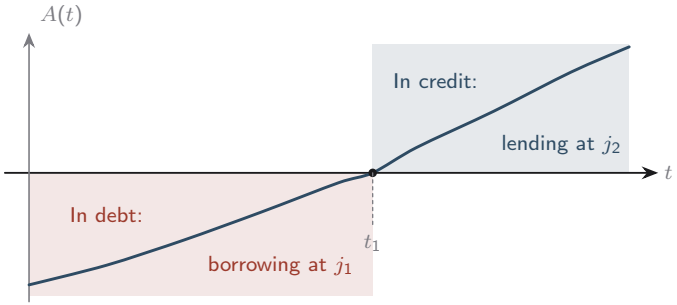
Both rates appear in Project E: 6.25% on the debt, 4% on the surplus. We compare them on a common date, despite the projects finishing on different times.
What has changed from Lecture 21 is that we can no longer get there by discounting at a single rate; the balance must be rolled forward year by year, at whichever rate applies.

SECTION 3

The discounted payback period

One sign change

In many practical problems the net cashflow changes sign only once, from negative to positive. The balance in the investor's account then changes sign at a unique time t_1, or is always negative. Of course, in the latter case, the project is not viable.



The warm-up was exactly this picture, sampled at three integer times.

Discounted payback period

DEFINITION 23.3 · Discounted payback period

The **discounted payback period** (DPP), if it exists. is the smallest t_1 such that

$$A(t_1) \;=\; \sum_{s \le t_1} (1+j_1)^{t_1-s} c_s + \int_0^{t_1} (1+j_1)^{t_1-s} \rho(s)\,\mathrm{d}s \;\geq\; 0.$$

This is the t_1 from the picture.

> KEY IDEA
>
> Note that t_1 depends only on j_1, the rate at which the investor can *borrowing*, as until t_1 the account was never in credit and the lending rate had nothing to act on.

If $A(T) < 0$, where T is the time the project ends — equivalently if $\mathrm{NPV}(j_1) < 0$ — then the project has no discounted payback period and is not profitable.

The undiscounted payback period

If the interest j_1 is ignored, that is if we set $j_1 = 0$ in

$$A(t_1) \;=\; \sum_{s \le t_1} (1+j_1)^{t_1-s} c_s + \int_0^{t_1} (1+j_1)^{t_1-s} \rho(s)\,\mathrm{d}s \;\ge\; 0,$$

the resulting period is called simply the **payback period**.

> CAUTION
>
> Its use instead of the discounted payback period often leads to erroneous results and is therefore **not to be recommended**. It pretends the borrowed money costs nothing, so it always reports a shorter, and sometimes a spurious, payback.

Profit after the payback period

If the project is viable, so that a discounted payback period t_1 exists, then the accumulated profit when the project ends at time T is

$$A(T) \;=\; (1+j_2)^{T-t_1}A(t_1) + \sum_{t\geq t_1}(1+j_2)^{T-t}c_t + \int_{t_1}^{T}(1+j_2)^{T-t}\rho(t)\,\mathrm{d}t,$$

since the net cashflow is accumulated at rate j_2 once the discounted payback period has elapsed.

> KEY IDEA
>
> All else being equal, the project with the *shortest* DPP is generally recommended, as it generates profit sooner, and thus, less is at risk for less time.

Lecture 23 on one slide

TWO RATES, AND THE DATE BETWEEN THEM

$$A(t_1) \geq 0 \text{ first at } t_1 \quad (\text{computed at } j_1 \text{ alone}),$$

$$A(T) = (1+j_2)^{T-t_1} A(t_1) + \sum_{t \geq t_1} (1+j_2)^{T-t} c_t + \int_{t_1}^{T} (1+j_2)^{T-t} \rho(t)\, \mathrm{d}t.$$

With $j_1 \neq j_2$, NPV and yield lose their meaning. Instead, roll the balance forward year by year at whichever rate applies, and compare accumulated profits at a common date. Where a single sign change existss, the payback date is set by the borrowing rate, and everything after it is set by the lending rate.

Exercises

EXERCISE 23.4

Repeat the warm-up assuming no capital repayments are permitted, so surpluses must be deposited at 3%. Compare with the warm-up and explain the difference.

EXERCISE 23.5

Why did the lending rate j_2 play no part in Project D, and what would have to change for it to matter?

EXERCISE 23.6

Explain why the discounted payback period depends on j_1 but not on j_2, and why the accumulated profit depends on both.

Solutions are in the backup slides.

END OF LECTURE 23

The date the balance turns is now defined. Next: four ways of actually finding it.

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Solution 1

With no repayments the debt of 50 accumulates at 8% throughout, reaching

$$50(1.08)^3 \;=\; 62.986,$$

while the three receipts are deposited at 3% as they arrive:

$$20(1.03)^2 + 20(1.03) + 20 \;=\; 21.218 + 20.600 + 20 \;=\; 61.818.$$

Now $61.818 < 62.986$, so the project makes a *loss* of 1.168, against a profit of 1.942 in the warm-up.

The whole difference is the gap between j_1 and j_2: being forced to hold cash at 3% while paying 8% on the debt costs more than the project earns. The cashflows are identical; only the financing terms changed.

Solution 2

Project D has no positive balance at any time. There is a single outlay at the start and a single receipt at the very end, and that receipt is used at once to clear the loan, so the account is in debt from the first day to the last.

For j_2 to matter, there would have to be a surplus held on deposit at some point. That would require an earlier receipt, and one large enough to exceed the interest then due, since a receipt that merely covers the interest leaves the balance unchanged rather than positive.

Project E is exactly that case: its receipt of 30 at the end of year 2 exceeds the 5.9375 of interest plus the 5 of outlay, so 19.063 goes on deposit and 4% begins to act.

Solution 3

Before t_1 the account has never been in credit, so no money has ever been lent and j_2 has had nothing to act on: the balance up to t_1 is governed entirely by j_1. The date at which that balance first reaches zero is therefore a function of j_1 alone.

After t_1 the account is in credit and grows at j_2. The final profit therefore depends on *both* rates — on j_1 through the date t_1 and the balance reached there, and on j_2 through the accumulation thereafter.

A useful check: raising j_1 pushes t_1 later and lowers the profit; raising j_2 leaves t_1 untouched and raises the profit.