LECTURE 22

# Comparing projects

## Why the higher yield need not win

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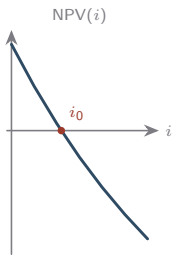
# One project, settled




**FROM LECTURE 21**

With $i$ the rate at which the investor may lend or borrow, and the yield $i_0$ a simple sign change, the project is profitable *if and only if*

$$(i - i_0) \cdot \mathrm{NPV}'(i_0) \; > \; 0.$$

For the usual decreasing NPV this just says $i < i_0$.

That settles whether *one* project is worth doing. It does not settle which of *two* is better.

# Today

1. Comparing two projects
2. Two ways to build a system
3. Two loans

## Warm-up

*EXERCISE 22.1*

*Two ventures have net present values*

$$\mathrm{NPV}_P(i) \;=\; -100 + 130\,\mathrm{v}, \qquad \mathrm{NPV}_Q(i) \;=\; -100 + 145\,\mathrm{v}^2.$$

*Find the yield on each. Which venture would you rather have?*

## Warm-up

*EXERCISE 22.1*

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*Find the yield on each. Which venture would you rather have?*

Setting each to zero, $\mathrm{v} = \frac{100}{130}$ and $\mathrm{v}^2 = \frac{100}{145}$, so

$$i_P \;=\; 1.3 - 1 \;=\; 30\%, \qquad i_Q \;=\; \sqrt{1.45} - 1 \;=\; 20.4\%.$$

> ASIDE
>
> Venture $P$ has a much higher yield, and the natural answer to the second question is $P$.

SECTION 1

# Comparing two projects

## The comparison of two investment projects

Suppose an investor is comparing the merits of two possible investments or business ventures, which we call projects A and B. We assume that the borrowing power of the investor is **not limited**.

Let $\text{NPV}_A(i)$ and $\text{NPV}_B(i)$ denote the respective net present value functions, and let $i_A$ and $i_B$ denote the respective yields, which we assume to exist.

Recall that yield of a Project $P$ exists if $\text{NPV}(i)$ has a unique root.

> CAUTION
>
> It might be thought that the investor should always select the project with the higher yield. This is *not* necessarily the best strategy.

## The right criterion

A better criterion is the profit at time $T$, where $T$ is the date when the later of the two projects ends. Equivalently, the net present value, calculated at the rate of interest $i_1$ **at which the investor may actually lend or borrow**.

> PROJECT $A$ IS MORE PROFITABLE THAN PROJECT $B$ AT RATE $i_1$ IF
>
> $$\mathrm{NPV}_A(i_1) > \mathrm{NPV}_B(i_1).$$

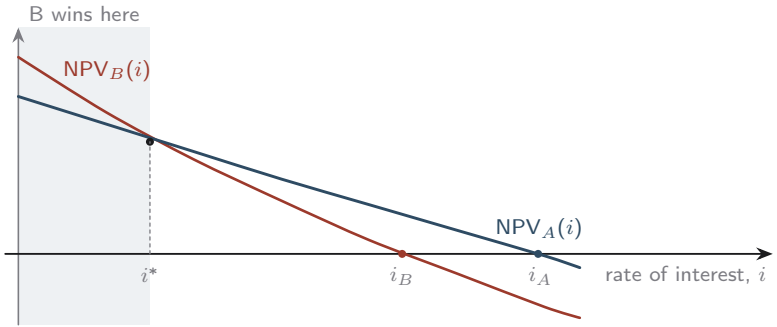
The point of this lecture is that

$$i_A > i_B \qquad \text{need } \textit{not} \text{ imply} \qquad \mathrm{NPV}_A(i_1) > \mathrm{NPV}_B(i_1).$$

Why? Because two decreasing curves can cross.

In other words, it is entirely possible that for $i \neq j$, $\mathrm{NPV}_A(i) > \mathrm{NPV}_B(i)$ but $\mathrm{NPV}_A(j) < \mathrm{NPV}_B(j)$.

# The argument, illustrated




Project $A$ has the higher yield as $i_A > i_B$, but the two curves do *cross over*. The rate $i^*$ at which they intersect is called the **cross-over rate**. For every $i_1 < i^*$, we have $\text{NPV}_B(i_1) > \text{NPV}_A(i_1)$ and for rates in that particular region, Project $B$, which happens to have the *lower* yield, is the more profitable.

## The warm-up, resolved

The two ventures of the warm-up cross where $130\mathrm{v} = 145\mathrm{v}^2$, that is at $\mathrm{v} = \frac{130}{145}$, so

$$i^* \;=\; \frac{145}{130} - 1 \;=\; 11.54\%.$$

For any rate below $i^*$, Project $Q$ is the better venture. For any rate above $i^*$, Project $P$ is preferred. For instance at 10%, Project $Q$ has a higher value as

$$\mathrm{NPV}_P(0.10) \;=\; -100 + \frac{130}{1.1} \;=\; 18.18, \qquad \mathrm{NPV}_Q(0.10) \;=\; -100 + \frac{145}{1.21} \;=\; 19.83.$$

YIELD IS INTRINSIC TO THE PROJECT, AND DOES NOT FACTOR IN THE INVESTOR

If the investor's choice is limited to 10%, $Q$ is preferable, despite yields of $i_P = 30\%$ against $i_Q = 20.4\%$. The yield is a property of the project alone; the profit depends on the project *and* the rate at which the investor.

SECTION 2

# Two ways to build a system

# The question

EXAMPLE 22.2

A software company is to set up a new computer system for a client. It considers whether to contract out the work (Project A) or do it in-house (Project B). The company can borrow or lend at 20% per annum. Would you advise it to invest in either project, and if so which is the more profitable?

| **Project A** | | | **Project B** | | |
|---|---|---|---|---|---|
| Description | Timing | Amount | Description | Timing | Amount |
| Contractor fees | Start Y1 | ⟨150⟩ | New equipment | Start Y1 | ⟨325⟩ |
| Contractor fees | Start Y2 | ⟨250⟩ | Staff costs | Through Y1 | ⟨75⟩ |
| Contractor fees | Start Y3 | ⟨250⟩ | Staff costs | Through Y2 | ⟨90⟩ |
| Sales | End Y3 | 1,000 | Staff costs | Through Y3 | ⟨120⟩ |
| | | | Sales | End Y3 | 1,000 |

All amounts in €1,000.00s; brackets indicate expenditure. "*Throughout*" indicates a continuous cashflow.

## Project A, from Lecture 10

Project A is the project we priced and solved in Lecture 10. Its cashflows are discrete, so with $v = 1/1.2 = 0.8\dot{3}$ and amounts in €0.00s,

$$\text{NPV}_A(i) \;=\; -150 - 250v - 250v^2 + 1000v^3, \qquad \text{NPV}_A(0.2) \;=\; 46.76.$$

> LECTURE 10
>
> Interpolating between $\text{NPV}_A(0.25) = 2$ and $\text{NPV}_A(0.26) = -5.977$ gave the yield $i_A \approx 0.2525$.

So, let us analyse Project B.

## Project B: a continuous outgo

The staff costs are paid *through* each year, so they are continuous payment streams. These are at rates 75 during year 1, 90 during year 2 and 120 during year 3. Each year's stream is worth $\bar{a}_{\overline{1}|}$ times its rate, valued at the *start* of that year, so

$$\mathrm{NPV}_B(i) \;=\; -325 - \left(75 + 90\mathrm{v} + 120\mathrm{v}^2\right)\bar{a}_{\overline{1}|} + 1000\mathrm{v}^3.$$

KEY IDEA

From Lecture 7, $\bar{a}_{\overline{1}|} = \dfrac{1-\mathrm{v}}{\delta} = \dfrac{d}{\delta}$, being the one-year continuous annuity. The factors $1, \mathrm{v}, \mathrm{v}^2$ then discount each year's block back to time 0.

## Project B at 20%

With $v = 0.8\dot{3}$ and $\delta = \ln(1.2) = 0.1823216$,

$$\bar{a}_{\overline{1}|} = \frac{1 - 0.8\dot{3}}{\ln(1.2)} = 0.914136, \qquad 75 + 90(0.8\dot{3}) + 120(0.8\dot{3})^2 = 233.3333.$$

Therefore

$$\text{NPV}_B(0.2) = -325 - 233.3333 \times 0.914136 + 1000(0.8\dot{3})^3 = 40.41.$$

Since $\text{NPV}_A(0.2) = 46.76 > 0$ and $\text{NPV}_B(0.2) = 40.41 > 0$, both projects are profitable. At this given rate, Project A is the *more* profitable one.

## The yield on Project B

We know $\text{NPV}_B(0.2) = 40.41 > 0$. Trying $i = 0.25$, where $v = 0.8$ and $\delta = \ln(1.25)$,

$$\text{NPV}_B(0.25) \;=\; -325 - 223.8 \times 0.896284 + 512 \;=\; -13.59 \;<\; 0.$$

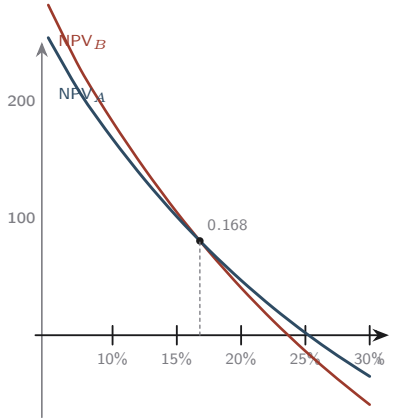
Linear interpolation then gives

$$i_B \;\approx\; 0.2 + \frac{40.41}{40.41 + 13.59}(0.05) \;=\; 0.237.$$

So $0.2 < i_B < i_A$. Both yields exceed the rate at which the company may lend or borrow, confirming that both are worthwhile.

> CAUTION
>
> $Thebracket$[0.2, 0.25] is wide, so this estimate is fairly rough. The true value is 0.2366.

# How the borrowing rate decides it




The curves cross at about 16.8%.

- At $i = 10\%$: $\text{NPV}_B = 182.1$ against $\text{NPV}_A = 167.4$, so **B** is better, despite its *lower* yield.
- At $i = 24\%$, where $i_B < 0.24 < i_A$, B produces a loss, A a profit.

The rate at which the company can borrow or lend is the decisive factor.

# Summary of project evaluations

KEY IDEA

- Compute the NPV of each project from its cashflow model.
- Compute the yield of each, and check profitability at the borrowing/lending rate.
- Choose the project with the highest NPV *at that rate*.

The first two answer if the projects are worth doing at all. It is only the third that ranks them.

SECTION 3

# Two loans

# The question

EXAMPLE 22.3

An investor considers whether to invest in either or both of the following loans.

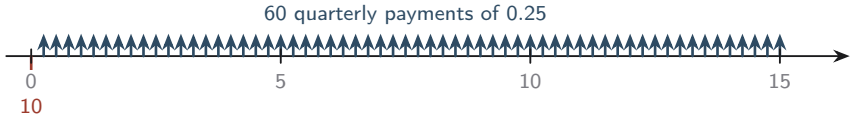
LOAN $X$ For a purchase price of €10,000.00, the investor receives €1,000.00 per annum payable quarterly in arrears for 15 years.

LOAN $Y$ For a purchase price of €11,000.00, the investor receives €605.00 annually in arrears for 18 years, and the return of the initial investment at the end.

The investor may borrow or lend at 4% per annum. Is either loan worth investing, and if so which is more profitable?

We work throughout in €1,000.00s.

# Loan X

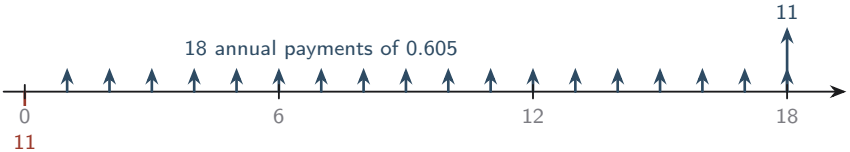



A level annuity of 1 per annum payable quarterly in arrears for 15 years, so $\text{NPV}_X(i) = -10 + a^{(4)}_{\overline{15}|}$. At $i = 0.04$, $i^{(4)} = 0.039414$ and $\text{v}^{15} = 0.555265$, giving

$$a^{(4)}_{\overline{15}|} \;=\; 11.2838, \qquad \text{NPV}_X(0.04) \;=\; 1.2838.$$

Interpolating between 5% and 6% gives $i_X \approx 0.0589$.

# Loan Y




Annual payments, a longer term, and a return of capital at the end:

$$\text{NPV}_Y(i) \;=\; -11 + 0.605\, a_{\overline{18}|} + 11\text{v}^{18}.$$

At $i = 0.04$, $a_{\overline{18}|} = 12.659297$ and $\text{v}^{18} = 0.493628$, giving $\text{NPV}_Y(0.04) = 2.08878$.

# The yield on Loan Y, exactly

Interpolating between 5% and 6% gives $i_Y \approx 0.0552$, but here we can do better.

KEY IDEA

Loan Y returns exactly the capital it cost, so it is a bond redeemable *at par*. By Lecture 16 its yield is simply its coupon rate, $0.605/11 = 0.055$. Indeed at $i = 0.055$ we have

$$0.605\, a_{\overline{18}|} \;=\; 11\left(1 - \mathrm{v}^{18}\right),$$

so

$$\mathrm{NPV}_Y(0.055) \;=\; -11 + 11 - 11\mathrm{v}^{18} + 11\mathrm{v}^{18} \;=\; 0$$

exactly.

So $i_Y = 5.5\%$ exactly, and the interpolated 0.0552 was high by two ten-thousandths.

# The verdict

| | Loan X | Loan Y |
|---|---|---|
| Yield | 0.0589 | 0.0550 |
| NPV at 4% | 1.2838 | 2.0888 |

1. Is either loan worth investing? **Yes, both**, as each yield exceeds 4%.
2. Which is more profitable? **Loan Y**, even though its yield is the *lower* of the two.

CAUTION

Again the borrowing rate decides. Both NPVs decrease in $i$, and $i_Y < 0.057 < i_X$, so at 5.7% Loan X is still profitable ($\text{NPV}_X = 0.115$) while Loan Y makes a loss ($\text{NPV}_Y = -0.244$).

# Lecture 22 on one slide

**RANK BY PROFIT, NOT BY YIELD**

$$\text{A is more profitable than B at rate } i_1 \iff \text{NPV}_A(i_1) > \text{NPV}_B(i_1).$$

$$i_A > i_B \not\Longrightarrow \text{NPV}_A(i_1) > \text{NPV}_B(i_1) \text{ for every } i_1.$$

Remember, the curves may cross over

The depends on the project *alone*; the ranking takes the investor into account. In other words, for an *unrestricted* investor, the project with a higher yield is better, but in general, an investor has constraints on the rate at which money can actually be borrowed or lent. *Below the cross-over rate*, the project with the lower yield wins.

# Exercises

*EXERCISE 22.4*

*A loan costs* €5,000.00 *and pays* €300.00 *annually in arrears for* 20 *years, with the* €5,000.00 *returned at the end. Write down its yield without calculating, and justify it.*

*EXERCISE 22.5*

*In the software example, verify that* $\text{NPV}_A$ *and* $\text{NPV}_B$ *agree at* $i^* = 0.168$*, and say which project you would choose at* $i = 15\%$*.*

*EXERCISE 22.6*

*Two projects have the same yield* $i_0$*. Can one still be preferable to the other? Justify your answer with a sketch.*

Solutions are in the backup slides.

# END OF LECTURE 22

Both comparisons assumed one rate for borrowing and lending. That is rarely the case. Next: what happens when a loan costs more than a deposit earns.

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## Solution 1

The loan returns exactly its purchase price, so it is a bond redeemable at par, and its yield is the coupon rate:

$$\frac{300}{5000} \;=\; 6\%.$$

The reason no calculation is needed: at $i = 0.06$ the income of €300.00 a year is exactly the interest due on €5,000.00, so the capital is neither being repaid nor eroded. It is returned intact at the end, and the investor has earned precisely 6% throughout.

Formally, $300\, a_{\overline{20}|} = 5000\left(1 - v^{20}\right)$ at 6%, so
$\mathrm{NPV}(0.06) = -5000 + 5000 - 5000v^{20} + 5000v^{20} = 0.$

## Solution 2

At $i = 0.168$ both net present values come to 80.36 per €0.00, so the curves do indeed meet there.

At $i = 15\%$, which is *below* $i^*$, we have

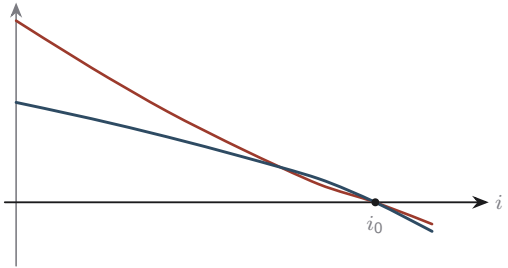
$$\text{NPV}_B(0.15) \;=\; 104.8 \qquad \text{against} \qquad \text{NPV}_A(0.15) \;=\; 101.1,$$

so choose **Project B**, even though $i_A = 0.2525 > 0.2366 = i_B$.

The pattern is the general one: B's outgo is spread through the three years while A's falls in three lumps at the year starts, so B suffers less when money is cheap and more when it is dear.

## Solution 3

Yes. The yield fixes only where a curve crosses the axis; it says nothing about where the curve is anywhere else.




Two projects sharing a yield still have different NPV curves, and at every rate below $i_0$ the upper one is worth more. Equal yields mean the two are equally *marginal*, not equally *good*.