LECTURE 19

# Securities with no redemption date

Equities, property, and income that grows for ever

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
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# Recall and outlook

$$P_0 \;=\; D\, a_{\overline{T}|}^{(m)} + R\mathrm{v}^T$$

Even when $T$ was the borrower's to choose, it lay between $[T_1, T_2]$.

Our securities so far always ended. Today, we examine two securities with no $T$ and no $R$ at all. The capital is never returned, so the whole value must come from an income stream that never stops. We will have to make some assumptions.

# Today

1. Equities
2. The dividend growth model
3. Property

## Warm-up

*EXERCISE 19.1*

*An investor requires* 9% *p. a. What is the value of an income of* €1.80 *a year, payable annually in arrears, in perpetuity?*

We discussed this in Lecture 6, briefly. Section 2 will deal with what happens when that income is allowed to grow.

## Warm-up

*EXERCISE 19.1*

*An investor requires* 9% *p. a. What is the value of an income of* €1.80 *a year, payable annually in arrears, in perpetuity?*

A level perpetuity in arrears is worth $1/i$ per unit of income (Lecture 6), so

$$1.80\, a_{\overline{\infty}|} = \frac{1.80}{0.09} = €20.00.$$

> ASIDE
>
> Recall why: €20.00 in the bank at 9% earns €1.80 a year, which can be withdrawn forever while the capital stays intact.

This is the whole idea of this lecture. An equity or a property are exactly this, except that the income grows.

SECTION 1

# Equities

# What an equity is

DEFINITION 19.2 · Equity

**Equities**, or **ordinary shares**, are securities issued by companies which entitle holders to receive all of the net profits of the company after interest on loans and fixed-interest stocks is paid.

Shareholders *own* the company and have voting rights. Two features distinguish an equity from every security so far:

- **Redemption.** There is no redemption rate and no redemption date. Capital is "redeemed" by selling the shares to someone else, and the profit or loss is the difference between the sale price and the purchase price.
- **Income.** There is no fixed rate of income. Part of the net profit is distributed annually as **dividends**; the rest is retained as reserves.

# Why equities are harder to value

Dividends fluctuate from year to year, and payment is at the directors' discretion, and can even be zero.

- Total return = dividend income + capital gain (or loss). Both parts are volatile.
- There is a risk of dividend reductions or even outright default. The latter means the capital is lost entirely.
- Investors buy equities expecting dividends and capital values to increase over time.

**OUR MODELING OF STOCKS IS NOT THE "REAL" PICTURE**

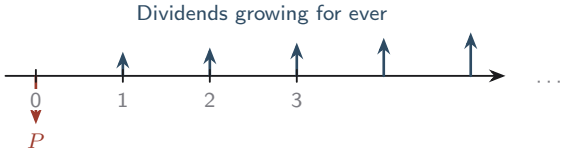
Future cashflows are *unknowable*, and can only properly be modelled **stochastically**. This is outside the scope of this module. We will assume a growth rate for the dividends, which gives a **deterministic** cashflow model instead.

# The dividend model

Suppose an equity has value $P$ immediately after the latest dividend payment $D$. Assume dividends are paid annually, and that the dividend at time $t$ is estimated to be $D_t$. We value the equity assuming the dividends continue *in perpetuity*, and without explicit allowance for default. This gives us

$$P = \sum_{t=1}^{\infty} D_t \, \mathrm{v}^t,$$

where $\mathrm{v} = (1+i)^{-1}$ and $i$ is the return on an investment at price $P$.



SECTION 2

# The dividend growth model

## Constant dividend growth

If a constant dividend growth rate $g$ is assumed, then $D_t = D(1+g)^t$ and

$$P = \sum_{t=1}^{\infty} D_t \, \mathrm{v}^t.$$

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THE TRICK

Write $\tilde{v} = \dfrac{1+g}{1+i}$. This is a discount factor for a *new* rate $j$. Setting $\tilde{v} = \dfrac{1}{1+j}$ gives

$$1+j \;=\; \frac{1+i}{1+g} \quad\Longrightarrow\quad j \;=\; \frac{(1+i)-(1+g)}{1+g} \;=\; \frac{i-g}{1+g}.$$

So growth rate $g$ and discount rate $i$ combine into discounting at the single *net* rate $j$.

## When does the sum converge?

The geometric series $\sum_{t\geq 1} \tilde{v}^t$ converges exactly when $\tilde{v} < 1$, that is when $j > 0$ (we assume $1 + j \geq 0$). Unpacking the condition gives us

$$j > 0 \iff \frac{1+i}{1+g} > 1 \iff 1 + i > 1 + g \iff i > g,$$

again under the assumption $1 + g > 0$. Summed up: $\boxed{\tilde{v} < 1 \iff -1 < g < i}$.

**CAUTION**

If $g \geq i$, the series diverges and the model returns an infinite price. That is not "an arithmetic failure": If dividend stream is growing forever as fast as the discount rate or even faster would indeed be an unbounded amount of vlaue.

You could compare two such streams (e.g., take difference or ratio), but you could not value it.

## The valuation formula

With $j > 0$ the sum is a perpetuity at rate $j$, that is Lecture 6's $a_{\overline{\infty}|} = 1/j$

$$P \;=\; D\sum_{t=1}^{\infty} \tilde{\mathrm{v}}^{\,t} \;=\; D\,\frac{\tilde{\mathrm{v}}}{1-\tilde{\mathrm{v}}} \;=\; D\,\frac{\frac{1}{1+j}}{1-\frac{1}{1+j}} \;=\; \frac{D}{(1+j)-1} \;=\; \frac{D}{j},$$

where we just cleared the fractions in the last step. Hence

$$\boxed{P \;=\; D\,a_{\overline{\infty}|} \text{ at rate } j \quad = \quad \frac{D}{j} \;=\; D\left(\frac{1+g}{i-g}\right)}$$

EXAMPLE 19.3

The warm-up share has just paid €1.80, with $g = 3\%$ and $i = 9\%$. Then $j = 0.06/1.03 = 0.0582524$ and $P = 1.80/0.0582524 = €30.90$. This is against €20.00 with no growth at all.

# Ex-dividend

At certain times close to the dividend payment date, an equity may be offered for sale *excluding* the next dividend. This allows for the fact that there may not be time between the sale date and the payment date for the company to adjust its records so that the buyer receives it.

DEFINITION 19.4 · Ex-dividend

An equity offered for sale without the next dividend is called **ex-dividend**, or **xd**.

> KEY IDEA
>
> Valuing an ex-dividend stock requires no new principles. We simply omit the dividend the buyer will not receive, and value what is left.

# An ex-dividend purchase

EXAMPLE 19.5

On 30 April 2020 an investor purchased 1000 shares in a company *ex dividend*. Dividends are paid annually and the first dividend of €2.50 per share was paid five months after the date of purchase. Dividends are expected to rise by 4% each calendar year. The investor pays tax at 30% on 31 March on all income received during the previous **calendar** year. Calculate the price paid if the expected net yield is 10% per annum.

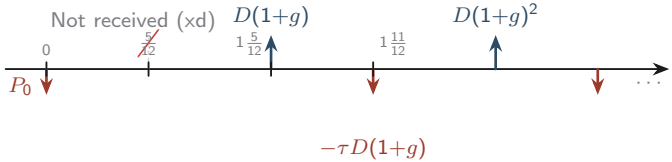
Everything given in €1,000.00s.

CAUTION

Bought *ex dividend*, so the investor does not receive that first payment of $D = 2.50$ at $t = \frac{5}{12}$. It goes to the seller. The first dividend they do receive is a year later.

# The dates

For $k \in \mathbb{N} \setminus \{0\}$, the dividend per share received at $t = k + \frac{5}{12}$ is $D(1+g)^k$ with $g = 0.04$, and the tax at $\tau = 0.3$ on it is paid at $t = k + \frac{11}{12}$.




The price is the present value, at $i = 0.1$, of the net dividends the investor *does* receive.

# The calculation

If $P_0$ is the price paid per share, then

$$P_0 = \sum_{k=1}^{\infty} D(1+g)^k \mathsf{v}^{k+5/12} - \sum_{k=1}^{\infty} \tau D(1+g)^k \mathsf{v}^{k+11/12}$$

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since $\mathsf{v}^{k+11/12} = \mathsf{v}^{k+5/12}\mathsf{v}^{1/2}$

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## The calculation

If $P_0$ is the price paid per share, then

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since $\mathsf{v}^{k+11/12} = \mathsf{v}^{k+5/12}\mathsf{v}^{1/2}$, so the tax terms differ from the dividend terms by the constant factor $\tau\sqrt{\mathsf{v}}$. Numerically,

$$2.5 \times \left(1 - 0.3\,(1.1)^{-0.5}\right) \times (1.1)^{-5/12} \times \frac{1.04}{0.06} = 29.73376,$$

so the total price for 1000 shares is €29,733.76.

SECTION 3

# Property

## Property as an investment

**Property** is an investment in land and buildings. Valuing it by discounting future income follows very similar principles to equities. Both need an assumption about the growth of future income, both have income broadly linked to inflation over the long term, and in both cases we use a deterministic approach instead of the more correct stochastic.

### THE TWO DIFFERENCES

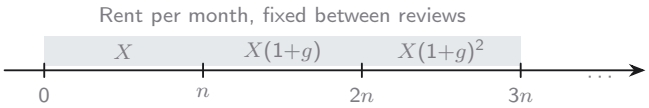
1. Property rents are generally fixed for a number of years at a time, and reviewed periodically.
2. Some contracts are fixed term. After a certain period the income ceases and ownership passes back to the original owner.

The first one dictates how the arithmetic works. Changes happen at reviews (discrete times), not continually.

# The question

EXAMPLE 19.6

A property pays rent monthly in arrears and has rent reviews every $n$ years. The most recent review has just been completed and the rent for the next $n$ years will be €$X$ per month. Rents are expected to increase by a rate $g$ compound at future rent reviews in perpetuity. Find the price required to yield $i \geq g/n$ p. a. effective.




Between reviews the rent is level, so each block is an ordinary $m$-thly annuity with $m = 12$.

## Building the sum

Let $P$ be the price. The first block pays $€X$ monthly for $n$ years, worth $12X\, a^{(12)}_{\overline{n}|}$ at time 0. The second is the same shape but $(1+g)$ times larger and $n$ years later. Thus, it contributes $12X\,(1+g)\mathrm{v}^{n}\, a^{(12)}_{\overline{n}|}$, and so on. Hence

$$\begin{aligned} P &= 12X a^{(12)}_{\overline{n}|} + 12X(1+g)\mathrm{v}^{n} a^{(12)}_{\overline{n}|} + 12X(1+g)^2\mathrm{v}^{2n} a^{(12)}_{\overline{n}|} + \cdots \\ &= 12X\, a^{(12)}_{\overline{n}|} \sum_{k=0}^{\infty} \left((1+g)\mathrm{v}^{n}\right)^{k}. \end{aligned}$$

> **CAUTION**
>
> This is again a geometric series. Before summing it *to infinity*, we must check that the common ratio $(1+g)\mathrm{v}^{n}$ is less than 1 (otherwise, it will diverge).

## Checking convergence

We need $(1+g)\mathrm{v}^n < 1$, that is $(1+i)^n > 1+g$. By the binomial theorem, for $n \geq 2$ and $i > 0$,

$$(1+i)^n \;=\; 1 + \binom{n}{1} i + \underbrace{\binom{n}{2} i^2 + \cdots + \binom{n}{n-1} i^{n-1} + i^n}_{\text{All positive}} \;>\; 1 + ni \;\geq\; 1 + g.$$

The last step follows as $i \geq g/n$ yields $ni \geq g$. Thus the series converges.

> **CAUTION**
>
> The first inequality is strict only for $n \geq 2$. At $n = 1$, $(1+i)^1 = 1 + ni$ exactly, and $i \geq g$ permits $i = g$, making the ratio 1 and the series divergent. This is the equity's failure again.

## The price

Summing the geometric series with first term 1 and ratio $(1+g)\mathrm{v}^n$ gives us

$$P \;=\; 12X\, a_{\overline{n}|}^{(12)} \sum_{k=0}^{\infty} \left((1+g)\mathrm{v}^n\right)^k \;=\; 12X\, a_{\overline{n}|}^{(12)} \, \frac{1}{1-(1+g)\mathrm{v}^n}$$

## The price

Summing the geometric series with first term 1 and ratio $(1+g)\mathrm{v}^n$ gives us

$$P \;=\; 12X\, a^{(12)}_{\overline{n}|} \sum_{k=0}^{\infty} ((1+g)\mathrm{v}^n)^k \;=\; \boxed{12X\, a^{(12)}_{\overline{n}|}\, \frac{(1+i)^n}{(1+i)^n-(1+g)}},$$

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where we multiplied the top and bottom by $(1+i)^n$ in the last step.

KEY IDEA

Compare with the equity formula $P = D\left(\frac{1+g}{i-g}\right)$. Both are growing perpetuities but the growth happenning every $n$ years here rather than every year. Hence $(1+i)^n$ replaces $(1+i)$ and the rent block $12Xa^{(12)}_{\overline{n}|}$ replaces the single dividend $D$.

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# Lecture 19 on one slide

## INCOME THAT NEVER STOPS, AND GROWS

Equity: $P = \dfrac{D}{j} = D\,\dfrac{1+g}{i-g}, \qquad j = \dfrac{i-g}{1+g}, \qquad$ valid for $-1 < g < i.$

Property: $P = 12X\, a_{\overline{n}|}^{(12)}\, \dfrac{(1+i)^n}{(1+i)^n - (1+g)}, \qquad$ valid for $(1+i)^n > 1+g.$

No redemption date and no redemption price, so the value is the whole income stream. Growth and discounting combine into one net rate. If that rate is not positive, the model returns infinity and the assumption must be revisited.

# Exercises

*EXERCISE 19.7*

*A share has just paid a dividend of* €2.40*, growing at* 4% *per annum, for an investor requiring* 10%*. Find the price, first via* $j$ *and then via* $D\frac{1+g}{i-g}$*.*

*EXERCISE 19.8*

*For the same share, the investor now pays income tax at* 20% *deducted at each dividend date. Find the price, and explain why it is exactly* 80% *of the previous answer.*

*EXERCISE 19.9*

*A property has just been reviewed and will pay* €2,000.00 *per month for five years, with reviews every five years and growth of* 15% *at each. Find the price to yield* 7% *per annum.*

Solutions are in the backup slides.

# END OF LECTURE 19

Both formulas assumed the income grows. Next: the reason it does, and what a yield means once prices themselves are moving.

ilia.pirashvili@universityofgalway.ac.ie

## Solution 1

Via the net rate,

$$j = \frac{i-g}{1+g} = \frac{0.10-0.04}{1.04} = 0.0576923, \qquad P = \frac{D}{j} = \frac{2.40}{0.0576923} = €41.60.$$

Directly,

$$D\,\frac{1+g}{i-g} = 2.40 \times \frac{1.04}{0.06} = €41.60,$$

is the same number. The two expressions differ only by clearing a fraction.

Note the size of the growth assumption. With no growth the share would be worth $2.40/0.10 = €24.00$; assuming 4% growth for ever raises that by nearly three quarters.

## Solution 2

Every dividend is multiplied by $(1-\tau) = 0.8$, and a constant factor comes straight out of the sum:

$$P \;=\; \sum_{k\geq 1}(1-\tau)D(1+g)^k \mathsf{v}^k \;=\; (1-\tau)\,D\,\frac{1+g}{i-g} \;=\; 0.8 \times 41.60 \;=\; €33.28.$$

It is exactly 80% because the tax is a constant proportion of every payment, deducted at the same moment the payment arrives. The entire cashflow is therefore scaled by 0.8, and present values scale with it.

> CAUTION
>
> This fails as soon as the tax is paid on a *different* date from the dividend, as in the ex-dividend example. There the factor was $1 - \tau\sqrt{\mathsf{v}}$, not $1 - \tau$.

## Solution 3

Here $X = 2000$, $n = 5$, $g = 0.15$ and $i = 0.07$. First check convergence: $(1.07)^5 = 1.402552 > 1.15$, so the series sums. Then

$$i^{(12)} = 0.067850, \qquad v^5 = 0.712986, \qquad a_{\overline{5}|}^{(12)} = \frac{1 - 0.712986}{0.067850} = 4.230139.$$

Hence

$$P = 12 \times 2000 \times 4.230139 \times \frac{1.402552}{1.402552 - 1.15} = 101{,}523 \times 5.553523 = €563{,}812.00.$$

The rent is €24,000.00 a year, so the property sells for over twenty-three years' rent. This only makes sense because the rent is assumed to keep rising.