LECTURE 18

The worst case in practice

Three problems on one rule

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19 October 2026 · Winter 2026

Recall from last time

$i^{(m)} > (1-\tau_1)\frac{D}{R}$	assume T_2
$i^{(m)} < (1-\tau_1)\frac{D}{R}$	assume T_1
$i^{(m)} = (1-\tau_1)\frac{D}{R}$	$P_0 = R$

If the price is given, compare P_0 with R.

The theory is complete. Today we identify the cashflows and take care over the dates, since three different taxes fall on three different days.

Today

1. The maximum price
2. The minimum net yield
3. A past exam question

Warm-up

EXERCISE 18.1

A stock bought on 1 January 2021 is redeemable at 115% *at the borrower's option in any year between 2039 and 2043 but only on 1 October. Write down* T_1 *and* T_2.

Warm-up

EXERCISE 18.1

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Measuring from 1 January 2021, the earliest redemption is on 1 October 2039 and the latest on 1 October 2043, so

$$T_1 \;=\; 18 + \tfrac{9}{12} \;=\; 18.75, \qquad T_2 \;=\; 22 + \tfrac{9}{12} \;=\; 22.75.$$

The date $T_1 = 18.75$ is the one used in Lectures 16 and 17, where this stock was priced at €64.19 for a net yield of 10%. Now the borrower has four further years in which to choose.

EXERCISE 18.1

A stock bought on 1 January 2021 is redeemable at 115% *at the borrower's option in any year between 2039 and 2043 but only on 1 October. Write down* T_1 *and* T_2*.*

Measuring from 1 January 2021, the earliest redemption is on 1 October 2039 and the latest on 1 October 2043, so

$$T_1 = 18 + \tfrac{9}{12} = 18.75, \qquad T_2 = 22 + \tfrac{9}{12} = 22.75.$$

ASIDE

Both are integer multiples of $\frac{1}{4}$, so the symbols $a^{(4)}_{\overline{18.75|}}$ and $a^{(4)}_{\overline{22.75|}}$ are well defined for quarterly coupons. Had redemption fallen on 1 November, T would have been $18.8\overline{3}$, and $4T$ would not be an integer.

SECTION 1

The maximum price

The question

EXAMPLE 18.2

A fixed-interest stock pays a coupon of 8% per annum quarterly, redeemable at 115% at the borrower's option on any 1 October between 2039 and 2043. Calculate the price an investor, liable to income tax at 30% and capital gains tax at 40%, would pay on 1 January 2021 for an effective net yield of *at least* 10% per annum.

So $i_0 = 0.1$, $D = 0.08$, $R = 1.15$, $m = 4$, $\tau_1 = 0.3$, $\tau_2 = 0.4$, and $T_1 = 18.75 \leq T \leq 22.75 = T_2$ from the warm-up.

The test picks the date

Before capital gains tax, the price at rate i and term T is

$$P(i,T) \;=\; (1-\tau_1)D\,a^{(4)}_{\overline{T}|} + R\mathrm{v}^T \;=\; R - \left(i^{(4)} - (1-\tau_1)\frac{D}{R}\right) R\,a^{(4)}_{\overline{T}|}.$$

Since from Lecture 16 we already have

$$i^{(4)} \;=\; 4\left((1.1)^{1/4}-1\right) \;=\; 0.096455 \quad > \quad 0.048696 \;=\; \frac{0.7\times 0.08}{1.15} \;=\; (1-\tau_1)\frac{D}{R},$$

there will be capital gain, and we must assume $T = T_2 = 22.75$, the *latest* possible redemption date. We also simplify the above, as we will need it in a second

$$P(i,T_2) \;=\; R - (0.096455 - 0.04866)\,R\,a^{(4)}_{\overline{22.75}|} \;=\; 1.15 - 0.054923\,a^{(4)}_{\overline{22.75}|}.$$

Allowing for capital gains tax

Only now does the second tax enter. As in Lecture 16, the liability at redemption has present value $\tau_2(R - P_0)\mathsf{v}^{T_2}$, so the price P_0 per unit nominal satisfies

$$P_0 \;=\; P(i_0, T_2) - \tau_2 \left(R - P_0\right) \mathsf{v}^{T_2}.$$

Solving this as in Lecture 16 gives

$$P_0 \;=\; \frac{P(i_0, T_2) - \tau_2 R\, \mathsf{v}^{T_2}}{1 - \tau_2 \mathsf{v}^{T_2}},$$

where $P(i_0, T_2)$ is the price from the previous slide at $i_0 = 0.1$.

> KEY IDEA
>
> The test picks the date, and only then is the formula for P_0 applied at that date. Doing these in the other order would be a mistake.

The calculation

With $\mathrm{v}^{22.75} = (1.1)^{-22.75} = 0.114371$ and $a^{(4)}_{\overline{22.75}|} = \dfrac{1 - 0.114371}{0.096454756} = 9.181806$, the price before capital gains tax is

$$P(i_0, T_2) \;=\; 1.15 - 0.054923\, a^{(4)}_{\overline{22.75}|} \;=\; 1.15 - 0.054923 \times 9.181806 \;=\; 0.645708.$$

Since $1 - 0.4\,\mathrm{v}^{22.75} = 0.954252$, allowing for capital gains tax gives

$$P_0 \;=\; \frac{0.645708 - 0.4 \times 1.15 \times 0.114371}{0.954252} \;=\; \frac{0.645708 - 0.052611}{0.954252} \;=\; 0.6215.$$

Per €100.00 nominal the price is €62.15.

What ensuring against the worst case cost

- In Lecture 16 the same stock, *fixed* to redeem at $T = 18.75$, was priced at €64.19. The investor must pay €2.04 less for the same yield to allow for redemption up to four years later.
- The €62.15 is the *maximum* price consistent with the guarantee. Paying more would leave the investor short of 10% if the borrower waits until 2043.

EXAMPLE 18.3 · What the investor actually gets

Redeemed	2039	2040	2041	2042	2043
Net yield	10.34%	10.24%	10.15%	10.07%	10.00%

SECTION 2

The minimum net yield

The same stock, the other way round

EXAMPLE 18.4

Consider the same stock. An investor liable to income tax at 30% and capital gains tax at 40% can acquire it on 1 January 2021 for €70.00 per €100.00 nominal. Calculate the minimum effective net yield per annum.

Since $R = 115 > 70 = P_0$ there will be a capital gain, so the worst case is again $T_2 = 22.75$. The equation of value per €100.00 nominal is

$$70 \;=\; (1-\tau_1)D\,a^{(4)}_{\overline{T_2}|} + R\mathrm{v}^{T_2} - \tau_2\,(R-P_0)\,\mathrm{v}^{T_2} \;=\; 5.6\,a^{(4)}_{\overline{22.75}|} + 97\,\mathrm{v}^{22.75},$$

since $115 - 0.4 \times 45 = 97$.

This is identical in shape to the warm-up of Lecture 17, with 18.75 replaced by 22.75.

Two trial rates

Take $\mathrm{NPV}(i) = 5.6\, a^{(4)}_{\overline{22.75}|} + 97\, v^{22.75} - 70$, remembering that $i^{(4)}$ must be recomputed at each rate.

For $i_1 = 0.085$ we have $i^{(4)} = 0.082418$ and $v^{22.75} = 0.156305$, so

$$\mathrm{NPV}(0.085) \;=\; 5.6\,\frac{1 - 0.156305}{0.082418} + 97\,(0.156305) - 70 \;=\; +2.4879.$$

Two trial rates

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For $i_1 = 0.085$ we have $i^{(4)} = 0.082418$ and $v^{22.75} = 0.156305$, so

$$\text{NPV}(0.085) \;=\; 5.6\,\frac{1 - 0.156305}{0.082418} + 97\,(0.156305) - 70 \;=\; +2.4879.$$

For $i_2 = 0.09$ we have $i^{(4)} = 0.087113$ and $v^{22.75} = 0.140782$, so

$$\text{NPV}(0.09) \;=\; 5.6\,\frac{1 - 0.140782}{0.087113} + 97\,(0.140782) - 70 \;=\; -1.1097.$$

Interpolating

$$i_0 \;\approx\; 0.085 + \left(\frac{2.4879}{2.4879 + 1.1097}\right)(0.09 - 0.085) \;=\; 0.088.$$

KEY IDEA

The investor is guaranteed at least about 8.8% per annum. In Lecture 17 the same price of €70.00, on a bond fixed to redeem in 2039, gave 9.1%, so the borrower's four extra years of discretion cost the investor about 0.3% of guaranteed yield.

Solving numerically gives 0.08841, so the interpolation is accurate to about three decimal places.

SECTION 3

A past exam question

Question 2, 2022/23

EXAMPLE 18.5

On 1 October 2022 a company issued a bond, redeemable at par on any date from 1 October 2032 to 1 October 2042 inclusive, at the company's option. The bond pays coupons at 5% per annum half-yearly in arrears. The investor pays income tax at 25% on 31 December on coupons received in that calendar year, and capital gains tax at 45% on 31 March on any gain earned over the previous calendar year.

(i) What price per €100.00 nominal gives a yield of at least 9% per annum?

(ii) If the price were €45.00 per €100.00 nominal, what minimum yield would be obtained?

The data, and the difficulty

Measuring time in years from 1 October 2022,

$D = 0.05$	coupon rate p.a.	$m = 2$	payable half-yearly
$R = 1$	at par	$i_0 = 0.09$	minimum yield
$\tau_1 = 0.25$	income tax	$\tau_2 = 0.45$	capital gains tax
$T_1 = 10$	1 Oct 2032	$T_2 = 20$	1 Oct 2042

CAUTION

Three different lags are in play, and each must be converted into a time. The coupons are half-yearly, the income tax falls at a calendar year end, and the capital gains tax on the following 31 March. It is advised to sketch the timeline before calculating.

Reading the dates off



Coupons of $\frac{D}{2}$ arrive at $t = 0.5, 1, \ldots, 20$, income tax of $\tau_1 D$ is paid at $t = 1.25, 2.25, \ldots, 20.25$, and capital gains tax is paid once, at $t = 20.5$.

Reading the dates off



Coupons of $\frac{D}{2}$ arrive at $t = 0.5, 1, \ldots, 20$, income tax of $\tau_1 D$ is paid at $t = 1.25, 2.25, \ldots, 20.25$, and capital gains tax is paid once, at $t = 20.5$.

Reading the dates off



Coupons of $\frac{D}{2}$ arrive at $t = 0.5, 1, \ldots, 20$, income tax of $\tau_1 D$ is paid at $t = 1.25, 2.25, \ldots, 20.25$, and capital gains tax is paid once, at $t = 20.5$.

Why those three patterns

- **Coupons.** Half-yearly in arrears from 1 Oct 2022, so $\frac{D}{2}$ at each $t = 0.5, 1, 1.5, \ldots$ The first falls on 1 April 2023, which is *not* in calendar 2022, so 2022 attracts no tax.
- **Income tax.** Each calendar year contains two coupons totalling D, and the tax $\tau_1 D$ is paid on 31 December of that year, a quarter of a year after the second coupon. This gives twenty payments, at $t = 1.25, \ldots, 20.25$.
- **Capital gains tax.** The gain arises at redemption, 1 Oct 2042, and is taxed on 31 March 2043, six months later, so at $t = 20.5$.

CAUTION

The tax at $t = 1.25, \ldots, 20.25$ is $\tau_1 D$, not $2\tau_1 D$, because D is the *annual* coupon, so the year's two payments of $\frac{D}{2}$ already add to D.

The capital gains test

First decide whether there is a gain. We have

$$i^{(2)} \;=\; 2\left((1.09)^{1/2}-1\right) \;=\; 0.0880613, \qquad (1-\tau_1)\frac{D}{R} \;=\; 0.75\times\frac{0.05}{1} \;=\; 0.0375,$$

and since $0.0880613 > 0.0375$ there will be a capital gain, so the investor assumes the *latest* redemption, $T = T_2 = 20$.

> CAUTION
>
> The test as proved assumes tax paid *at* the coupon dates. Deferring it makes the coupons slightly more valuable, so the test is only approximate here. However, the margin is more than twofold, so the conclusion is safe.
>
> If it were close, we would simply have to do the full calculation.

(i) The equation of value

Let P_0 be the price per €100.00 nominal. Reading straight off the timeline,

$$P_0 \;=\; \underbrace{5\,a^{(2)}_{\overline{20}|}}_{\text{coupons}} + \underbrace{100\,v^{20}}_{\text{redemption}} - \underbrace{v^{0.25}\,(0.25\times 5)\,a_{\overline{20}|}}_{\text{income tax}} - \underbrace{0.45\,(100-P_0)\,v^{20.5}}_{\text{capital gains tax}}.$$

To evaluate it,

$$a^{(2)}_{\overline{20}|} \;=\; \frac{1-(1.09)^{-20}}{0.0880613} \;=\; 9.329514, \qquad a_{\overline{20}|} \;=\; \frac{1-(1.09)^{-20}}{0.09} \;=\; 9.128546.$$

Note that both annuities appear. The coupons are half-yearly, but the tax on them is annual, with one instalment covering both of that year's coupons.

(i) Solving

$$\begin{aligned} P_0 &= 5 \times 9.329514 + 100 \times (1.09)^{-20} \\ &\quad - (1.09)^{-0.25}(1.25) \times 9.128546 - 0.45\,(100 - P_0) \times (1.09)^{-20.5} \\ &= 46.647568 + 17.843089 - 11.167475 - 7.690761 + 0.0769076\,P_0. \end{aligned}$$

Collecting the P_0 terms on the left,

$$(0.9230924)\,P_0 = 45.632421 \qquad \Longrightarrow \qquad P_0 = €49.43.$$

KEY IDEA

The P_0 on the right is the circularity of Lecture 16. Gathering terms is the same as applying that lecture's formula to the price 53.32 before capital gains tax, now with $v^{20.5}$ since the tax is paid at $t = 20.5$.

(ii) A price of €45.00

As a sanity check, $45 < 49.43$ and €49.43 was the price giving exactly 9%. Paying *less* must give *more*, so we expect $i_0 > 9\%$.

The equation of value per €100.00 nominal is now

$$45 \;=\; 5\,a^{(2)}_{\overline{20}|} + 100\,v^{20} - 1.25\,v^{0.25}a_{\overline{20}|} - 24.75\,v^{20.5},$$

since $0.45 \times (100 - 45) = 24.75$.

> KEY IDEA
>
> With the price known, the circularity is gone and the capital gains tax is a definite €24.75, exactly as in Section 2.

(ii) Two trial rates

Put $\text{NPV}(i) = 5\, a^{(2)}_{\overline{20}|} + 100\, \text{v}^{20} - 1.25\, \text{v}^{0.25} a_{\overline{20}|} - 24.75\, \text{v}^{20.5} - 45$. At $i_1 = 0.09$ every quantity has already been computed, so

$$\text{NPV}(0.09) \;=\; 46.647568 + 17.843089 - 11.167475 - 4.229918 - 45 \;=\; +4.093264.$$

(ii) Two trial rates

Put $\mathrm{NPV}(i) = 5\,a^{(2)}_{\overline{20}|} + 100\,\mathrm{v}^{20} - 1.25\,\mathrm{v}^{0.25}a_{\overline{20}|} - 24.75\,\mathrm{v}^{20.5} - 45$. At $i_1 = 0.09$ every quantity has already been computed, so

$$\mathrm{NPV}(0.09) \;=\; 46.647568 + 17.843089 - 11.167475 - 4.229918 - 45 \;=\; +4.093264.$$

At $i_2 = 0.10$ all four terms must be recomputed, including $i^{(2)}$, $a_{\overline{20}|}$ and three separate powers of v, so

$$\mathrm{NPV}(0.10) \;=\; 43.606662 + 14.864363 - 10.391380 - 3.507722 - 45 \;=\; -0.428078.$$

(ii) Interpolating

$$i_0 \;\approx\; 0.09 + \left(\frac{4.093264}{4.093264 + 0.428078}\right)(0.10 - 0.09) \;=\; 0.099,$$

so the minimum yield is about 9.9% per annum.

> KEY IDEA
>
> NPV(0.09) is nearly ten times |NPV(0.10)|, so the root sits close to 10%, and indeed the exact value is 0.0990. As always, this is a minimum, since redemption before 2042 would give more.

The sanity check is confirmed, since €45.00 is cheaper than €49.43 and 9.9% exceeds 9%.

Lecture 18 on one slide

THE ORDER OF WORK NEVER CHANGES

1. Convert every date into a time, and draw the cashflow.
2. Run the capital gains test (or, if the price is given, compare P_0 with R) and fix T at T_1 or T_2.
3. Write the equation of value at that T, with every tax at its own date.
4. Solve for P_0 if the price is unknown, or interpolate if the rate is unknown.

One of the biggest difficulty in the exam will be finishing on **time**. As such, practice is key.

Exercises

EXERCISE 18.6

In the first example the yield was exactly 10% *at* T_2 *and higher at every earlier date. Why must equality occur at* T_2 *and nowhere else?*

EXERCISE 18.7

In the exam question, explain why $a_{\overline{20}|}$ *appears in the income-tax term but* $a^{(2)}_{\overline{20}|}$ *appears in the coupon term.*

EXERCISE 18.8

In the exam question, €49.43 *gave exactly* 9% *and* €45.00 *gave* 9.9%*. Without calculating, what would* €55.00 *give?*

Solutions are in the backup slides.

END OF LECTURE 18

Every security so far has had a redemption date, even an uncertain one. Next we meet one that has none at all, and pays an income that grows.

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Solution 1

The price paid was chosen so that the investor earns exactly i_0 when redemption is at T_2, so equality there holds by construction.

For $T < T_2$ we showed in Lecture 17 that, when there is a capital gain, the price at a given rate decreases as T increases. So at any earlier redemption the bond is worth strictly more than the investor paid. Since the price is strictly decreasing in i, the rate that restores equality must be strictly greater than i_0.

Hence there is equality at T_2 and strict inequality everywhere else, which is exactly what the table of realised yields shows.

Solution 2

The two annuities count different things.

The *coupons* are paid twice a year, in arrears, at $t = 0.5, 1, 1.5, \ldots, 20$. That is a level annuity payable m-thly with $m = 2$, so its value is $a^{(2)}_{\overline{20}|}$, which consists of forty payments of $\frac{D}{2}$.

The *income tax*, however, is paid *once* a year, on 31 December, in a single instalment covering both of that year's coupons. That is an ordinary annual annuity of twenty payments, hence $a_{\overline{20}|}$, shifted by $v^{0.25}$ because each payment falls a quarter of a year after the corresponding anniversary of issue.

The same cashflow is taxed at a different frequency from the one at which it arrives.

Solution 3

Less than 9%.

The cashflow is fixed, with the same coupons, the same redemption and the same taxes, so the only thing that changes is what the investor pays for it. Paying more for an unchanged stream must earn less. Indeed €49.43 bought exactly 9%, and €55.00 is dearer still.

> CAUTION
>
> One small complication pulls the other way. At a higher price the capital gain is smaller, so the capital gains tax is smaller too, which pushes the yield *up* a little. That effect is far outweighed by the higher price. Working it out gives about 8.05%.