LECTURE 17

When the borrower chooses

Pricing a bond whose redemption date is not yet decided

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Reading the same equation in multiple ways

$$P_0 \;=\; (1-\tau_1) D\, a^{(m)}_{\overline{T}|} + R\mathsf{v}^T - \tau_2 \left(R - P_0\right) \mathsf{v}^T$$

The price equation, given i, T, τ_1, τ_2, D, R

If we are given all but i (in one of its forms, here v), we can solve for i, which we will do today. Then consider the case when T is unknown.

Today

1. Finding the yield
2. Securities with no fixed redemption date
3. Which redemption date is the worst case?

Warm-up

EXERCISE 17.1

An investor liable to income tax at 30% *at source and capital gains tax at* 40% *buys, for* €70.00 *per* €100.00 *nominal, a bond with coupons of 8% p.a. quarterly, redeemable at 115% in 18.75 years (Lecture 16). Write down (not yet solve) the equation of value for the net yield.*

The point of the exercise is what happens to the capital gains tax once the price is known.

EXERCISE 17.1

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Per €100.00 nominal, with $P_0 = 70$ given, we have

$$70 \;=\; (1-0.3)\,8\,a^{(4)}_{\overline{18.75}|} + 115\,\mathrm{v}^{18.75} - 0.4\,(115-70)\,\mathrm{v}^{18.75} \;=\; 5.6\,a^{(4)}_{\overline{18.75}|} + 97\,\mathrm{v}^{18.75},$$

since $115 - 0.4 \times 45 = 115 - 18 = 97$.

Warm-up

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$$70 \;=\; 5.6\, a^{(4)}_{\overline{18.75}|} + 97 v^{18.7}.$$

KEY IDEA

As P_0 is known, the circularity of Lecture 16 vanishes. The tax of €18.00 is a definite number, and it simply reduces the redemption proceeds from €115.00 to €97.00.

Capital-gain test is not needed here. With the price given, compare it with R directly. There is a gain as $70 < 115$.

SECTION 1

Finding the yield

The other direction

An investor liable to capital gains tax may wish to determine the net yield on a transaction purchased at a given price.

KEY IDEA

Determine the price on two different net yield bases and estimate the actual yield by interpolation. Equivalently, compute NPV at two rates and interpolate (Lecture 10).

CAUTION

Added difficulty: $i^{(m)}$ itself depends on i. Each trial rate therefore needs its own $i^{(m)}$ computed *before* the annuity value. This makes interpolation more timeconsuming.

In our formulation, $i^{(m)}$ is contained in $a^{(m)}_{\overline{T}|}$ and i in v. Knowing one determines the other.

Two trial rates

Take $\text{NPV}(i) = 5.6\, a^{(4)}_{\overline{18.75}|} + 97\, v^{18.75} - 70$ from the warm-up.

$i_1 = 0.09$

We have $i^{(4)} = 4\left((1.09)^{1/4} - 1\right) = 0.0871127$, so

$$\text{NPV}(0.09) \;=\; 5.6\,\frac{1-(1.09)^{-18.75}}{0.0871127} + 97\,(1.09)^{-18.75} - 70 \;=\; 0.7859\,.$$

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$i_2 = 0.095$

We have $i^{(4)} = 0.0917917$, so

$$\text{NPV}(0.095) \;=\; 5.6\,\frac{1-(1.095)^{-18.75}}{0.0917917} + 97\,(1.095)^{-18.75} - 70 \;=\; -2.4279\,.$$

Interpolating

The sign has changed, so the yield lies between 9% and 9.5%, and much nearer the lower end since NPV(0.09) is the smaller in magnitude. Hence

$$i_0 \;\approx\; 0.09 + \left(\frac{0.7859}{0.7859 + 2.4279}\right)(0.095 - 0.09) \;=\; 0.091.$$

KEY IDEA

The investor earns about 9.1% per annum net. This is below the 10% of Lecture 16, which is consistent as €70.00 is more than the €64.19 that a 10% yield required. Paying more for the same cashflow means earning less.

Solving numerically, the solution is 0.09119..., so the estimate is good to three decimal places.

SECTION 2

Securities with no fixed redemption date

The terms of issue

Sometimes a security is issued without a fixed redemption date. The terms of issue may provide that the borrower can redeem **at the borrower's option** at any interest date on or after some specified date; or at any interest date on or between two specified dates; or on any one of a series of dates between them.

- The latest possible redemption date is the **final redemption date**. If there is no such date, the bond is **undated**.
- A loan may also be redeemable at the option of the *lender*, but this is less common.

Throughout this lecture, the choice belongs to the borrower.

This leads to the interesting subject of **calls**, and it is indeed possible to value them without predicting them. This is the topic of the so called *Black-Scholes equation*. — Not part of this module.

What the investor can still say

An investor who wishes to purchase such a loan cannot, at the time of purchase, know how the market will move. As such, it's not possible to say for certain when the borrower will repay. The investor therefore *cannot know the yield*.

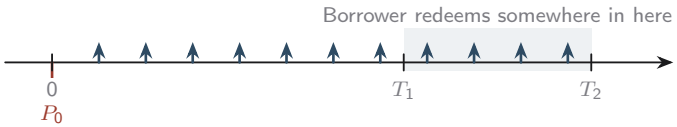
However, using the capital gains test, the investor can determine either of:

- The **maximum price** to be paid for a specified minimum net yield;
- The **minimum net yield** for a specified price.

> KEY IDEA
>
> Both are guarantees against the worst case. The investor gives up knowing the answer exactly, in exchange for a bound that holds whatever the borrower does.

The set-up

Consider a fixed-interest security paying coupons of D p. a., payable m-thly in arrears, with redemption amount R. Its outstanding term T is chosen by the borrower subject to $T_1 \leq T \leq T_2$, where T_1 and T_2 are integer multiples of $\frac{1}{m}$.



Suppose an investor liable to income tax at τ_1 wishes to achieve a net annual yield of **at least** i_0. What is the maximum price to pay?

SECTION 3

Which redemption date is the worst case?

Two forms of the price

At effective annual rate i, the present value per unit nominal of the future net payments is

$$P(i,T) \;=\; (1-\tau_1)D\,a_{\overline{T}|}^{(m)} + R\mathrm{v}^T \;=\; R - \left(i^{(m)} - (1-\tau_1)\frac{D}{R}\right) R\,a_{\overline{T}|}^{(m)},$$

the second being the rearrangement from Lecture 16.

TWO CONSEQUENCES

Both equations have their uses. We can read off the following from them:

FIRST $\downarrow P$ falls as $\uparrow i$ rises;

SECOND P's dependence on T is governed by the sign of the bracket, equivalently, the *net running yield* (Lecture 16), since $a_{\overline{T}|}^{(m)}$ decreases as T increases.

P's dependence on i

$$\boxed{P(i,T) \;=\; (1-\tau_1)D\,a^{(m)}_{\overline{T}|} + R\mathsf{v}^{T}.}$$

If $i_1 > i_2$ then $\mathsf{v}_1 < \mathsf{v}_2$ and so, for every $k \geq 1$, v^k is smaller at i_1 than at i_2. Hence both

$$a^{(m)}_{\overline{T}|} \;=\; \frac{1}{m}\left(\mathsf{v}^{1/m} + \mathsf{v}^{2/m} + \cdots\right) \qquad \text{and} \qquad \mathsf{v}^{T}$$

are smaller, and therefore so is P.

$P(i,T)$ IS A DECREASING FUNCTION OF i

For fixed T, $P(i_0,T) > P(i,T)$ if $i > i_0$. Read backwards, it says that less interest gives higher earning for the same cashflow. That is the fact used from here on.

P's dependence on T. Case 1: A capital gain

Suppose $i_0^{(m)} > (1-\tau_1)\frac{D}{R}$, so the investor will receive a capital gain at redemption. Look at the second form:

$$P(i_0, T) \;=\; R - \underbrace{\left(i_0^{(m)} - (1-\tau_1)\frac{D}{R}\right)}_{\text{Positive, and free of } T} R\, a_{\overline{T}|}^{(m)}.$$

$P(i, T)$ IS A DECREASING FUNCTION OF T, IF $i_0^{(m)} > (1-\tau_1)\frac{D}{R}$ (CAPTICAL GAIN)

For fixed i_0, as T rises, $a_{\overline{T}|}^{(m)}$ rises, so a larger positive quantity is *subtracted* from R.

What that means for the yield

Suppose the investor pays $P(i_0, T')$ for some $T_1 \leq T' \leq T_2$. If the borrower redeems *earlier* than T', then $P(i_0, T) > P(i_0, T')$. Meaning, at the shorter term the bond is worth more than was paid, and since P decreases in i, the rate restoring equality satisfies $i > i_0$.

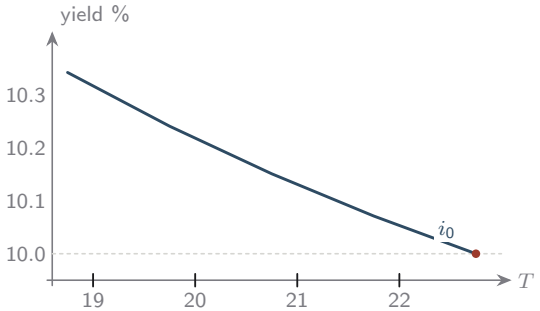
$$\text{Redemption earlier than } T' \implies i > i_0, \qquad \text{Later than } T' \implies i < i_0.$$

KEY IDEA

From the investor's viewpoint, *the sooner a capital gain is received the better*. So the danger is a *late* redemption. To ensure a yield of at least i_0, assume the worst case $T = T_2$ and pay $P_0 = P(i_0, T_2)$. This is the lowest of the candidate prices.

The guarantee, checked

The investor pays $P(i_0, T_2)$, with $i_0 = 10\%$ and $T_2 = 22.75$, then waits to see what the borrower does.



Whatever the borrower chooses, the yield is at least 10%, with equality only in the worst case $T = T_2$, and a bonus of up to 0.34% if redemption comes early.

P's dependence on T. Case 2: A capital loss

Now suppose $i_0^{(m)} < (1-\tau_1)\frac{D}{R}$, so there will be a capital *loss*. Writing the bracket the other way,

$$P(i_0, T) \;=\; R + \left((1-\tau_1)\frac{D}{R} - i_0^{(m)}\right) R\, a_{\overline{T}|}^{(m)}.$$

$P(i,T)$ IS AN INCREASING FUNCTION OF T, IF $i_0^{(m)} < (1-\tau_1)\frac{D}{R}$ (CAPTICAL LOSS)

For fixed i_0, as T rises, $a_{\overline{T}|}^{(m)}$ rises, so a larger positive quantity is *added* to R.

EVERY INEQUALITY OF CASE 1 REVERSES

For greater yield, you want to *defer* a loss as long as possible. The worst case is therefore the *earliest* redemption. That is, assume $T = T_1$ and pay $P_0 = P(i_0, T_1)$.

P's dependence on T. Case 3: Neither loss nor gain

If $i_0^{(m)} = (1-\tau_1)\frac{D}{R}$, the bracket in

$$P(i_0, T) \;=\; R - \left(i_0^{(m)} - (1-\tau_1)\frac{D}{R} \right) R\, a_{\overline{T}|}^{(m)} \;=\; R$$

is zero, so the whole subtracted term vanishes whatever $a_{\overline{T}|}^{(m)}$ may be, and we are left with

$$P(i_0, T) \;=\; R$$

for *every* T.

> **THE NET ANNUAL YIELD IS i_0 WHATEVER THE BORROWER DOES**
>
> It makes no difference when the security is redeemed as the coupons alone deliver exactly the required rate. There is no capital gain or loss whose timing could matter.

The rule, in one table

Test	At redemption	Assume
$i_0^{(m)} > (1-\tau_1)\frac{D}{R}$	Capital gain	Latest date, $T = T_2$
$i_0^{(m)} < (1-\tau_1)\frac{D}{R}$	Capital loss	Earliest date, $T = T_1$
$i_0^{(m)} = (1-\tau_1)\frac{D}{R}$	Neither	Any date; $P_0 = R$

A GAIN SHOULD ARRIVE EARLY AND A LOSS SHOULD ARRIVE LATE.

This makes the worst cases a *late gain* or an *early loss*. In both, the investor prices the bond at whichever date is worse and pays the lower price.

If the price is given instead

Suppose instead the price is given. The minimum net annual yield is obtained by again assuming the worst case for the investor.

$P_0 < R$ The investor receives a capital gain. The worst case is redemption at the *latest* possible date. If redemption is earlier, the yield will be greater.

$P_0 > R$ The investor receives a capital loss. The worst case is redemption at the *earliest* possible date.

$P_0 = R$ There is neither gain nor loss. The net annual yield is $i_0^{(m)} = (1 - \tau_1)\frac{D}{R}$ irrespectice of redemption date.

Does capital gains tax change anything?

KEY IDEA

No. A capital gains tax liability does not change any of these conclusions.

An investment with a gain *before* capital gains tax must still have a net gain *after* it, since the investor keeps the fraction $1 - \tau_2$ and $\tau_2 < 1$. The gain is smaller, but still a gain, so the worst case is still the latest redemption, and the reverse for a loss.

SO THE ORDER OF OPERATIONS IS FIXED

Run the capital gains test, choose T_1 or T_2, and only *then* apply Lecture 16's τ_2 machinery at the chosen date.

Lecture 17 on one slide

PRICE FOR THE WORST CASE

$i^{(m)} > (1-\tau_1)\dfrac{D}{R} \Rightarrow$ gain $\Rightarrow$ assume T_2, $\qquad i^{(m)} < (1-\tau_1)\dfrac{D}{R} \Rightarrow$ assume T_1

where $T_1 < T_2$. If the price is given, compare P_0 with R instead.

$P(i,T)$ falls as i rises; as T rises it falls if there is a gain and rises if there is a loss. A gain should come early and a loss late, so the worst case is a late gain or an early loss. The capital gains tax changes none of it.

Exercises

EXERCISE 17.2

A stock pays 6% *half-yearly coupons, redeemable at par at the borrower's option between* 10 *and* 15 *years. An investor pays* 25% *tax and needs* 8% *net. Find the price.*

EXERCISE 17.3

The same, but 10% *coupons, redeemable between* 10 *and* 20 *years,* 20% *tax,* 5% *net. Show the worst case is now the earliest date, and price it.*

EXERCISE 17.4

An investor pays €95.00 *per* €100.00 *nominal for a stock redeemable at par between* 8 *and* 12 *years. Which date gives the least yield?*

Solutions are in the backup slides.

END OF LECTURE 17

The worst case scenario is understood. Next, we will use it on examples.

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Solution 1

Run the test. At $i_0 = 8\%$,

$$i^{(2)} \;=\; 2\left((1.08)^{1/2} - 1\right) \;=\; 0.078461, \qquad (1-\tau_1)\frac{D}{R} \;=\; 0.75 \times 0.06 \;=\; 0.045.$$

Since $0.078461 > 0.045$ there is a capital gain, so the worst case is the *latest* redemption and we assume $T_2 = 15$.

With $\mathrm{v}^{15} = 1.08^{-15} = 0.315242$ and $a^{(2)}_{\overline{15}|} = \dfrac{1 - 0.315242}{0.078461} = 8.727375$,

$$P_0 \;=\; 0.75 \times 6 \times 8.727375 + 100 \times 0.315242 \;=\; €70.80.$$

At $T_1 = 10$ the price would have been €77.11, thus higher. It is therefore unsafe. Pay that and a redemption in 2039 leaves the investor short of 8%.

Solution 2

At $i_0 = 5\%$,

$$i^{(2)} \;=\; 2\left((1.05)^{1/2} - 1\right) \;=\; 0.049390, \qquad (1-\tau_1)\frac{D}{R} \;=\; 0.8 \times 0.10 \;=\; 0.08.$$

Since $0.049390 < 0.08$ there is a capital *loss*: The net coupons alone deliver more than the investor requires, so the bond must be bought above par and the excess lost at redemption. A loss is best deferred, so the worst case is the *earliest* redemption, $T_1 = 10$.

With $\mathrm{v}^{10} = 0.613913$ and $a^{(2)}_{\overline{10}|} = 7.817079$,

$$P_0 \;=\; 0.8 \times 10 \times 7.817079 + 100 \times 0.613913 \;=\; €\,123.93,$$

a price above par, as a loss requires.

Solution 3

Here the price is given, so no rate comparison is needed: Compare P_0 with R directly. Since

$$P_0 \;=\; 95 \;<\; 100 \;=\; R,$$

the investor will make a capital gain at redemption.

A gain is worth more the sooner it arrives, so the *worst* outcome for the investor is the one that delays it longest: Redemption at the latest possible date, $T = 12$. The yield computed on that basis is the minimum, and any earlier redemption gives more.