LECTURE 16

Tax on income, tax on gains

Delayed income tanx and capital gains tax

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Recap

$$P_0 = (1-\tau_1)D\, a^{(m)}_{\overline{T}|} + R\mathrm{v}^T$$

$$P_0 = D\, a^{(m)}_{\overline{T}|} + R\mathrm{v}^T - \tau_1 D\, \mathrm{v}^k a_{\overline{T}|}$$

Last lecture we considered tax at source and tax paid k years in arrears, but always on the annuity. Today, we consider deferred and non-full tax years, and even a new type of tax: Capital gains.

Today

1. Income tax paid late
2. Capital gains tax
3. A worked example

Warm-up

EXERCISE 16.1

The bond of last lecture (15) paid a coupon of 8% *p. a. quarterly and was redeemable at* 115% *in* 18.75 *years. Find the price per* €100.00 *nominal for an investor paying no tax at all who requires* 10% *p.a.*

Recall our calculations form last time: $i^{(4)} = 0.096454756$ and $a_{\overline{18.75}|}^{(4)} = 8.63150$.

Warm-up

EXERCISE 16.1

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With no tax the coupons are taken gross, so

$$P \;=\; 8\, a^{(4)}_{\overline{18.75}|} + 115\, \mathrm{v}^{18.75} \;=\; 69.05 + 19.26 \;=\; €88.31.$$

> ASIDE
>
> The same bond taxed at 30% at source was worth €67.59. Income tax alone therefore costs this investor €20.72 from the €88.31 value, or about a quarter.

SECTION 1

Income tax paid late

The question

EXAMPLE 16.2 · The same bond

We have a coupon of 8% per annum quarterly, redeemable at 115% on 1 October 2039. An investor pays tax at 30% on 31 March on income received in the previous calendar year.

What price should the bond have on 1 January 2021 for an effective net yield of 10%?

We have $T = 18.75$, $D = 0.08$, $R = 1.15$, $i = 0.1$, $\tau_1 = 0.3$. The only thing that changes is the *timing* of the tax.

> CAUTION
>
> Lecture 15's formula, $-\tau_1 D \mathrm{v}^k a_{\overline{T}|}$, assumed T was a whole number of years, so every tax year was full. Here $T = 18.75$, and so the last yeas is not full.

Getting the tax dates right

Measure time from 1 January 2021.



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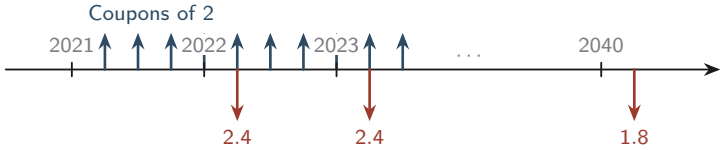
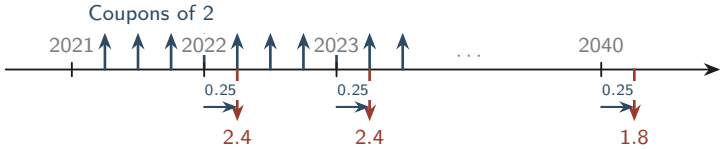


Getting the tax dates right

Measure time from 1 January 2021. Coupons arrive quarterly at $t = 0.25, 0.5, \ldots, 18.75$; tax is paid each 31 March, that is at $t = 1.25, 2.25, \ldots$



Income received during calendar year k is taxed on 31 March of year $k + 1$. This is a lag of a quarter past the year end.

A deferred annuity and a stray term

For the coupons, in the same notation as before,

- Year k, for $k \in \{1, \ldots, 18\}$ gives income D, taxed on 31 March of year $k+1$, i.e. at time $t = k + 0.25$;
- Year 2039 gives an income of $\frac{3}{4}D$, taxed on 31 March 2040, i.e. at time $t = 19.25$.

A deferred annuity and a stray term

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- Year 2039 gives an income of $\frac{3}{4}D$, taxed on 31 March 2040, i.e. at time $t = 19.25$.

The eighteen full years contribute $\tau_1 D$ at $t = 1.25, 2.25, \dots, 18.25$, and the short year contributes $\tau_1 \frac{3}{4} D$ at $t = 19.25$. Their present value is

$$\tau_1 D \left(\mathrm{v}^{1.25} + \mathrm{v}^{2.25} + \cdots + \mathrm{v}^{18.25}\right) + \tfrac{3}{4}\tau_1 D\, \mathrm{v}^{19.25} \;=\; \tau_1 D\, \mathrm{v}^{0.25} \left(a_{\overline{18}|} + \tfrac{3}{4}\mathrm{v}^{19}\right).$$

The calculation

Therefore the price per €100.00 nominal is

$$\begin{aligned} P_0 &= 8\, a^{(4)}_{\overline{18.75|}} + 115\, v^{18.75} - (0.3)\, 8\, v^{0.25} \left(a_{\overline{18|}} + \tfrac{3}{4} v^{19} \right) \\ &= 8 \left(\frac{1 - (1.1)^{-18.75}}{0.096454756} \right) + 115\, (1.1)^{-18.75} \\ &\quad - 2.4\, (1.1)^{-0.25} \left(\frac{1 - (1.1)^{-18}}{0.1} + \tfrac{3}{4} (1.1)^{-19} \right) \\ &= 69.0520 + 19.2568 - 2.4 \times 8.12804592 \;=\; €68.80. \end{aligned}$$

Three tax treatments compared

The same bond, the same 30% tax, the same 10% required yield but with different tax dates:

Tax treatment	price per €100.00 nominal
No tax at all	88.31
Deducted at source, with each coupon	67.59
Paid each 31 March	68.80

KEY IDEA

Deferring the tax bill by a few months is worth €1.21 per €100.00 nominal as money paid later is cheaper.

SECTION 2

Capital gains tax

A second kind of tax

DEFINITION 16.3 · Capital gains tax

If the price paid for a bond is less than the redemption price (or the sale price, if it is sold earlier) the investor has made a **capital gain**, and **capital gains tax** is levied on that gain.



In contrast to income tax, this is normally payable *once only*, at the date of sale or redemption.

Will there be a gain at all?

Suppose the investor is liable to income tax at τ_1 and capital gains tax at τ_2. To find the price we must first decide whether there *will* be a gain. Ignore the capital gains tax for a moment and let P be the price the investor would then pay. If $P < R$ there is a gain, and we consider $R - P$:

$$\begin{aligned} R - P &= R - \left((1-\tau_1) D\, a_{\overline{T}|}^{(m)} + R\mathrm{v}^T \right) = R\left(1 - \mathrm{v}^T\right) - (1-\tau_1) D\, a_{\overline{T}|}^{(m)} \\ &= R\, i^{(m)} a_{\overline{T}|}^{(m)} - (1-\tau_1) D\, a_{\overline{T}|}^{(m)} = \left(i^{(m)} - (1-\tau_1)\frac{D}{R} \right) R\, a_{\overline{T}|}^{(m)}, \end{aligned}$$

using $1 - \mathrm{v}^T = i^{(m)}\, a_{\overline{T}|}^{(m)}$, which is the definition of $a_{\overline{T}|}^{(m)}$ rearranged.

Here, we are using the slighly simpler income-tax-at-source , but the general logic is the same.

The test

Since $R\, a_{\overline{T}|}^{(m)} > 0$, the sign of $R - P$ is the sign of the bracket, thus

$$\boxed{P < R \iff i^{(m)} > (1 - \tau_1)\frac{D}{R}}$$

WHAT THE TEST COMPARES

$(1 - \tau_1)\frac{D}{R}$ is the **net running yield**. It is the net coupon income per unit of *redemption*. If the investor's required rate $i^{(m)}$ exceeds what the coupons alone deliver, the shortfall must be made up by buying below R; meaning capital gain.

Note that the test involves no annuity values at all.

This formula is true for the income-tax-at-source, not in general, but the approach is identical.

The price depends on the tax, and the tax on the price

Suppose there is a gain, so capital gains tax is due. Let P be the price the investor would pay with income tax *only*, however that tax is levied, and let P_0 be the price once capital gains tax is paid too. At redemption there is an additional liability of

$$\tau_2 (R - P_0),$$

worth $\tau_2 (R - P_0) \mathsf{v}^T$ today. The investor pays for it by offering less, so

$$P_0 \;=\; P - \tau_2 (R - P_0) \mathsf{v}^T.$$

The gain depends on the price and the price on the gain, so P_0 appears on both sides. That is harmless: it appears only to the first power, so this is a linear equation, and linear equations are easy to solve.

Nothing here depends on how the income tax is paid: P may come from tax at source, tax paid late, or anything else we have met.

Solving for P_0

Expanding, and gathering the P_0 terms on the left,

$$\begin{aligned} P_0 &= P - \tau_2 R \mathsf{v}^T + \tau_2 \mathsf{v}^T P_0 \\ \left(1 - \tau_2 \mathsf{v}^T\right) P_0 &= P - \tau_2 R \mathsf{v}^T \end{aligned}$$

so that

$$\boxed{P_0 = \frac{P - \tau_2 R \mathsf{v}^T}{1 - \tau_2 \mathsf{v}^T}}$$

TWO STEPS

First price the bond as if there were no capital gains tax; this gives P. Then P_0 is one line of arithmetic away.

What capital gains tax does to the price

Subtract the equation for P_0 from R:

$$R - P_0 \;=\; (R - P) + \tau_2 \,(R - P_0)\, \mathsf{v}^T \quad \Longrightarrow \quad (R - P_0)\,\big(1 - \tau_2 \mathsf{v}^T\big) \;=\; R - P.$$

Dividing by $1 - \tau_2 \mathsf{v}^T$, which lies strictly between 0 and 1,

$$\boxed{R - P_0 \;=\; \frac{R - P}{1 - \tau_2 \mathsf{v}^T}} \qquad \text{and so} \qquad P - P_0 \;=\; \frac{\tau_2 \mathsf{v}^T}{1 - \tau_2 \mathsf{v}^T}\,(R - P).$$

CAPITAL GAINS TAX LOWERS THE PRICE, AS IT MUST

When $P < R$, the right-hand side is positive, so $P_0 < P$. A lower price means a larger gain, which is why the gain is divided by a number below 1. Also, $R - P_0$ and $R - P$ always have the same sign, so the gain test needs only P.

When there is no gain

Suppose instead $P \geq R$. There is no capital gain and no liability at redemption, and

$$P_0 \;=\; P,$$

whatever the income tax is. For tax at source, $P = (1-\tau_1)D\, a\overline{T|}^{(m)} + R\mathsf{v}^T$, exactly the formula from Lecture 15, and $P \geq R$ is the case $(1-\tau_1)\frac{D}{R} \geq i^{(m)}$ of the test.

> ALWAYS RUN THE TEST *FIRST*
>
> If $P \geq R$, the formula above gives $R - P_0 \leq 0$ and $P_0 \geq P$. Thus, it would treat a loss as a tax refund, and pay more than P.

The taxed-at-source case

For income tax at source, $P = (1-\tau_1)D\,a^{(m)}_{\overline{T}|} + R\mathsf{v}^T$, so
$P - \tau_2 R\mathsf{v}^T = (1-\tau_1)D\,a^{(m)}_{\overline{T}|} + (1-\tau_2)R\mathsf{v}^T$, and

$$\boxed{P_0 \;=\; \frac{1-\tau_1}{1-\tau_2\mathsf{v}^T}D\,a^{(m)}_{\overline{T}|} + \frac{1-\tau_2}{1-\tau_2\mathsf{v}^T}R\mathsf{v}^T}$$

Lecture 15's formula, but each term picking up a factor. It is used in the example below.

ALTERNATIVE EXPRESSION

Using $R - P = \left(i^{(m)} - (1-\tau_1)\frac{D}{R}\right) R\,a^{(m)}_{\overline{T}|}$ in $P_0 = R - \frac{R-P}{1-\tau_2\mathsf{v}^T}$ gives a cross-check, not a formula to memorise:

$$P_0 \;=\; R - \frac{1}{1-\tau_2\mathsf{v}^T}\left(i^{(m)} - (1-\tau_1)\frac{D}{R}\right) R\,a^{(m)}_{\overline{T}|}.$$

SECTION 3

A worked example

The question

EXAMPLE 16.4

A fixed-interest bond pays a coupon of 8% per annum quarterly, with the next coupon due on 31 December 2020, redeemable at 115% on 1 October 2039. Calculate the price an investor, liable to income tax of 30% at source and capital gains tax at 40% on any gain, would pay on 1 January 2021 for an effective net yield of 10% per annum.

$T =$ $D =$ $R =$

$i =$ $\tau_1 =$ $\tau_2 =$

The question

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$T = 18.75$ redemption date $\quad D = \quad R =$

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The question

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$T = 18.75$ redemption date $\quad D = 0.08$ coupon rate $\quad R =$

$i =$ $\quad \tau_1 =$ $\quad \tau_2 =$

The question

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$T = 18.75$ redemption date $\quad D = 0.08$ coupon rate $\quad R = 1.15$ redemption price

$i =$ $\quad \tau_1 =$ $\quad \tau_2 =$

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$T = 18.75$ redemption date	$D = 0.08$ coupon rate	$R = 1.15$ redemption price
$i = 0.1$ yield p.a.	$\tau_1 =$	$\tau_2 =$

The question

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$T = 18.75$ redemption date $\quad D = 0.08$ coupon rate $\quad R = 1.15$ redemption price
$i = 0.1$ yield p.a. $\quad \tau_1 = 0.3$ income tax $\quad \tau_2 =$

The question

EXAMPLE 16.4

A fixed-interest bond pays a coupon of 8% per annum quarterly, with the next coupon due on 31 December 2020, redeemable at 115% on 1 October 2039. Calculate the price an investor, liable to income tax of 30% at source and capital gains tax at 40% on any gain, would pay on 1 January 2021 for an effective net yield of 10% per annum.

$T = 18.75$ redemption date	$D = 0.08$ coupon rate	$R = 1.15$ redemption price
$i = 0.1$ yield p.a.	$\tau_1 = 0.3$ income tax	$\tau_2 = 0.4$ capital gains tax

The test

$T = 18.75$ redemption date $\quad D = 0.08$ coupon rate $\quad R = 1.15$ redemption price
$i = 0.1$ yield p.a. $\quad \tau_1 = 0.3$ income tax $\quad \tau_2 = 0.4$ capital gains tax

Recall $i^{(4)} = 4\left((1.1)^{1/4} - 1\right) = 0.096454756$, and compute the net running yield

$$(1-\tau_1)\frac{D}{R} \;=\; \frac{(1-0.3)\times 0.08}{1.15} \;=\; 0.048695652.$$

We test if there will be a capital gain. Since

$$i^{(4)} \;=\; 0.096454756 \;>\; 0.048695652 \;=\; (1-\tau_1)\frac{D}{R},$$

there will be one, indeed comfortably. The required yield is nearly double what the net coupons provide, so a large part of the return must come from buying cheaply.

The calculation

With $a\overline{18.75|}^{(4)} = 8.63150002$, $v^{18.75} = 0.16745077$ and $1 - 0.4v^{18.75} = 0.933020$,

$$\begin{aligned} P_0 &= \left(\frac{1-0.3}{1-0.4\,v^{18.75}}\right) 8\, a\overline{18.75|}^{(4)} + \left(\frac{1-0.4}{1-0.4\,v^{18.75}}\right) 115\, v^{18.75} \\ &= \frac{0.7}{0.933020} \times 8 \times 8.63150002 \;+\; \frac{0.6}{0.933020} \times 115 \times 0.16745077 \\ &= €64.19 \quad \text{per } €100.00 \text{ nominal.} \end{aligned}$$

The same figure can also be derived from the alternative expression: $115 - \frac{0.047759104 \times 115 \times 8.63150002}{0.933020} = 115 - 50.81$.

Checking the answer

The same bond, taxed only at income, was worth €67.59. The capital gains tax reduced the price to €64.19. This is no surprise as the investor receives less but still demands 10%.

EXAMPLE 16.5 · The circle closes

The gain is $115 - 64.19 = €50.81$, so the tax due at redemption is $0.4 \times 50.81 = €20.32$, whose present value is $20.32 \times 0.16745 = €3.40$. And indeed

$$67.59 - 3.40 \;=\; €64.19.$$

It's a sanity check confirming that P_0 is consistent with its implied tax. It's called a fixed point of a circular equation.

THIS IS **NOT** A SHORT-HAND: P_0 GIVES P, BUT P DOES NOT GIVE P_0

Note that above, we had to know €64.19. We have $(115 - 67.59) \times 0.4 \times 0.16745 = €3.18$, giving us $67.59 - 3.18 = 64.32 \neq 64.19$. We have

$$P \;=\; P_0 + P_0 \times \tau_2 \times \mathrm{v}^T \qquad \textbf{not} \qquad P_0 \;=\; P - P \times \tau_2 \times \mathrm{v}^T$$

Lecture 16 on one slide

TEST FIRST, THEN PRICE

$$i^{(m)} > (1-\tau_1)\frac{D}{R} \iff \text{Capital gain}$$

$$P_0 = \frac{(1-\tau_1)D\,a_{\overline{T}|}^{(m)} + (1-\tau_2)R\mathsf{v}^T}{1-\tau_2\mathsf{v}^T} \qquad \text{If there is a gain; otherwise } P_0 = P.$$

Income tax paid late: Factor the lag out as v^k and patch any part year separately.
Capital gains tax: The price appears on both sides of the equation, so gather and divide.

Exercises

EXERCISE 16.6

A bond pays coupons of 5% *p. a. half-yearly, redeemable at* 110% *in twelve years, for an investor paying* 25% *income tax and requiring* 8% *net. Show there is a capital gain.*

EXERCISE 16.7

For the same bond and investor, now also paying capital gains tax at 35%*, find the price per* €100.00 *nominal and compare it with the price ignoring that tax.*

EXERCISE 16.8

A bond pays coupons of 9% *per annum half-yearly, redeemable at par in ten years, for an investor paying* 20% *income tax and requiring* 6% *net. Show there is no gain, and price it.*

Solutions are in the backup slides.

END OF LECTURE 16

The price is known once the tax is. Next, we will examine the reverse question: Given the price, what does the investor earn; and then a borrower who will not say when the bond matures.

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Solution 1

The test compares the required nominal rate with the net running yield. At $i = 8\%$,

$$i^{(2)} \;=\; 2\left((1.08)^{1/2} - 1\right) \;=\; 0.078461, \qquad (1-\tau_1)\frac{D}{R} \;=\; \frac{0.75 \times 0.05}{1.10} \;=\; 0.034091.$$

Since $0.078461 > 0.034091$ there is a capital gain.

In words: the coupons, after tax, deliver only about 3.4% per unit of redemption money, against a required 7.85%. The rest of the return has to come from buying below €110.00. Thus, a gain at redemption is unavoidable.

Solution 2

With $v^{12} = 1.08^{-12} = 0.397114$ and $a^{(2)}_{\overline{12}|} = \dfrac{1 - 0.397114}{0.078461} = 7.683900$, the gain formula gives

$$P_0 \;=\; \frac{0.75 \times 5 \times 7.683900 + 0.65 \times 110 \times 0.397114}{1 - 0.35 \times 0.397114} \;=\; €66.44.$$

Ignoring capital gains tax the price would have been

$$0.75 \times 5 \times 7.683900 + 110 \times 0.397114 \;=\; €72.50,$$

so the tax costs the investor over €6.00 of price. It has to, as the investor keeps only 65% of a gain of some €43.00, and must buy more cheaply to make up the difference.

Solution 3

Run the test first. At $i = 6\%$,

$$i^{(2)} \;=\; 2\left((1.06)^{1/2} - 1\right) \;=\; 0.059126, \qquad (1-\tau_1)\frac{D}{R} \;=\; 0.8 \times 0.09 \;=\; 0.072.$$

Since $0.059126 < 0.072$ there is *no* capital gain, so no capital gains tax is due and the Lecture 15 formula applies unchanged:

$$P_0 \;=\; P \;=\; 0.8 \times 9 \times a_{\overline{10}|}^{(2)} + 100\,\mathrm{v}^{10} \;=\; €109.62.$$

A price above par, which is consistent with there being no gain, since the investor pays more than the €100.00 that comes back. Applying the τ_2 formula here would have produced a lower price still, taxing a loss.