LECTURE 15

Bonds and their prices

Bonds: A type of annuity with tax

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Everything is in place

$$a_{\overline{n}|}^{(m)} = \frac{1 - \mathsf{v}^n}{i^{(m)}} \qquad \text{whenever } nm \in \mathbb{N}$$

$$\mathsf{PV} = \sum_t c_t \mathsf{v}^t, \qquad \mathsf{NPV}(i_0) = 0$$

We now introduce a new object, a bond. In a certain sense, this is the inverse of a loan. Mathematically we already got everything we need.

Today

1. Securities
2. The price of a fixed-interest bond
3. Income tax
4. A worked example

Warm-up

EXERCISE 15.1

A contract pays €3.00 at the end of each of the next three years. It also pays €100.00 at the end of the third. What is it worth today at $i = 5\%$*?*

Warm-up

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We have

$$3\,a_{\overline{3}|} + 100\,v^3 \;=\; 3 \times 2.723248 + 100 \times 0.863838 \;=\; 8.17 + 86.38 \;=\; €94.55.$$

> ASIDE
>
> It is worth less than the €100.00 at the end, because 3% a year of income does not compensate for the 5% interest. Here, you are effectively the lender.

That is a bond. Everything in this lecture is to calcualte this price. But there is an added difficulty, tax.

SECTION 1

Securities

What a security is

DEFINITION 15.2 · Security

A **security** is a tradeable financial instrument. That is, a financial contract that can be bought and sold.

Examples include government bonds, shares and stock, corporate bonds, and certificates of deposit. Two natural and related questions arise:

- What price P per unit nominal should an investor pay to secure a net yield of i per annum?
- If an investor pays price P per unit nominal, what net yield per annum will be obtained?

These are Lecture 10's two directions: value from a rate, or rate from a value. To answer either, we need the issue price, the interest rate, the tax, the redemption price and the redemption date.

Fixed-interest securities

DEFINITION 15.3 · Fixed-interest security

A **fixed-interest security** is a loan floated on a stock exchange by a government or company to raise money, issued in bonds of stated nominal amount. Holders receive a lump sum at a specified date, with regular fixed **coupon** payments until then.

- The **nominal amount** is the face value of the bond.
- The **coupon rate** is the nominal rate D per annum payable to holders.
- The **redemption price** is the rate R per unit nominal payable at redemption.

The word *rate* should be understood more as the word *ratio*, and not be confused with a *stream*. Meaning, you input X and get back $R \times X$, not R per annum.

CAUTION

The nominal amount is a label on the contract, not what the bond costs. That is generally determined by market forces. Everything is quoted *per unit nominal*.

Redemption at, above and below par

$$R = 1 \text{ at par}, \qquad R > 1 \text{ above par}, \qquad R < 1 \text{ below par}.$$



Redemption above par is said to be *at a premium*, below par *at a discount*. Either way, a bond is just a cashflow.

Example: A government bond

EXAMPLE 15.4

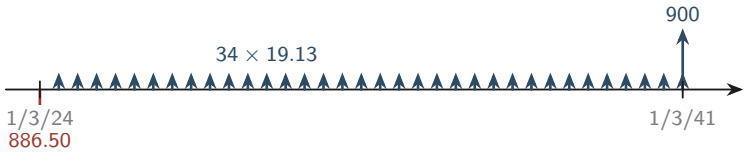
An investor is allocated €900.00 of a fixed-interest government bond issued under the following terms.

Issue date	1 March 2024
Issue size (total nominal)	€2,000,000,000.00
Coupon rate	4.25% p.a. payable half-yearly
Redemption date	1 March 2041
Redemption price	€100.00 per €100.00 nominal
Issue price	€98.50 per €100.00 nominal

Here $R = 1$ (redemption at par) and $D = 0.0425$. The investor's holding is €900.00 *nominal*, not €900.00 of outlay.

What the investor pays and receives

The investor initially pays out $0.985 \times 900 = €886.50$, then receives $\frac{0.0425}{2} \times 900 = €19.13$ every six months for seventeen years — that is 34 coupons — and €900.00 on redemption, with the final coupon.



After the issue date the investor can buy or sell bonds on the stock exchange. This makes the *price* a market driven question, and generally (very) hard to predict.

Strips

Sometimes the interest (coupon) and the redemption proceeds can be bought and sold separately. This creates two bonds:

- A **zero-coupon bond**: No interest payments, just the lump sum;
- A **bond redeemable at zero**: The coupons only, no redemption payment.

These are **strips**. Each payment can be owned by a different investor, and stripped payments can be recombined into the bond.

> PRICING IS UNAFFECTED
>
> The present values add, so the bond is worth the sum of its strips. If it were not, an investor would buy the cheap side, sell the expensive one, and take the difference at no risk. This is Lecture 1's no-arbitrage argument again.

Why value differs between investors, and why strips happen

In practice investors pay tax on the proceeds of investments, and each investor may be subject to a different taxation regime. There can be different rates on income from coupons and on capital gains.

EXAMPLE 15.5

If several investors bid for a bond, all aiming for the same net rate of return, then the guarantee of higher net payments means that the investor with the *lowest* rate of income tax can make the highest bid.

So the same bond is genuinely worth different amounts to different buyers. This is why the rest of the lecture is about tax. Without it there would be one price, and we'd be done.

SECTION 2

The price of a fixed-interest bond

The price is a present value

The price P_0 per unit nominal an investor is willing to pay is the present value of all future coupon and redemption payments, after tax, discounted at the investor's required yield.

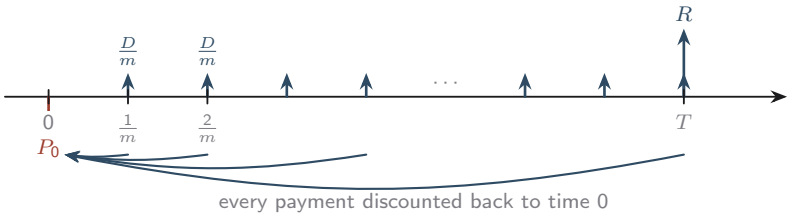
Take the simplest case. Let D be the coupon rate per annum payable m-thly in arrears, redeemable at R at time T. The coupons are payments of $\frac{D}{m}$ at times $\frac{1}{m}, \ldots, T$. This is an annuity of D per annum payable m-thly, worth $D\, a^{(m)}_{\overline{T}|}$.

COUPONS PAYABLE m-THLY IN ARREARS **AND** REDEMPTION (NO-TAX CASE)

$$P_0 \;=\; D\, a^{(m)}_{\overline{T}|} + R\mathrm{v}^T.$$

The warm-up was this, with $m = 1$, $D = 0.03$, $R = 1$, $T = 3$.

The price, drawn



The coupons collapse into $D\, a_{\overline{T}|}^{(m)}$ and the lump sum into $R\mathrm{v}^T$. Thus, the total value is the sum of these as both are now valued at time 0.

Per €100.00nominal

Multiplying through by 100, the price per €100.00 nominal is

$$P \;=\; 100D\, a_{\overline{T}|}^{(m)} + 100R\mathrm{v}^{T} \qquad (\,=\; 100P_0).$$

KEY IDEA

Read this from the POV of Lecture 10's two directions:

- Given i, it *computes* the price P;
- Given P, it is an equation of value whose solution i is the **gross yield**.
  If the payments have been adjusted for tax, the yield obtained is the **net yield**.

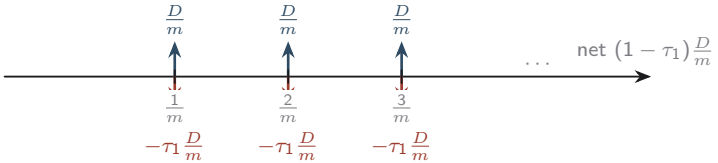
In the second direction there is no formula for i, so we are back to linear interpolation. But, since $i^{(m)}$ itself depends on i, it must be recomputed at each trial rate.

SECTION 3

Income tax

Tax paid when the coupon is paid

Suppose an investor is liable to income tax at rate τ_1 on the coupons, due at the time the coupons are paid.



This gives two cashflows at each coupon date, and we can simply take their difference. So the price per unit nominal, at effective rate i, being now the *net* yield, is

$$P_0 \;=\; (1-\tau_1)D\, a_{\overline{T}|}^{(m)} + R\mathrm{v}^T.$$

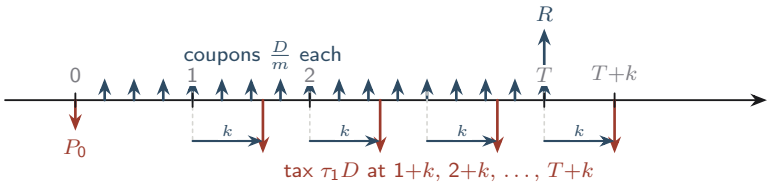
Tax paid later

In some countries the tax is paid at a later date, for example at the calendar year end, in April, or some other fixed date. This is an added step, but nothing new. It is still a cashflow and we follow the usual procedure of identifying the cashflow amounts and dates, and get the equation of value as in Lecture 10.

Suppose $T \in \mathbb{N}$ and the annual income tax on the bond is paid in a single instalment, due k years after the final coupon of **each** year. Note, k is usually a rational number $0 \leq k \leq 1$.

> KEY IDEA
>
> Each year the investor receives coupons totalling D per unit nominal, so the tax bill for year r is $\tau_1 D$. It is settled at time $r + k$, not at time r.

The whole cashflow, with tax lagging behind



The coupons run to T and the redemption arrives at T, but the last tax instalment is not paid until $T + k$. That is, k steps after the bond itself has matured.

The equation of value

Reading the payments off the diagram, for a given net yield i, we get

$$P_0 \;=\; D\,a^{(m)}_{\overline{T}|} + R\mathsf{v}^{T} - \tau_1 D\left(\mathsf{v}^{k+1} + \cdots + \mathsf{v}^{k+T}\right) \;=\; D\,a^{(m)}_{\overline{T}|} + R\mathsf{v}^{T} - \tau_1 D\mathsf{v}^{k} a_{\overline{T}|},$$

since $\mathsf{v}^{k+1} + \cdots + \mathsf{v}^{k+T} = \mathsf{v}^{k}\left(\mathsf{v} + \cdots + \mathsf{v}^{T}\right) = \mathsf{v}^{k} a_{\overline{T}|}$.

> KEY IDEA
>
> The coupons are taken *gross* here, and the tax subtracted as a separate annuity of T payments deferred by k. Recall the deferred annuity of Lecture 9, ${}_{k|}a_{\overline{T}|} = \mathsf{v}^{k} a_{\overline{T}|}$.

Paying the same tax later is worth more to the investor, so this price is higher than the tax-at-source price.

SECTION 4

A worked example

The question

EXAMPLE 15.6

A fixed-interest bond pays a coupon of 8% per annum quarterly, with the next coupon due on 31 December 2020. The bond is redeemable at 115% on 1 October 2039. Calculate the price an investor, liable to income tax of 30% at source, would pay on 1 January 2021 to receive an effective net yield of 10% per annum.

> CAUTION
>
> Note the dates: The coupon of 31 December 2020 is paid *before* the purchase on 1 January 2021, so the buyer does not receive it. The first coupon they do receive is a quarter later.

Reading off the data

Let P_0 be the price per €100.00 nominal. We are given

$T = 18.75$	Redemption date	1 Jan 2021 to 1 Oct 2039
$D = 0.08$	Coupon rate	Nominal, quarterly
$R = 1.15$	Redemption price per unit nominal	115%
$i = 0.1$	Yield p.a.	Net
$\tau_1 = 0.3$	Rate of income tax	

KEY IDEA

$T = 18.75$ is not an integer, but $18.75 \times 4 = 75$ is. Thus, there are exactly 75 quarterly coupons and $a^{(4)}_{\overline{18.75}|}$ is defined by Lecture 14.

The calculation

The equation of value gives the price per €100.00 nominal:

$$\begin{aligned} P_0 &= (1-0.3)\,8\,a_{\overline{18.75}|}^{(4)} + 100\cdot 1.15\,\mathrm{v}^{18.75} \\ &= 5.6\left(\frac{1-\mathrm{v}^{18.75}}{i^{(4)}}\right) + 115\,\mathrm{v}^{18.75} \\ &= 5.6\left(\frac{1-(1.1)^{-18.75}}{0.096454756}\right) + 115\,(1.1)^{-18.75} \\ &= 5.6\times 8.63150002 + 115\times 0.16745077 \;=\; €67.59, \end{aligned}$$

where $i^{(4)} = 4\left((1.1)^{1/4}-1\right) = 0.096454756$.

Interpreting the answer

The investor pays €67.59 per €100.00 nominal, and receives €2.00 of coupon each quarter, which reduces to €1.40 after tax, plus €115.00 at redemption.

	Value at 10%	Share of price
75 net coupons of €1.40	48.34	71.5%
Redemption of €115.00 at T	19.26	28.5%
Price	67.59	

KEY IDEA

The price is well below par. The required 10% is far above the 8% coupon, and tax widens the gap. Untaxed, the same bond would be worth €88.31.

Lecture 15 on one slide

A BOND IS A CASHFLOW, AND ITS VALUE IS ITS PRICE

$$\underbrace{P_0 = \underbrace{D\,a_{\overline{T}|}^{(m)}}_{\text{Coupons}} + \underbrace{R\mathsf{v}^T}_{\text{Redemption}}}_{\text{Gross without tax}}, \qquad \underbrace{P_0 = \underbrace{(1-\tau_1)D\,a_{\overline{T}|}^{(m)}}_{\text{Taxed coupons}} + \underbrace{\mathsf{v}^T}_{\text{Redemption}}}_{\text{Net with coupons taxed at source}}$$

$$\underbrace{P_0 = D\,a_{\overline{T}|}^{(m)} + R\mathsf{v}^T - \tau_1 D\,\mathsf{v}^k a_{\overline{T}|}}_{\text{Net with tax payed } k \text{ years in arrears}}.$$

Everything is per unit nominal. Given i, the formula computes a price; given a price it is an equation of value for the yield.

Exercises

EXERCISE 15.7

A bond pays coupons of 6% *per annum half-yearly, redeemable at par in ten years. Find the price per* €100.00 *nominal for a gross yield of* 7%*.*

EXERCISE 15.8

For the same bond, price it for an investor paying income tax at 40% *at source and requiring* 7% *net; then again with the tax paid half a year after each year's final coupon.*

EXERCISE 15.9

A bond is stripped into its coupons and its redemption payment. Why must the two strips be worth the bond?

Solutions are in the backup slides.

END OF LECTURE 15

The price falls with tax, and rises when it is deferred. Next, we will see what happens when the gain itself is taxed. This will make the price appears on both sides of its own equation.

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Solution 1

Here $D = 0.06$, $R = 1$, $T = 10$, $m = 2$ and $i = 0.07$. First the nominal rate that goes underneath the annuity:

$$i^{(2)} \;=\; 2\left((1.07)^{1/2} - 1\right) \;=\; 0.0688161, \qquad \mathrm{v}^{10} \;=\; (1.07)^{-10} \;=\; 0.5083493,$$

$$a^{(2)}_{\overline{10}|} \;=\; \frac{1 - 0.5083493}{0.0688161} \;=\; 7.144415.$$

Hence, per €100.00 nominal,

$$P \;=\; 6 \times 7.144415 + 100 \times 0.5083493 \;=\; €93.70.$$

Below par, as it must be: the required 7% exceeds the 6% coupon, so the shortfall has to be made up by buying cheaply.

Solution 2

Tax at source. The coupons are worth $(1 - 0.4) \times 6 = 3.6$ per annum net, and the redemption is untouched:

$$P \;=\; 3.6 \times 7.144415 + 50.83493 \;=\; €76.55.$$

Tax paid at $k = 0.5$. Now take the coupons gross and subtract the tax as a deferred annuity, with $a_{\overline{10}|} = 7.023582$ and $v^{0.5} = 0.9667365$:

$$P \;=\; 42.866 + 50.835 - 0.4 \times 6 \times 0.9667365 \times 7.023582 \;=\; €77.41.$$

Higher than €76.55, and it must be: the same tax is paid later, so its present value is smaller, and the investor can pay more and still earn 7%.

Solution 3

Present values add. The coupon strip is worth $D\, a_{\overline{T}|}^{(m)}$ and the redemption strip $R\mathrm{v}^{T}$; their sum is exactly the price of the whole bond.

Suppose it were not. If the two strips together sold for *less* than the bond, an investor could buy both, recombine them, sell the bond, and pocket the difference. This needs no capital tied up and no risk. If they sold for *more*, the same trade run backwards would do it: Buy the bond, strip it, sell the pieces.

It is not that there is a magic law forbidding these things to exist. They do. But such trades continue until the prices agree, which in a free market they *eventually* will. This is the no-arbitrage argument of Lecture 1, the one that forced markets to be consistent, applied to a bond instead of to a deposit account.