LECTURE 13

Interest and capital

What each repayment is really paying for

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Recall

$$X_t = b_t + f_t$$

$$L_t = \sum_{k=t+1}^{n} X_k \mathsf{v}^{k-t}$$

$$L_t = L_0(1+i)^t - \sum_{k=1}^{t} X_k (1+i)^{t-k}$$

In last lecture, we understood which parts of a loan are interest and which parts are capital being backpayed by looking forwards or backwards.

Now, we want to look at whole blocks of them at once.

And ask, when *exactly* the capital overtakes the loan.

Today

1. One instalment
2. A block of instalments
3. The ten-year loan

Warm-up

EXERCISE 13.1

A loan of €10,000.00 *is to be repaid over ten years by a level annuity payable monthly in arrears, at an interest rate of* 1% *per month effective. Find the monthly repayment, and split the first one into interest and capital.*

The rate is quoted per *month*, so take the month as the unit of time and it is already effective.

Warm-up: solution

With the month as the unit, $i = 0.01$ and $n = 120$. The equation of value gives

$$10{,}000 \;=\; X\, a_{\overline{120}|} \qquad \Longrightarrow \qquad X \;=\; \frac{10{,}000 \times 0.01}{1-(1.01)^{-120}} \;=\; \frac{100}{0.697005} \;=\; €143.47.$$

The first month's interest is charged on the whole €10,000.00, so

$$b_1 \;=\; iL_0 \;=\; €100.00, \qquad f_1 \;=\; X - b_1 \;=\; €43.47.$$

> ASIDE
>
> Over the ten years the borrower hands over $120 \times 143.47 = €17{,}216.40$ for a loan of €10,000.00. Of the very first payment, barely a third went towards the loan.

SECTION 1

One instalment

Splitting a single repayment

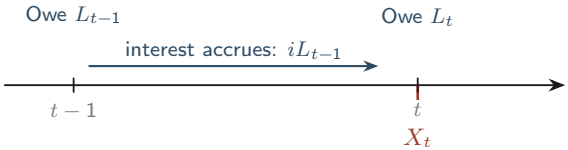
If we want to split the instalment X_t payed at time t in interest and capital backpay, look at the previous payment time $t-1$. After paying X_{t-1}, the outstanding loan is L_{t-1}. The interest due at time t is thus

$$b_t \;=\; iL_{t-1},$$

and this is the interest of X_t. The capital repaid may then be found either as

$$f_t \;=\; X_t - iL_{t-1} \qquad \text{or as} \qquad f_t \;=\; L_{t-1} - L_t.$$

The same statement as a picture



KEY IDEA

Over the period the debt grows by iL_{t-1}. The payment X_t settles that first, and whatever is left over reduces the debt. The two expressions for f_t agree and thus

$$X_t - iL_{t-1} \;=\; L_{t-1} - L_t.$$

The capital repayments form a geometric progression

From the previous lecture (12), $f_t = X\mathsf{v}^{n-t+1}$ for *level instalments*, and one step earlier $f_{t-1} = X\mathsf{v}^{n-t+2}$. As $\mathsf{v}^{-1} = 1+i$,

$$f_t \;=\; X\mathsf{v}^{n-t+1} \;=\; X\mathsf{v}^{n-t+2}(1+i) \;=\; f_{t-1}(1+i).$$

Iterating the argument gives us

$$\boxed{f_t \;=\; f_{t-1}(1+i) \;=\; f_{t-2}(1+i)^2 \;=\; \cdots \;=\; f_1(1+i)^{t-1}}$$

KEY IDEA

The capital content grows at exactly the rate of interest, from the first value $f_1 = X - iL_0$. Knowing *any one* capital repayment gives you all the others!

Another way to look at a loan repayment

You can think of it like this: Imagine you have a loan of €10,000.00 at effective interest rate of 1% per term. You can pay the €143.47 each term towards the loan.

But what if you paid only the interest, being €100.00 each turn (effectively made it a perpetuity for a while) and put the remaining €43.47 in a savings account at the same effective interest of 1% per turn?

The loan remains unchaining, but your savings account accrues at a rate of 1% per turn, until eventually it reaches €10,000.00, and then you pay back the full loan.

What we effectively showed is that these two approaches would agree (assuming identical payment times etc.).

This is called a *sinking fund* (not part of this module).

The warm-up loan, revisited

We already have $f_1 = €\,43.47$, so every capital repayment of the ten-year loan is known without a single balance being computed:

$$f_t \;=\; 43.47\,(1.01)^{\,t-1}.$$

Month t	1	12	60	119	120
f_t	43.47	48.50	78.19	140.64	142.05
b_t	100.00	94.97	65.28	2.83	1.42

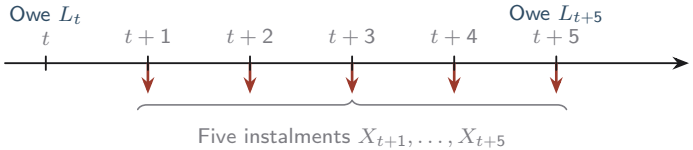
The interest column is $€\,143.47 - f_t$ since $b_t = X - f_t$.

SECTION 2

A block of instalments

The question

We can find the interest paid and capital repaid over *several* instalments at once in a similar way. Consider five instalments from $t+1$ to $t+5$, inclusive.



The loan outstanding just before the first of these is L_t. After the fifth, it is L_{t+5}. How much of the money handed over in between was interest, and how much capital?

Step 1: The total capital repaid

Capital repaid is, by definition, the amount by which the debt falls. In a single period $f_k = L_{k-1} - L_k$, so over the five periods

$$\sum_{k=t+1}^{t+5} f_k = \big(L_t - L_{t+1}\big) + \big(L_{t+1} - L_{t+2}\big) + \big(L_{t+2} - L_{t+3}\big) + \big(L_{t+3} - L_{t+4}\big) + \big(L_{t+4} - L_{t+5}\big).$$

We see that the intermediate terms cancel leaving the first and the last:

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$$+ \cancel{\left(L_{t+3} - L_{t+4}\right)} + \cancel{\left(L_{t+4} - L_{t+5}\right)}.$$

We see that the intermediate terms cancel leaving the first and the last:

$$\boxed{\sum_{k=t+1}^{t+5} f_k = L_t - L_{t+5}}$$

Step 2: The total paid

The five instalments are handed over in full and the total of capital *and* interest paid is

$$X_{t+1} + X_{t+2} + X_{t+3} + X_{t+4} + X_{t+5}.$$

But every instalment splits as $X_k = f_k + b_k$ and thus, their sum over the five payments is

$$\sum_{k=t+1}^{t+5} X_k \;=\; \underbrace{\sum_{k=t+1}^{t+5} f_k}_{\text{capital}} \;+\; \underbrace{\sum_{k=t+1}^{t+5} b_k}_{\text{interest}}.$$

We know the total on the left, and Step 1 gave us the first summand of the RHS.

Step 3: The interest

Rearranging, and substituting $\sum f_k = L_t - L_{t+5}$ gives us

$$\sum_{k=t+1}^{t+5} b_k \;=\; \big(X_{t+1} + X_{t+2} + X_{t+3} + X_{t+4} + X_{t+5}\big) - \big(L_t - L_{t+5}\big)$$

IN WORDS

Interest paid = Money handed over − Reduction in the debt. Note that we do not need individual interest b_k's or intermediate balance, just the start and end.

Nothing in the argument used level instalments or a constant rate. The same argument goes through any block of consecutive payments.

SECTION 3

The ten-year loan

The first year, prospectively

EXAMPLE 13.2

For the warm-up loan, €10,000.00 with 120 monthly instalments of €143.47 and $i = 1\%$ per month, find the capital repaid and the interest paid in the first year.

By Step 1 with $t = 0$ the capital repaid in the first twelve months is $L_0 - L_{12}$. Lecture 12's prospective method gives L_{12} as the value of the 108 payments still to come. Thus

$$L_{12} \;=\; X\, a_{\overline{108}|} \;=\; 143.47 \times 65.857790 \;=\; €9{,}448.62,$$

so the capital repaid is $10{,}000 - 9{,}448.62 = €551.38$.

The first year, the other way

Alternatively, use the geometric progression of Section 1. The twelve capital repayments are $f_1,\ f_1(1.01),\ \ldots,\ f_1(1.01)^{11}$, so

$$\sum_{k=1}^{12} f_k \;=\; f_1 \sum_{k=1}^{12} (1.01)^{k-1} \;=\; 43.47\, s_{\overline{12|}} \;=\; 43.47 \times 12.682503 \;=\; €551.31.$$

A WORD ABOUT THE SEVEN CENTS

The two routes gave €551.38 and €551.31. Neither is wrong, its just a rounding error and it goes to show that even carying 6 significant digits in L_{12} can give a small but noticible differnece. Carrying full precision gives €551.32.

Round *once*, at the end, not at every step. On a real loan the capital repayments are adjusted slightly so that they total exactly the sum lent, which is why a final instalment is often a few cents different from the others.

The first year against the last

The total paid in any year is $143.47 \times 12 = €1{,}721.64$. So, by Step 3, the interest paid in the first year is

$$1{,}721.64 - 551.38 \;=\; €1{,}170.26.$$

In the first year the interest *exceeds* the capital repaid, by more than two to one.

The first year against the last

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$$1{,}721.64 - 551.38 \;=\; €1{,}170.26.$$

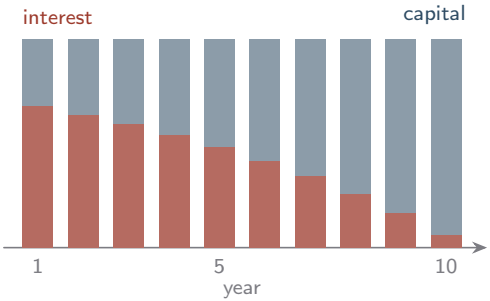
In the first year the interest *exceeds* the capital repaid, by more than two to one.

The capital repaid in the final year is whatever was still outstanding at the start of it, namely

$$L_{108} \;=\; X\, a_{\overline{12}|} \;=\; 143.47 \times 11.255077 \;=\; €1{,}614.77,$$

so the interest paid in the final year is $1{,}721.64 - 1{,}614.77 = €106.87$. There, the capital repaid exceeds the interest, by fifteen to one.

The whole loan, year by year



Every bar is the same height as the payments are level. What changes is the split.

Interest falls from €1,170.00 to €107.00. The capital rises from €551.00 to €1,615.00, by a factor $(1.01)^{12}$ each year. The two cross during year five.

When does capital overtake interest?

EXAMPLE 13.3

For which monthly repayment does the capital content first exceed the interest content?

Since $X = f_t + b_t$, we want the least integer t with

$$b_t < f_t \quad \Longleftrightarrow \quad X - f_t < f_t \quad \Longleftrightarrow \quad X < 2f_t.$$

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Now use $f_t = X\mathsf{v}^{n-t+1} = X\mathsf{v}^{121-t}$, so that X cancels:

$$X < 2X\mathsf{v}^{121-t} \quad \Longleftrightarrow \quad \tfrac{1}{2} < (1.01)^{-(121-t)} \quad \Longleftrightarrow \quad (1.01)^{121-t} < 2.$$

Solving for t

Taking logarithm and dividing by $\ln(1.01) > 0$, we get

$$121 - t \;<\; \frac{\ln 2}{\ln(1.01)} \;=\; 69.66, \qquad \Longleftrightarrow \qquad t \;>\; 121 - 69.66 \;=\; 51.34.$$

Hence $t = 52$. It is the 52nd monthly repayment at which the capital content first exceeds the interest content.

> KEY IDEA
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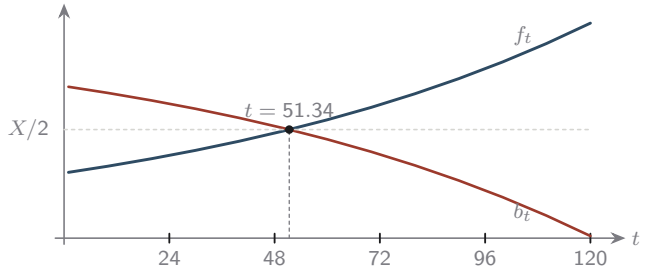
> Because X cancelled, no rounded quantity ever entered the calculation, and the answer depends only on the term and the rate, not on the €10,000.00. Double the loan and the crossover month is unchanged.

What that looks like



The two curves are mirror images about the line $X/2$, because $f_t + b_t = X$ for every t. They cross where each equals $X/2$, and the first *integer* month past that point is $t = 52$.

Lecture 13 on one slide

INTEREST IS WHAT YOU PAID THAT DID NOT REDUCE THE DEBT

$$b_t = iL_{t-1}, \qquad f_t = X_t - iL_{t-1} = L_{t-1} - L_t, \qquad f_t = f_1(1+i)^{t-1} \text{ (level)}$$

$$\sum_{k=t+1}^{t+r} f_k = L_t - L_{t+r}, \qquad \sum_{k=t+1}^{t+r} b_k = \sum_{k=t+1}^{t+r} X_k - \big(L_t - L_{t+r}\big)$$

The capital repayments telescope, so a whole block needs only its two end balances. Under level repayments the capital content grows by $(1+i)$ each period and the interest content falls to match.

Exercises

EXERCISE 13.4

A loan of €30,000.00 *is repaid by* 15 *level annual instalments in arrears at* $i = 7\%$*. Find the instalment, and the interest and capital content of the first one.*

EXERCISE 13.5

For the same loan, find the total capital repaid and the total interest paid over years 6 *to* 10 *inclusive, using* L_5 *and* L_{10} *only.*

EXERCISE 13.6

Still with the same loan, find the first instalment whose capital content exceeds its interest content, and check the answer does not depend on the €30,000.00*.*

Solutions are in the backup slides.

END OF LECTURE 13

One loan, taken apart. Next: the schedule in general — when the rate moves, and when the payments come faster than the years.

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Solution 1

At 7%, $a_{\overline{15}|} = 9.107914$, so the instalment is

$$X \;=\; \frac{30{,}000}{9.107914} \;=\; €\,3{,}293.84.$$

The first month's interest is charged on the whole loan, so

$$b_1 \;=\; iL_0 \;=\; 0.07 \times 30{,}000 \;=\; €\,2{,}100.00, \qquad f_1 \;=\; X - b_1 \;=\; €\,1{,}193.84.$$

As a check on f_1 from the closed form, $f_1 = X\mathrm{v}^{15} = 3293.84 \times 0.362446 = €\,1{,}193.84$, which agrees.

Solution 2

Use Step 1, which needs only the two end balances. Prospectively,

$$L_5 \;=\; X\, a_{\overline{10|}} \;=\; €\,23{,}134.55, \qquad L_{10} \;=\; X\, a_{\overline{5|}} \;=\; €\,13{,}505.39,$$

so the capital repaid over years 6 to 10 is

$$L_5 - L_{10} \;=\; €\,9{,}629.16.$$

Five instalments total $5X = €\,16{,}469.19$, so by Step 3 the interest paid over the same five years is

$$16{,}469.19 - 9{,}629.16 \;=\; €\,6{,}840.03.$$

No individual year's interest was needed at any point.

Solution 3

With $n = 15$, the clean route asks for the least integer t with $X < 2X\mathrm{v}^{16-t}$, that is $(1.07)^{16-t} < 2$:

$$16 - t \;<\; \frac{\ln 2}{\ln 1.07} \;=\; 10.24 \qquad \Longrightarrow \qquad t \;>\; 5.76,$$

so $t = 6$. Checking either side, $f_5 = X\mathrm{v}^{11} = 1564.88$ against $b_5 = 1728.96$, while $f_6 = X\mathrm{v}^{10} = 1674.42$ against $b_6 = 1619.42$.

The loan amount cancelled at the first step, so the answer depends only on n and i: the same fifteen-year loan at 7% crosses over at the sixth instalment whether it is for €30,000.00 or €30.00.