LECTURE 12

Loan repayment

Interest, capital, and remaining

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5 October 2026 · Winter 2026

From the outside to the inside



a loan, from outside

Thus far, we considered loans as single cash-flows and only equated the borrowed value with the repaid value. We could calculate instalment sizes and that was all we asked.

But what if we want to know how much of each payment covered the interest and how much the capital? This is today's topic.

Today

1. What a repayment is doing
2. The capital outstanding
3. The prospective method
4. The retrospective method

Warm-up

EXERCISE 12.1

A bank lends €1,000.00 *in return for three level payments of* X*, one at the end of each year. The bank charges an effective rate of interest of* 7% *per annum. Find* X*.*

Warm-up

EXERCISE 12.1

A bank lends €1,000.00 *in return for three level payments of* X*, one at the end of each year. The bank charges an effective rate of interest of* 7% *per annum. Find* X*.*

Three level payments in arrears, meaning $a_{\overline{3}|}$, with

$$1000 = X\, a_{\overline{3}|} \implies X = \frac{1000}{a_{\overline{3}|}} = \frac{1000 \times 0.07}{1-(1.07)^{-3}} = \frac{70}{0.183702} = €381.05.$$

In total, $3 \times 381.05 =$ €1,143.15 repaid. €1,000.00 was capital and €143.15 interest.

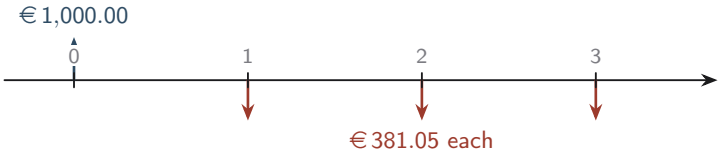
But how much of *each* payment was the interest and how much the capital being repaid?

SECTION 1

What a repayment is doing

What the three payments are doing

A very common transaction involving compound interest is a loan repaid by regular instalments, at a fixed rate, for a predetermined term. These three payments cover both the interest due *and* the €1,000.00 of capital.



The split is not fixed in advance by the contract, just whatever needs to be repaid to make the balance zero.

Working through it, year by year

1. The interest due at time 1 is €70.00. The payment is €381.05, leaves €311.05 to repay the capital, leaving $1000 - 311.05 = €688.95$ outstanding.
2. At time 2 the interest due is 7% of €688.95, that is €48.23. The borrower pays interest only on what is still outstanding, not on capital already repaid. That leaves $381.05 - 48.23 = €332.83$ for capital, and $688.95 - 332.83 = €356.12$ remains.
3. At time 3 the interest due is 7% of €356.12, that is €24.93, leaving $381.05 - 24.93 = €356.12$. This is precisely the outstanding capital. The loan is repaid exactly.

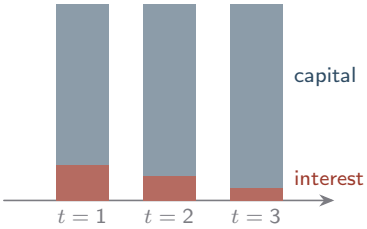
We repaid the interest *first*, from the balance, and then paid on the capital afterwards.

The schedule

t	Payment	Interest	Capital	Outstanding
0	—	—	—	1,000.00
1	381.05	70.00	311.05	688.95
2	381.05	48.23	332.83	356.12
3	381.05	24.93	356.12	0.00
	1,143.15	143.15	1,000.00	

The capital column sums to the loan and the interest column to the total interest. A useful check on any schedule.

The same table, drawn



Each repayment pays the interest due on the outstanding capital first. The remainder goes to the capital.

Where the repayments are level, the interest component decreases as capital is repaid. It has the consequence that the capital component increases.

It may be necessary to identify the separate elements, for tax purposes, for example.

SECTION 2

The capital outstanding

Notation

Let L_t be the loan outstanding at time $t = 0, 1, \ldots, n$, *immediately after* the repayment at t. The repayments are instalments of size X_t at time t, for $t = 1, \ldots, n$.

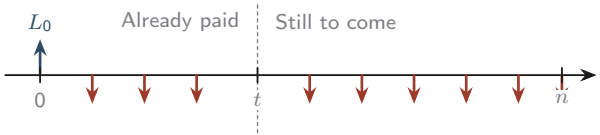
> CAUTION
>
> We are **not** assuming that all instalments are of the same amount.

Let i be the effective rate charged on the loan, f_t the capital repaid at t, and b_t the interest paid at t. This gives us $X_t = f_t + b_t$. The equation of value at time 0 is then

$$L_0 \;=\; X_1 \mathsf{v} + X_2 \mathsf{v}^2 + \cdots + X_n \mathsf{v}^n. \tag{1}$$

Two directions of travel

The loan outstanding at time t can be found **prospectively** or **retrospectively**.



Standing at t, there are two ways to say what is owed. We can look *forward* at what still needs to be paid, or *backward* at the loan and what has already been repaid. These must agree.

We will show by the end of the lecture that they do indeed.

SECTION 3

The prospective method

One step of the recursion

Start at the end. After the final instalment, the loan must be repaid exactly. Thus, X_n must cover whatever remains of the capital after repaying at $n-1$, together with the interest on it. That is, $b_n = iL_{n-1}$ and $f_n = L_{n-1}$, giving $X_n = (1+i)L_{n-1}$.

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Start at the end. After the final instalment, the loan must be repaid exactly. Thus, X_n must cover whatever remains of the capital after repaying at $n-1$, together with the interest on it. That is, $b_n = iL_{n-1}$ and $f_n = L_{n-1}$, giving $X_n = (1+i)L_{n-1}$.

The same holds one step at a time. At any time $t+1$ with $0 \leq t \leq n-1$, the capital repaid is $L_t - L_{t+1}$, and the interest is iL_t. Thus

$$X_{t+1} \;=\; f_{t+1} + b_{t+1} \;=\; (L_t - L_{t+1}) + iL_t \;=\; (1+i)L_t - L_{t+1},$$

$$\boxed{L_t \;=\; (L_{t+1} + X_{t+1})\,\mathsf{v}}$$

In words: What you owe now is next year's payment, plus what you will still owe after that, discounted one year.

Unwinding the recursion

We have $L_t = (L_{t+1} + X_{t+1})\mathrm{v}$, and equally $L_{t+1} = (L_{t+2} + X_{t+2})\mathrm{v}$, and so on. Substituting repeatedly until we reach $L_n = 0$:

Unwinding the recursion

We have $L_t = (L_{t+1} + X_{t+1})\mathsf{v}$, and equally $L_{t+1} = (L_{t+2} + X_{t+2})\mathsf{v}$, and so on. Substituting repeatedly until we reach $L_n = 0$:

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$$L_t = L_{t+1}\mathsf{v} + X_{t+1}\mathsf{v}$$

Expanding brackets

Unwinding the recursion

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$$L_t = X_{t+1}\mathsf{v} + X_{t+2}\mathsf{v}^2 + L_{t+2}\mathsf{v}^2$$

Reordering

Unwinding the recursion

We have $L_t = (L_{t+1} + X_{t+1})\mathrm{v}$, and equally $L_{t+1} = (L_{t+2} + X_{t+2})\mathrm{v}$, and so on. Substituting repeatedly until we reach $L_n = 0$:

$$L_t = X_{t+1}\mathrm{v} + X_{t+2}\mathrm{v}^2 + \boxed{L_{t+2}}\,\mathrm{v}^2$$

$L_{t+2} = (L_{t+3} + X_{t+3})\mathrm{v}$

Unwinding the recursion

We have $L_t = (L_{t+1} + X_{t+1})\mathsf{v}$, and equally $L_{t+1} = (L_{t+2} + X_{t+2})\mathsf{v}$, and so on. Substituting repeatedly until we reach $L_n = 0$:

$$L_t = X_{t+1}\mathsf{v} + X_{t+2}\mathsf{v}^2 + (L_{t+3} + X_{t+3})\mathsf{v}^3$$

Unwinding the recursion

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Expanding brackets

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Unwinding the recursion

We have $L_t = (L_{t+1} + X_{t+1})\mathsf{v}$, and equally $L_{t+1} = (L_{t+2} + X_{t+2})\mathsf{v}$, and so on. Substituting repeatedly until we reach $L_n = 0$:

$$L_t = X_{t+1}\mathsf{v} + X_{t+2}\mathsf{v}^2 + \cdots + X_{t+k}\mathsf{v}^k + \boxed{L_{t+k}}\,\mathsf{v}^k$$

Where $k = n - t$

Unwinding the recursion

We have $L_t = (L_{t+1} + X_{t+1})\mathsf{v}$, and equally $L_{t+1} = (L_{t+2} + X_{t+2})\mathsf{v}$, and so on. Substituting repeatedly until we reach $L_n = 0$:

$$\boxed{L_t = X_{t+1}\mathsf{v} + X_{t+2}\mathsf{v}^2 + \cdots + X_n\mathsf{v}^{n-t}}$$

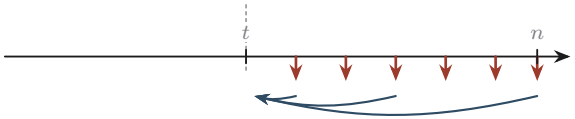
The remainder term $L_{t+k}\mathsf{v}^k$ is carried along at every stage and vanishes only at the last step, where $L_n = 0$.

The prospective method

PROSPECTIVE METHOD

The loan outstanding at time t is the discounted value at time t of the future repayment instalments:

$$L_t = X_{t+1}\mathsf{v} + X_{t+2}\mathsf{v}^2 + \cdots + X_n\mathsf{v}^{n-t}.$$



Discount the future instalments back to t

The prospective method

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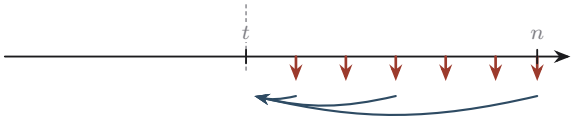

Discount the future instalments back to t

This is just Equation (1) shifted to time t, which is clear in hindsight: At time t the outstanding loan is L_t, just as at time 0 it was L_0. Refinancing it with a new loan of size L_t changes nothing, so the formula has the same form, with L_t replacing L_0. Or simpler: *Money has no memory.*

Why the prospective formula is just a shift (as an example)

t	Payment	Interest	Capital	Outstanding
0	—	—	—	1,000.00
1	142.38	70.00	72.38	927.62
2	142.38	64.93	77.44	850.18
3	142.38	59.51	82.87	767.31
4	142.38	53.71	88.67	678.65
5	142.38	47.51	94.87	583.78
6	142.38	40.86	101.51	482.26
7	142.38	33.76	108.62	373.64
8	142.38	26.16	116.22	257.42
9	142.38	18.02	124.36	133.06
10	142.38	9.31	133.06	0.00
	1,423.78	423.78	1,000.00	

t	Payment	Interest	Capital	Outstanding
0	—	—	—	678.65
1	142.38	47.51	94.87	583.78
2	142.38	40.86	101.51	482.26
3	142.38	33.76	108.62	373.64
4	142.38	26.16	116.22	257.42
5	142.38	18.02	124.36	133.06
6	142.38	9.31	133.06	0.00
	854.27	175.62	678.65	

Level instalments

If all instalments are equal, $X_t = X$ for $t = 1, \dots, n$, the sum is an ordinary annuity of $n - t$ payments:

$$L_t \;=\; X\left(\mathsf{v} + \mathsf{v}^2 + \cdots + \mathsf{v}^{n-t}\right) \;=\; X\, a_{\overline{n-t}|}.$$

EXAMPLE 12.2

In the warm-up loan, with $X = 381.05$, $n = 3$ and $i = 7\%$,

$$L_1 \;=\; X\, a_{\overline{2}|} \;=\; 381.05 \times 1.808018 \;=\; €688.95, \qquad L_2 \;=\; X\, a_{\overline{1}|} \;=\; €356.12,$$

we get the "outstanding" column of the schedule we constructed without having to build the schedule at all.

Interest and capital, in closed form

Still with level instalments, the interest paid at time t is

$$b_t \;=\; iL_{t-1} \;=\; i\,X\,a_{\overline{n-t+1}|} \;=\; X\left(1-\mathsf{v}^{\,n-t+1}\right),$$

using $i\,a_{\overline{m}|} = 1-\mathsf{v}^m$. The capital repaid at time t is

$$f_t \;=\; L_{t-1}-L_t \;=\; X\left(a_{\overline{n-t+1}|}-a_{\overline{n-t}|}\right) \;=\; X\mathsf{v}^{\,n-t+1}.$$

Interest and capital, in closed form

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We could also get to it from $b_t+f_t=X_t$, and indeed, we observe

$$b_t+f_t \;=\; X\left(1-\mathsf{v}^{\,n-t+1}\right)+X\mathsf{v}^{\,n-t+1} \;=\; X \;=\; X_t \quad \text{(level payments)}.$$

Interest and capital, in closed form

Still with level instalments, the interest paid at time t is

$$b_t \;=\; iL_{t-1} \;=\; i\,X\,a_{\overline{n-t+1}|} \;=\; X\left(1-\mathrm{v}^{n-t+1}\right),$$

using $i\,a_{\overline{m}|} = 1-\mathrm{v}^m$. The capital repaid at time t is

$$f_t \;=\; L_{t-1}-L_t \;=\; X\left(a_{\overline{n-t+1}|}-a_{\overline{n-t}|}\right) \;=\; X\mathrm{v}^{n-t+1}.$$

KEY IDEA

So $f_{t+1} = X\mathrm{v}^{n-t} = (1+i)f_t$. Under level repayments, the capital content grows by a factor of exactly $(1+i)$ each period. The interest content falls to match. In the schedule, $311.05 \times 1.07 = 332.83$ and $332.83 \times 1.07 = 356.12$.

SECTION 4

The retrospective method

Starting from the beginning instead

At $t = 1$ the interest due is $b_1 = iL_0$, so the capital repaid is $f_1 = X_1 - iL_0$ and the outstanding loan is

$$L_1 \;=\; L_0 - f_1 \;=\; L_0 - (X_1 - iL_0) \;=\; L_0(1+i) - X_1.$$

In general, at time $t \geq 1$ the interest due is $b_t = iL_{t-1}$, so $f_t = X_t - iL_{t-1}$, giving

$$\boxed{L_t \;=\; L_{t-1} - f_t \;=\; L_{t-1}(1+i) - X_t}$$

KEY IDEA

Roll the balance forward one period at interest, then subtract what was paid. This is closer to how a bank's ledger actually works.

Unrolling the other way

We have $L_t = L_{t-1}(1+i) - X_t$, and equally $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$, and so on. Working back from time t to time 0, we get:

Unrolling the other way

We have $L_t = L_{t-1}(1+i) - X_t$, and equally $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$, and so on. Working back from time t to time 0, we get:

$$L_t = \boxed{L_{t-1}(1+i) - X_t}$$

Unrolling the other way

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Unrolling the other way

We have $L_t = L_{t-1}(1+i) - X_t$, and equally $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$, and so on. Working back from time t to time 0, we get:

$$L_t = \big(L_{t-2}(1+i) - X_{t-1}\big)(1+i) - X_t$$

Unrolling the other way

We have $L_t = L_{t-1}(1+i) - X_t$, and equally $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$, and so on. Working back from time t to time 0, we get:

$$L_t = L_{t-2}(1+i)^2 - X_{t-1}(1+i) - X_t$$

Expanding brackets

Unrolling the other way

We have $L_t = L_{t-1}(1+i) - X_t$, and equally $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$, and so on. Working back from time t to time 0, we get:

$$L_t = L_{t-2}(1+i)^2 - X_{t-1}(1+i) - X_t$$

$$L_{t-2} = L_{t-3}(1+i) - X_{t-2}$$



Unrolling the other way

We have $L_t = L_{t-1}(1+i) - X_t$, and equally $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$, and so on. Working back from time t to time 0, we get:

$$L_t = \big(L_{t-3}(1+i) - X_{t-2}\big)(1+i)^2 - X_{t-1}(1+i) - X_t$$

Unrolling the other way

We have $L_t = L_{t-1}(1+i) - X_t$, and equally $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$, and so on. Working back from time t to time 0, we get:

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Expanding brackets

Unrolling the other way

We have $L_t = L_{t-1}(1+i) - X_t$, and equally $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$, and so on. Working back from time t to time 0, we get:

$$L_t = L_{t-j}(1+i)^j - X_{t-j+1}(1+i)^{j-1} - \cdots - X_{t-1}(1+i) - X_t$$

Here, $j = t$

Unrolling the other way

We have $L_t = L_{t-1}(1+i) - X_t$, and equally $L_{t-1} = L_{t-2}(1+i) - X_{t-1}$, and so on. Working back from time t to time 0, we get:

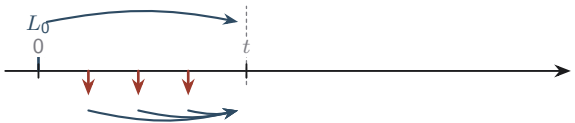
$$\boxed{L_t = L_0(1+i)^t - \left(X_1(1+i)^{t-1} + X_2(1+i)^{t-2} + \cdots + X_{t-1}(1+i) + X_t\right)}$$

Each instalment is accumulated to t by the number of periods in the borrower's hands. The last one, paid at t itself, thus carries no factor at all.

The retrospective method

RETROSPECTIVE METHOD

The loan outstanding at time t is the accumulated value at time t of the original loan, less the accumulated value at time t of the repayments made to date.



Accumulate the loan and the payments forward to t

CAUTION

This is the only option when the future instalments are not yet known. This can happen for loans whose rates are reset each year.

Level instalments again

If $X_t = X$ throughout, the accumulated repayments are themselves an ordinary annuity:

$$L_t \;=\; L_0(1+i)^t - X\left((1+i)^{t-1} + (1+i)^{t-2} + \cdots + (1+i) + 1\right) \;=\; L_0(1+i)^t - X\, s_{\overline{t}|}.$$

EXAMPLE 12.3

For the warm-up loan at $t = 2$,

$$L_2 \;=\; 1000\,(1.07)^2 - X\, s_{\overline{2}|} \;=\; 1144.90 - 381.05 \times 2.07 \;=\; 1144.90 - 788.78 \;=\; €\,356.12,$$

agreeing with the prospective answer $X\, a_{\overline{1}|} = €\,356.12$.

Why the two agree

Substitute the equation of value $L_0 = \sum_{k=1}^{n} X_k \mathrm{v}^k$ into the retrospective formula, to get

$$L_t \;=\; (1+i)^t \left[\sum_{k=1}^{n} X_k \mathrm{v}^k - \sum_{k=1}^{t} X_k \mathrm{v}^k \right] \;=\; (1+i)^t \sum_{k=t+1}^{n} X_k \mathrm{v}^k \;=\; \sum_{k=t+1}^{n} X_k \mathrm{v}^{k-t},$$

where the middle step uses $X_k(1+i)^{t-k} = (1+i)^t X_k \mathrm{v}^k$. The last sum is the prospective formula.

> KEY IDEA
>
> Both methods are the *same* sum, split at t. The retrospective one prices the terms before t, the prospective one the terms after it, and the equation of value says the two halves account for the whole loan.

Which method should you use?

As they both agree, you should use them depending on where you evaluate:

- **Prospective** is shorter near the *end*, when few payments remain, and is the natural choice for level instalments, since $L_t = Xa_{\overline{n-t}|}$ is a single annuity value.
- **Retrospective** is shorter near the *start*, and is the only option if the future instalments are not yet known.

> KEY IDEA
>
> Neither result depends on the rate being constant. If the rate from time $r-1$ to r is i_r, a single factor $(1+i)^k$ becomes the *product* of the factors for the years actually traversed. The principle of consistency from Lecture 1 is what governs this.

Lecture 12 on one slide

ONE INSTALMENT, AND ONE BALANCE

$$X_t \;=\; \underbrace{iL_{t-1}}_{b_t\ \text{(interest)}} \;+\; \underbrace{L_{t-1}-L_t}_{f_t\ \text{capital}}$$

$$L_t \;=\; \sum_{k=t+1}^{n} X_k v^{k-t} \;=\; L_0(1+i)^t - \sum_{k=1}^{t} X_k (1+i)^{t-k}$$

Interest first, capital second. The two formulas are one sum split at t, so they agree. With level instalments, we have $L_t = X a_{\overline{n-t}|} = L_0(1+i)^t - X s_{\overline{t}|}$.

Exercises

EXERCISE 12.4

A loan of €20,000.00 is repaid by five level annual instalments in arrears at $i = 6\%$. Find the instalment, and L_2 both prospectively and retrospectively.

EXERCISE 12.5

For the same loan, split the third instalment into interest and capital, and check the two sum to it.

EXERCISE 12.6

A loan of €50,000.00 is repaid by ten level annual instalments in arrears at $i = 5\%$. Find the capital repaid in the first and last years, and explain their ratio.

Solutions are in the backup slides.

END OF LECTURE 12

It is now clear how a single instalment is split. Next, a whole block of them at once, without touching a single balance in between.

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Solution 1

At 6%, $a_{\overline{5}|} = 4.212364$, so

$$X \;=\; \frac{20{,}000}{4.212364} \;=\; €4{,}747.93.$$

Prospectively, three payments remain after time 2:

$$L_2 \;=\; X\, a_{\overline{3}|} \;=\; 4747.93 \times 2.673012 \;=\; €12{,}691.27.$$

Retrospectively, the loan and the two payments made are accumulated to time 2:

$$L_2 \;=\; 20{,}000(1.06)^2 - X\, s_{\overline{2}|} \;=\; 22{,}472.00 - 4747.93 \times 2.06 \;=\; €12{,}691.27.$$

The same number, as the equation of value guarantees.

Solution 2

With $n = 5$ and $t = 3$, the closed forms give

$$b_3 \;=\; X\left(1 - \mathrm{v}^3\right) \;=\; 4747.93 \times 0.160381 \;=\; €\,761.48,$$

$$f_3 \;=\; X\mathrm{v}^3 \;=\; 4747.93 \times 0.839619 \;=\; €\,3{,}986.45.$$

Their sum is $761.48 + 3986.45 = €\,4{,}747.93$. This is the instalment since $X\left(1 - \mathrm{v}^3\right) + X\mathrm{v}^3 = X$ identically.

The same answer from the balance: $L_2 = €\,12{,}691.27$ from Solution 1, and $b_3 = 0.06 \times 12{,}691.27 = €\,761.48$.

Solution 3

At 5%, $a_{\overline{10}|} = 7.721735$, so $X = 50{,}000/7.721735 = €6{,}475.23$. Using $f_t = X\mathsf{v}^{n-t+1}$,

$$f_1 \;=\; X\mathsf{v}^{10} \;=\; €3{,}975.23, \qquad f_{10} \;=\; X\mathsf{v} \;=\; €6{,}166.88.$$

The ratio is

$$\frac{f_{10}}{f_1} \;=\; \frac{X\mathsf{v}}{X\mathsf{v}^{10}} \;=\; \mathsf{v}^{-9} \;=\; (1+i)^9 \;=\; 1.05^9 \;=\; 1.5513,$$

which needs no further calculation. The capital content grows by a factor $(1+i)$ each year, so over the nine years between the first instalment and the last it grows by $(1+i)^9$.