LECTURE 11

Cashflows that may not arrive

Factoring risk (probability) as a second discount

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Where we are, and where we are going

$$\mathrm{NPV}(i) \;=\; \sum_t \mathrm{v}^t c_t + \int_0^T \mathrm{v}^t \rho(t)\,\mathrm{d}t$$

$$\mathrm{NPV}(i_0) \;=\; 0$$

$$i_0 \;\approx\; i_1 + \frac{\mathrm{NPV}(i_1)}{\mathrm{NPV}(i_1) - \mathrm{NPV}(i_2)}\,(i_2 - i_1)$$

Every payment so far has been *certain* to be paied in full, and on the promised date. But irl, risk is one of the most important factors and today, we will introduce it in our cashflow models.

Today

1. Allowing for uncertainty
2. A project with uncertain receipts
3. Solving the yield equation

Warm-up

EXERCISE 11.1

An uncertain (probability 0.8*) payment of* €1,000.00 *is due at time* 3 *with* $i = 6\%$.

(a) *What is the expected present value of the payment?*

(b) *At what rate would a certain* €1,000.00 *at time* 3 *have that same present value?*

Warm-up

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(a) Discount, then weight by the chance of arriving:

$$1000 \times 0.8 \times 1.06^{-3} \;=\; 800 \times 0.839619 \;=\; €671.70.$$

(b) We need $1000\,(1+i^*)^{-3} = 671.70$, so

$$(1+i^*)^3 \;=\; \frac{1000}{671.70} \;=\; 1.48877, \qquad i^* \;=\; 1.48877^{1/3} - 1 \;=\; 14.185\%.$$

Thus "6% interest with probability 0.8 has the same expected value as "14.2%, guaranteed".

SECTION 1

Allowing for uncertainty

Two ways to compensate for risk

There are two obvious ways of factoring in risk in a cashflow:

1. Apply a probability of payment or receipt to the cashflow at *each* time step.
2. Use a higher rate of discount.

They sound different but as we will show, in many cases, they agree. We saw it in a simple case in the warm-up.

Attaching probabilities

To factor in the probability in our cashflow we modify the equation of value from

$$\sum_t (1+i)^{-t} c_t + \int_0^T (1+i)^{-t} \rho(t)\,\mathrm{d}t \;=\; 0$$

to

$$\sum_t \; p_t \; (1+i)^{-t} c_t + \int_0^T \; \mathrm{p(t)} \; (1+i)^{-t} \rho(t)\,\mathrm{d}t \;=\; 0,$$

where p_t and $p(t)$ are the probabilities of a cashflow at time t.

p_t and $p(t)$ compare to each other like v_t and $\rho(t)$. Integrating the latter gives something comparable to the former.

> "DISCOUNTING" THE DISCOUNT
>
> We are effectively putting a modifier on the discount. Each term is now probability of arriving $\times$ discount factor $\times$ amount. Part (a) of the warm-up was one such term.

Probability as a second discount

For a force of interest δ, we may take the probability itself to be a discounting function,

$$p(t) \;=\; \mathrm{e}^{-\mu t}, \qquad \mu > 0,$$

so that the equation of value becomes

$$\sum_{t} \mathrm{e}^{-\delta t}\mathrm{e}^{-\mu t} c_t + \int_0^T \mathrm{e}^{-\delta t}\mathrm{e}^{-\mu t}\rho(t)\,\mathrm{d}t \;=\; 0.$$

> **WHY THAT SHAPE IS NATURAL**
>
> $p(0) = 1$; p decreases with time; and decays over $s + t$ as $p(s)p(t)$. A cashflow further away is less likely to materialise, and the chance decays at a constant rate. This is exactly the assumptions we made to model *growth* in Lecture 2.

Where such probabilities come from

Probabilities may often be estimated by considering the experience of similar cashflows.

> KEY IDEA
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> Pensions are such example of a cashflow, where life expectancy is factored in with $e^{-\mu t}$ a crude but workable model of that. This approach is developed in MA3992.

The second approach: Discount harder

In the modified equation of value the discounting function and the probability both depend on time, so they can be combined into a single time-dependent function.

Where there is insufficient information to produce the probabilities objectively, that combined function may be viewed as an **adjusted discounting function**, making an implicit allowance for the probability of the cashflow.

> BEHIND THE JARGON
>
> In practice this is what a "risk premium", or a "risk-adjusted discount rate", means. Rather than model what might go wrong, the analyst discounts harder and leaves the reason implicit.

In general, determining the risk or right discount it not a mean feat, but a core difficutly.

The two approaches coincide

Since $e^{-\delta t}e^{-\mu t} = e^{-(\delta+\mu)t}$, a probability of the same form as the discounting function combines with it as a single, larger rate:

$$\sum_t e^{-\delta^* t} c_t + \int_0^T e^{-\delta^* t}\rho(t)\,dt \;=\; 0, \qquad \text{where} \qquad \boxed{\delta^* \;=\; \delta + \mu}$$

KEY IDEA

The revised force exceeds the actual one, since $\mu > 0$ is needed for probability (to be below 1). Risk just contributes one more term to the exponent.

Warm-up: $e^{-3\mu} = 0.8$ gives $\mu = 0.0744$, so $\delta^* = 0.0583 + 0.0744 = 0.1327$ and $i^* = e^{0.1327} - 1 = 14.185\%$.

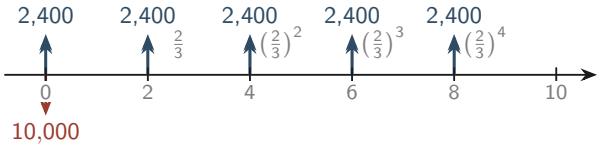
SECTION 2

A project with uncertain receipts

The question

EXAMPLE 11.2

For an initial investment of €10,000.00, an investor expects to receive €2,400.00 paid in advance every two years for the next ten years, with probability $\left(\frac{2}{3}\right)^{k-1}$ of receiving the kth payment. What is the expected yield per annum?



Five receipts, at times $0, 2, 4, 6, 8$. The first is certain. Each one after that is two-thirds as likely as the one before.

The equation of value

Working in €1,000.00s and attaching the probability to each receipt,

$$0 = -10 + 2.4\left(1 + \frac{2}{3}v^2 + \left(\frac{2}{3}\right)^2 v^4 + \left(\frac{2}{3}\right)^3 v^6 + \left(\frac{2}{3}\right)^4 v^8\right).$$

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The bracket is a geometric progression with first term 1 and common ratio $\frac{2}{3}v^2$, so by Lecture 5,

$$0 \;=\; -10 + 2.4\left(\frac{1 - \left(\frac{2}{3}v^2\right)^5}{1 - \frac{2}{3}v^2}\right), \qquad \text{provided } \tfrac{2}{3}v^2 \neq 1.$$

For $\frac{2}{3}v^2 \neq 1$ you would need to check what happens when you go to the limit.

A look before calculating

It pays to sanity-check before hunting for a root. Ignore discounting altogether and add up what the investor *expects* to receive:

$$2400\left(1+\tfrac{2}{3}+\tfrac{4}{9}+\tfrac{8}{27}+\tfrac{16}{81}\right) \;=\; 2400\times 2.604938 \;=\; €6{,}251.85.$$

CAUTION

That is less than the €10,000.00 paid out, and discounting can only reduce it further. Thus, yield cannot ever be positive: We are looking for a root with $i_0 < 0$.

Five payments of €2,400.00 would have totalled €12,000.00, but the probabilities remove nearly half of it.

SECTION 3

Solving the yield equation

A substitution for the interpolation

When looking for the yield, we have to solve NPV for i. Of course, in this example, we have v, which is just another form of (i). It only appears as a single expression in our case, namely v^2. Simplify by replacing the whole of it by a single letter: Let

$$x \;=\; \frac{2}{3}\mathsf{v}^2 \neq 1, \qquad \text{noting } x > 0,$$

and interpolate instead on

$$f(x) \;=\; -10 + 2.4\left(\frac{1-x^5}{1-x}\right).$$

KEY IDEA

This look a lot tidier and friendlir and solve this instead, and then use $\mathsf{v} = \sqrt{3/2x}$.

What f really is

The quotient is the geometric sum, which is not surprising. For $x \neq 1$, we have

$$f(x) \;=\; -10 + 2.4\left(1 + x + x^2 + x^3 + x^4\right).$$

KEY IDEA

Written this way, three things are immediate:

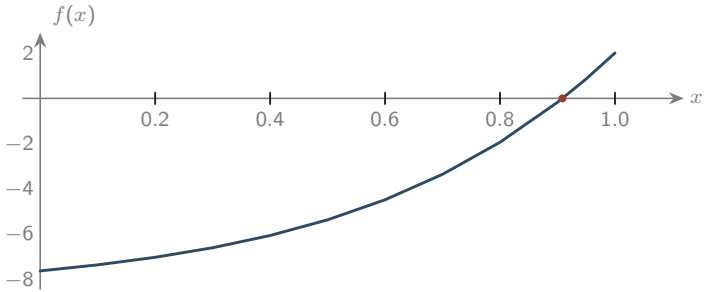
- f is a polynomial with positive coefficients, so it is *strictly increasing* for $x > 0$;
- $f(0) = -10 + 2.4 = -7.6 < 0$;
- $f(1) = -10 + 2.4 \times 5 = +2 > 0$.

So there is exactly one positive root, and it lies in $(0, 1)$.

Uniqueness in x gives uniqueness in i, since we do not allow for negative v.

Choosing the trial values

How should we guess x_1 and x_2? One way is to do a rough plot f.



The curve crosses between 0.9 and 0.95, so take $x_1 = 0.9$, $x_2 = 0.95$.

Without plotting

We can usually get a feel for how different functions behave, especially polynomials. Here, f increases, with $f(0) = -7.6$ and $f(1) = 2$, so the root is in $(0, 1)$; and because $f(1)$ is much nearer to zero than $f(0)$, the root must lie well over towards 1. One evaluation confirms it:

$$f(0.9) \;=\; -10 + 2.4\,(1 + 0.9 + 0.81 + 0.729 + 0.6561) \;=\; -10 + 2.4 \times 4.0951.$$

The two trial values

For $x_1 = 0.9$,

$$f(0.9) \;=\; -10 + 2.4\left(\frac{1-(0.9)^5}{1-0.9}\right) \;=\; -10 + 2.4 \times 4.0951 \;=\; -0.17176 \;<\; 0.$$

For $x_2 = 0.95$,

$$f(0.95) \;=\; -10 + 2.4\left(\frac{1-(0.95)^5}{1-0.95}\right) \;=\; -10 + 2.4 \times 4.52438 \;=\; +0.858515 \;>\; 0.$$

CAUTION

Here $f(x_1) < 0$ and $f(x_2) > 0$, the opposite way round from Lecture 10's NPV, which fell. Of course, the algorithm does not care.

Interpolating

$$x_0 \approx x_1 + \frac{f(x_1)}{f(x_1) - f(x_2)}(x_2 - x_1) = 0.9 + \frac{-0.17176}{-0.17176 - 0.858515} \times (0.05)$$

$$= 0.9 + \frac{-0.17176}{-1.030275} \times 0.05 = 0.9 + 0.166713 \times 0.05 = 0.9083356.$$

Both signs cancel in the fraction, leaving a number in $(0, 1)$, as it must, since the root lies between x_1 and x_2. The estimate sits about a sixth of the way across, $|f(x_1)|$ being the smaller of the two.

The exact root is $x_0 = 0.9087055\ldots$.

Translating back to a yield

We have $x_0 \approx 0.9083356$, and $x = \frac{2}{3}\mathsf{v}^2$, so

$$\mathsf{v}^2 \;=\; \tfrac{3}{2}x \quad\Longrightarrow\quad \mathsf{v} \;=\; \left(\tfrac{3}{2}x\right)^{1/2} \quad\Longrightarrow\quad 1+i \;=\; \frac{1}{\mathsf{v}} \;=\; \left(\tfrac{3}{2}x\right)^{-1/2},$$

and therefore the expected yield per annum is

$$i_0 \;\approx\; \left(\tfrac{3}{2}x_0\right)^{-1/2} - 1 \;=\; -14.33\%.$$

KEY IDEA

A negative yield is a legitimate root. It satisfies $i_0 > -1$, so $\delta_0 = \ln(1+i_0)$ exists. It says the investor expects to *lose* about 14% a year. This is unsurprising, since the probabilities cut the expected receipts to €6,251.85 against €10,000.00 paid out.

This is not a malformed example. IRL, knowing what the exact projected loss is can still be very helpful.

Lecture 11 on one slide

UNCERTAINTY IS ONE MORE TERM IN THE EXPONENT

$$\sum_t p_t \, \mathsf{v}^t c_t + \int_0^T p(t) \, \mathsf{v}^t \rho(t) \, \mathrm{d}t \;=\; 0, \qquad p(t) \;=\; \mathsf{e}^{-\mu t} \;\Longrightarrow\; \delta^* \;=\; \delta + \mu.$$

Attach probabilities can often be reduced to discounting harder, especially if p is of the same form as v. When solving equations (e.g., yield), see if a common harder term can not be substituted out for a simpler.

Exercises

EXERCISE 11.3

A cashflow is discounted at a force $\delta = 0.06$*, each payment arriving with probability* $\mathrm{e}^{-\mu t}$*,* $\mu = 0.02$*. Find* δ^* *and* i^**, and compare with* δ *alone.*

EXERCISE 11.4

Two analysts value one risky cashflow. One discounts at $\delta = 0.05$ *with survival probabilities* $\mathrm{e}^{-0.03t}$*; the other at* $\delta^* = 0.08$ *with none. Show that they must agree.*

EXERCISE 11.5

An investor pays €5,000.00 *now for* €1,500.00 *at the end of years* $2, 4, 6, 8$*. Find the yield, using a substitution.*

Solutions are in the backup slides.

END OF LECTURE 11

Next: Loan calculations and replayments.

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Solutions 1–2

1. Forces add, so $\delta^* = 0.06 + 0.02 = 0.08$ and

$$i^* \;=\; \mathrm{e}^{0.08} - 1 \;=\; 8.3287\%, \qquad \text{against} \qquad \mathrm{e}^{0.06} - 1 \;=\; 6.1837\%$$

from the force alone. The uncertainty costs a little over two percentage points of effective rate. Note that it is the *forces* that differ by exactly μ, not the effective rates.

2. In the first analyst's equation every term carries $\mathrm{e}^{-0.05t}\mathrm{e}^{-0.03t} = \mathrm{e}^{-0.08t}$, which is exactly the second analyst's discount factor, so the two equations agree term by term. For the first to be right, 5% must be the correct underlying force *and* the probabilities must really decay at $\mu = 0.03$; for the second, 8% must be the right all-in rate. The same number, but only the first is explicit about why.

Solution 3

The payments fall at even times only, so i enters through v^2. Put $x = v^2$; the equation of value becomes

$$g(x) \;=\; 1500\left(x + x^2 + x^3 + x^4\right) - 5000 \;=\; 0,$$

a polynomial with positive coefficients, so strictly increasing for $x > 0$: one root, and the yield is well defined. Since $g(0) = -5000$ and $g(1) = 1000$, it lies in $(0, 1)$, and near the top of it.

Now $g(0.9) = 1500(3.0951) - 5000 = -357.35$ and $g(0.95) = 1500(3.52438) - 5000 = +286.57$, so

$$x_0 \;\approx\; 0.9 + \frac{357.35}{357.35 + 286.57} \times 0.05 \;=\; 0.92775,$$

and $1 + i = x_0^{-1/2}$ gives $i_0 \approx 3.82\%$. (The exact root is 0.928386, giving 3.79%.)