LECTURE 10

The yield of a project

Reading the equations the other way round

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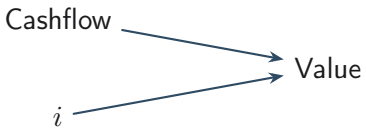
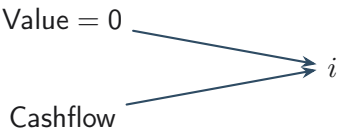
Thus far, we were been given the rate. Now, we solve for it — to evluate our project.



Lectures 3–9

SOLVE FOR THE RATE

In the preivous lectures, we determined the value of a cashflow based on the rate. But as is so often the case, we can use the *the same* formulas to *read backwards* and determine the rate from knowing the value of a cashflow.



Today

GOAL

The idea behind this is rather elegant: We ask *what rate evaluates our proect to zero value*. Whatever the answer, that is the **yield** of our project — how much it's actual value is.

Today

1. The value as a function of the rate
2. The equation of value and the yield
3. Finding the yield
4. The software project, finished

Warm-up

EXERCISE 10.1

An investor pays €10,000.00 *now and receives* €3,000.00 *at the end of each of the next five years. Is the investment worth it at* $i = 8\%$*?*

Warm-up

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Five payments in arrears makes the income worth $3000\, a_{\overline{5}|}$ at $t = 0$. As $a_{\overline{5}|} = 3.993$ at 8%,

$$-10{,}000 + 3000 \times 3.992710 \;=\; -10{,}000 + 11{,}978.13 \;=\; €1{,}978.13 \;>\; 0,$$

the project is worth doing at this rate.

> AS WE SAW BEFORE, THIS IS ALL NPV (NET PRESENT VALUE IS)...
> Money out is negative, money in positive, and we add them at a common date.

But 8% was handed to us. What if the investor's borrowing rate were 12%? Or 16%?

SECTION 1

The value as a function of the rate

Net present value

As in Lecture 3, a project has income and outgoings, and we take the *net* amount $c_t = b_t - a_t$ and $\rho(t) = \rho_2(t) - \rho_1(t)$ at each time t.

DEFINITION 10.2 · Net present value (specialisation from Lecture 3) for constant discount rate

The **net present value** $\mathsf{NPV}(i)$ of a project ending at time T, at an effective rate of interest i per unit time, is

$$\mathsf{NPV}(i) \;=\; \sum_t (1+i)^{-t} c_t + \int_0^T (1+i)^{-t} \rho(t)\,\mathrm{d}t.$$

The same thing can, of course, also be rewritten in the following ways:

$$\mathsf{NPV}(i) \;=\; \sum_t \mathsf{v}^t c_t + \int_0^T \mathsf{v}^t \rho(t)\,\mathrm{d}t \;=\; \sum_t \mathrm{e}^{-\delta t} c_t + \int_0^T \mathrm{e}^{-\delta t} \rho(t)\,\mathrm{d}t.$$

We allow $T = \infty$.

What changed

CHANGE OF PERSPECTIVE: A FUNCTION OF i

Up to now, the interest/discout was fixed, NPV a number, and the cashflow varried.

Now the cashflow is considered fixed, and NPV a function of i.

Two facts, used without proof: NPV(i) is defined for $i > -1$, and it is *continuous* in i. Note, for $i \leq -1$, ln$(1+i)$ is not defined.

THE FORCE IS CONSTANT

An easy-to-miss change is that, unlike in Lecture 3, NPV now has constant interest. That is to say, v, i or δ, however given, is a number, no longer a function of time.

Reason: Want to solve for i, and otherwise would not have a unique solution.

A project to follow through the lecture

EXAMPLE 10.3

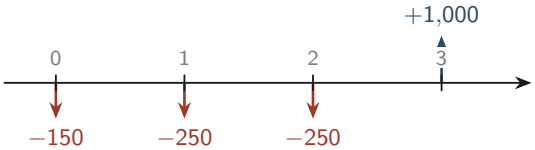
A software company will set up a computer system for a client, contracting out the work. The estimated cashflows are, in €1,000.00 s,

Description	Timing	Amount
Contractor fees	Start Y1	⟨150⟩
Contractor fees	Start Y2	⟨250⟩
Contractor fees	Start Y3	⟨250⟩
Sales	End Y3	1,000

Brackets indicate expenditure. Find the net present value at 20% per annum, the rate at which the company can lend or borrow.

"Start of year 1" is time 0, "start of year 2" is time 1, and "end of year 3" is time 3.

The project, on a line



With $\mathsf{v} = \dfrac{1}{1+i}$, the net present value is

$$\mathrm{NPV}(i) \;=\; -150 - 250\,\mathsf{v} - 250\,\mathsf{v}^2 + 1000\,\mathsf{v}^3.$$

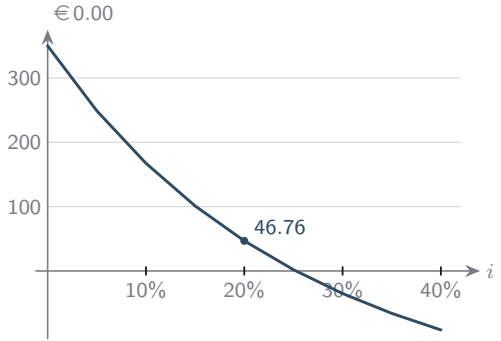
The arithmetic at 20%

At $i = 0.2$ we have $\mathsf{v} = \dfrac{1}{1.2} = 0.8\dot{3}$:

t	c_t	v^t	$\mathsf{v}^t c_t$
0	−150	1.000000	−150.00
1	−250	0.833333	−208.33
2	−250	0.694444	−173.61
3	1,000	0.578704	578.70
		NPV(0.2)	46.76

So NPV(0.2) = €46,759.00, very much positive. Meaning, at a borrowing rate of 20% the project is worth doing.

The same project at other rates



- At $i = 0$ nothing is discounted, so NPV(0) is simply the face values: $1000 - (150 + 250 + 250) = 350$.
- As i rises, the late million is discounted harder than the early outgo, so the curve falls.
- It changes sign near 25%. Above that, funding the project costs more than it earns.

SECTION 2

The equation of value and the yield

That crossing point is what we are after: Exactly where the present values of the negative and the positive cashflows are in equilibrium. Meaning, the rate i, for which $\mathrm{NPV}(i) = 0$.

DEFINITION 10.4 · Equation of value

The **equation of value** is the algebraic expression $\mathrm{NPV}(\delta) = 0$. Its roots are the force(s) of interest δ for which the total value of the net cashflows is 0:

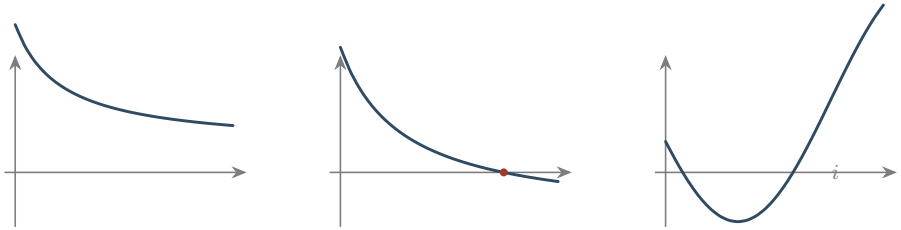
$$\sum_t \mathrm{e}^{-\delta t} c_t + \int_0^T \mathrm{e}^{-\delta t} \rho(t)\,\mathrm{d}t \;=\; 0.$$

Since $\mathrm{e}^{\delta} = 1 + i$ we may equally write it with $(1+i)^{-t}$. In that form it is also called the **yield equation**.

We can use i, δ, v interchangibly, so long as we pick the correct formula from page 4.

How many roots can it have?

For a given transaction the equation of value may have *no* roots, a *unique* root, or *several*.



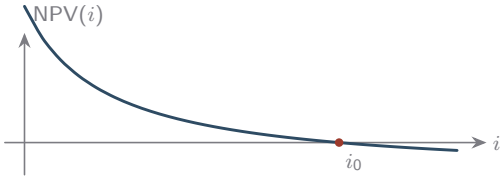
The left-hand case is easy to recognise: *If all the cashflows have the same sign, no solution exists* since terms can not cancel each other.

The yield

DEFINITION 10.5 · Yield

If the equation of value has a **unique root** δ_0, it is the **force of interest implied by the transaction**, and $i_0 = \mathrm{e}^{\delta_0} - 1$ is the **yield per unit time**.

Recall $1 + i = e^{\delta}$.



In finance the yield is also called the **internal rate of return**, or IRR. It is the same number, the root of $\mathrm{NPV}(i) = 0$.

The yield, intuitively

INSTEAD OF VALUING X, VALUE SOMETHING EQUIVALENT TO X

Yield is the interest rate that summarizes the *overall* return of a transaction into a single number, making comparisions easy. It is the answer to the following question:

If all the transaction's cash in- and outflows were paid in- and out of a hypothetical account, at what interest rate would this account have zero balance at the end?

Thus, yield represents the effective rate of return generated by the transaction's cashflows. In particular, it answers the following important question:

Is this project worth doing, or would I be better off investing in a risk-free asset (e.g., bonds)? If yield $>$ risk-free rate, the project is worth doing. Otherwise, it is not.

Of course, for real world projects, risk needs to be considered as well, but that comes in the next lecture.

When is the yield defined?

The yield is defined *if and only if* the yield equation has precisely one root greater than -1. Roots at or below -1 are inadmissible, since δ does not exist there. With two of them "the yield" is no longer well defined as its not a single number anymore.

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TWO WAYS TO KNOW THERE IS EXACTLY ONE

- **Sign change plus monotonicity.** If NPV is monotonic and $\exists, i_1, i_2$ with $\text{NPV}(i_1) \cdot \text{NPV}(i_2) < 0$, continuity gives a root between them and monotonicity forbids a second.
- **Spend first, earn later.** A unique solution exists if all negative net cashflows precede all positive ones, or vice versa.

The second covers many projects met in practice, at least approximately, where you make (several) larege investments at the start and then earn after.

SECTION 3

Finding the yield

There is no formula

The yield equation is a polynomial in v whose degree is the term of the project. Beyond a couple of payments there is no algebraic solution, so we estimate.

LINEAR INTERPOLATION

By trial and error, find $i_1 < i_2$, with $i_2 - i_1$ small, such that

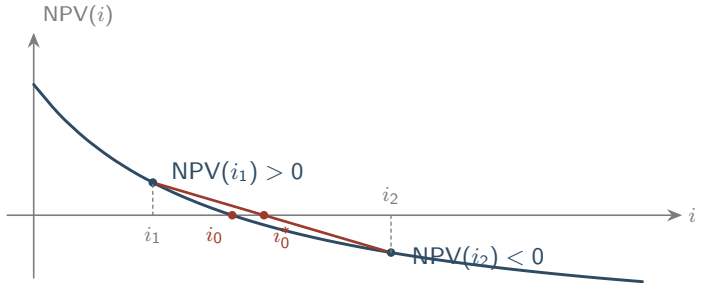
$$\mathrm{NPV}(i_1) \cdot \mathrm{NPV}(i_2) \; < \; 0.$$

Estimate a solution i_0 by the i-intercept of the straight line passing through $\big(i_1, \mathrm{NPV}(i_1)\big)$ and $\big(i_2, \mathrm{NPV}(i_2)\big)$. If $\mathrm{NPV}(i_0) \neq 0$, replace whichever i_1, i_2 satisfies $\mathrm{NPV}(i_i) \cdot \mathrm{NPV}(i_0) < 0$ and repeat. This converges to the real solution, and fast.

In other words: replace the curve by the chord, which we *can* solve, and hope that the two cross the axis at nearly the same place.

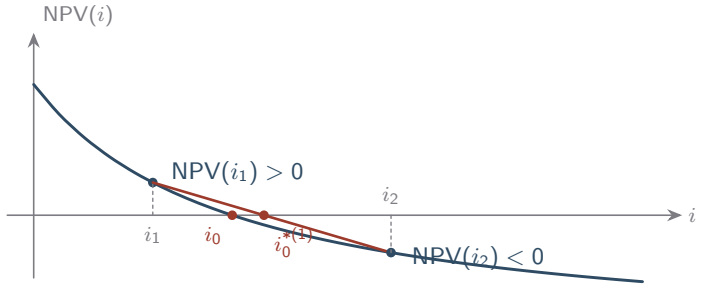
It is theoretically impossible to guarntee that we can find such i_1 and i_2, but in paractice, that is never a problem.

The picture



The chord meets the axis at i_0^*, a little to the right of the true root i_0.

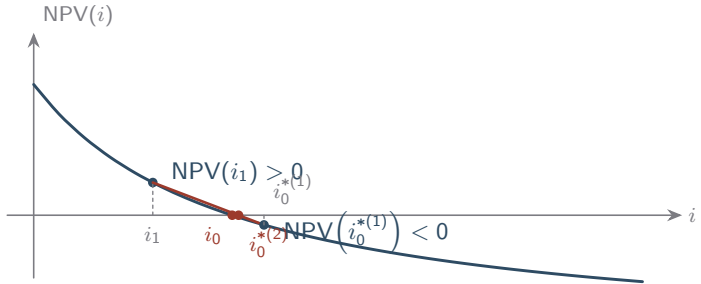
The formula in pictures



The first chord gives the intercept

$$i_0^{*(1)} = 1.740.$$

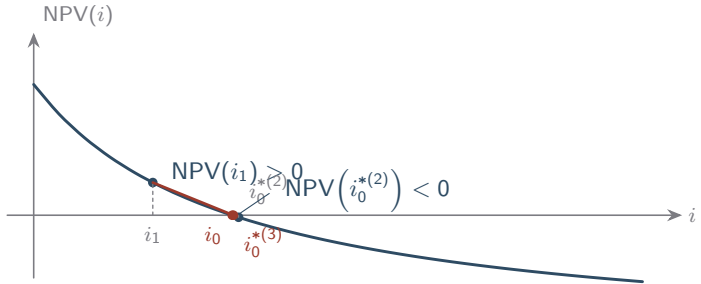
The formula in pictures



Since NPV$\left(i_0^{*(1)}\right) < 0$, replace the negative endpoint i_2 by $i_0^{*(1)}$. The new chord gives

$$i_0^{*(2)} = 1.548.$$

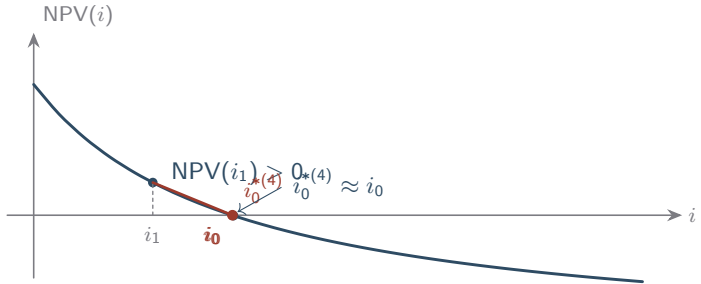
The formula in pictures



Again, $\mathrm{NPV}\left(i_0^{*(2)}\right) < 0$, so it becomes the new negative endpoint. The next chord gives

$$i_0^{*(3)} = 1.5096.$$

The formula in pictures



Repeating the same step gives

$$i_0^{*(4)} = 1.50192,$$

which is already very close to the true root $i_0 = 1.5$.

The formula in symbols and words

The slope of any line may be computed from any two distinct points. That is, for any line L and any $a_1 \neq a_2$, $b_1 \neq b_2$ in its domain,

$$\frac{L(a_1) - L(a_2)}{a_1 - a_2} = \frac{L(b_1) - L(b_2)}{b_1 - b_2}$$

The formula in symbols and words

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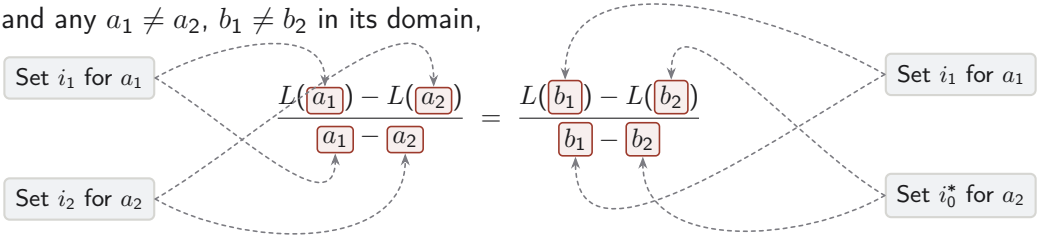

Let L be the line through $\big(i_1, \mathrm{NPV}(i_1)\big)$ and $\big(i_2, \mathrm{NPV}(i_2)\big)$, and i_0^* its i-intercept, i.e. $L(i_0^*) = 0$.

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$$\frac{L(i_1) - L(i_2)}{i_1 - i_2} = \frac{L(i_1) - L(i_0^*)}{i_1 - i_0^*}$$

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The slope of any line may be computed from any two distinct points. That is, for any line L and any $a_1 \neq a_2$, $b_1 \neq b_2$ in its domain,

$$\frac{\boxed{\text{NPV}(i_1)} - \boxed{\text{NPV}(i_2)}}{i_1 - i_2} = \frac{\boxed{\text{NPV}(i_1)} - L(i_0^*)}{i_1 - i_0^*}$$

$L(i_1) = \text{NPV}(i_1)$ and $L(i_2) = \text{NPV}(i_2)$ by definition of L_1, L_2

The formula in symbols and words

The slope of any line may be computed from any two distinct points. That is, for any line L and any $a_1 \neq a_2$, $b_1 \neq b_2$ in its domain,

$$\frac{\mathrm{NPV}(i_1) - \mathrm{NPV}(i_2)}{i_1 - i_2} = \frac{\mathrm{NPV}(i_1) - \boxed{0}}{i_1 - i_0^*}$$

Since i_0^* is L's intercept

The formula in symbols and words

The slope of any line may be computed from any two distinct points. That is, for any line L and any $a_1 \neq a_2$, $b_1 \neq b_2$ in its domain,

$$\frac{\mathrm{NPV}(i_1) - \mathrm{NPV}(i_2)}{i_1 - i_2} = \frac{\mathrm{NPV}(i_1)}{i_1 - i_0^*}$$

The formula in symbols and words

The slope of any line may be computed from any two distinct points. That is, for any line L and any $a_1 \neq a_2$, $b_1 \neq b_2$ in its domain,

$$i_1 - i_0^* = \frac{\mathsf{NPV}(i_1)}{\mathsf{NPV}(i_1) - \mathsf{NPV}(i_2)} (i_1 - i_2)$$

The formula in symbols and words

The slope of any line may be computed from any two distinct points. That is, for any line L and any $a_1 \neq a_2$, $b_1 \neq b_2$ in its domain,

$$i_0^* \;=\; i_1 - \frac{\mathsf{NPV}(i_1)}{\mathsf{NPV}(i_1) - \mathsf{NPV}(i_2)}\,(i_1 - i_2)$$

The formula in symbols and words

The slope of any line may be computed from any two distinct points. That is, for any line L and any $a_1 \neq a_2$, $b_1 \neq b_2$ in its domain,

$$i_0^* = i_1 + \frac{\mathrm{NPV}(i_1)}{\mathrm{NPV}(i_1) - \mathrm{NPV}(i_2)} (i_2 - i_1)$$

Since $(i_2 - i_1) = -(i_1 - i_2)$ the sign changes

Since L only approximates NPV near i_1, i_2, this makes i_0^* an estimate of the true root i_0, not an exact value.

The formula in symbols and words

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$$\boxed{i_0 \approx i_0^* = i_1 + \frac{\mathrm{NPV}(i_1)}{\mathrm{NPV}(i_1) - \mathrm{NPV}(i_2)} (i_2 - i_1)}$$

Since L only approximates NPV near i_1, i_2, this makes i_0^* an estimate of the true root i_0, not an exact value.

The interpolation formula – explained

$$i_0 \approx i_0^* = i_1 + \frac{\mathrm{NPV}(i_1)}{\mathrm{NPV}(i_1) - \mathrm{NPV}(i_2)}\,(i_2 - i_1)$$

A READING THAT MAKES IT MEMORABLE

With $\mathrm{NPV}(i_1) > 0$ and $\mathrm{NPV}(i_2) < 0$, we have

$$\frac{\mathrm{NPV}(i_1)}{\mathrm{NPV}(i_1) + |\mathrm{NPV}(i_2)|} \in (0, 1),$$

so i_0^* divides $[i_1, i_2]$ in the ratio of the two magnitudes, nearer whichever end has the smaller $|\,\mathrm{NPV}\,|$, exactly where you would put it by eye.

The warm-up project, revisited

EXERCISE 10.6

Find the yield on the warm-up project: €10,000.00 *paid now against* €3,000.00 *at the end of each of five years. Interpolate on* [0.15, 0.16].

The warm-up project, revisited

EXERCISE 10.6

Find the yield on the warm-up project: €10,000.00 *paid now against* €3,000.00 *at the end of each of five years. Interpolate on* [0.15, 0.16].

All the outgo precedes all the income, so the yield exists. Here $\mathrm{NPV}(i) = -10000 + 3000\, a_{\overline{5}|}$, and

$$\mathrm{NPV}(0.15) \;=\; -10000 + 3000(3.352155) \;=\; +56.47, \qquad \mathrm{NPV}(0.16) \;=\; -177.12,$$

$$i_0 \;\approx\; 0.15 + \frac{56.47}{56.47 + 177.12}\,(0.01) \;=\; 15.24\%.$$

So the 8% of the warm-up left a wide margin: the project pays for borrowing at anything under about 15.2%.

How wrong is it?

Look again at the picture. Between i_1 and i_2 the chord lies *above* the curve, so it reaches zero slightly later. For the usual convex NPV, linear interpolation therefore *overstates* the yield.

> KEY IDEA
>
> The error depends on how much the curve bends across $[i_1, i_2]$, so the remedy is to keep the bracket narrow. Halving the width roughly quarters the error.

We check this on the software project in a moment: brackets of width 0.10, 0.04 and 0.01 give errors of about 4.7×10^{-3}, 6.5×10^{-4} and 3.5×10^{-5}.

SECTION 4

The software project, finished

Two trial rates

All the negative cashflows (times $0, 1, 2$) precede the positive one (time 3), so a unique root exists. From the graph it is near 25%. For $i_1 = 0.25$, $\mathrm{v} = 0.8$:

$$\mathrm{NPV}(0.25) \;=\; -150 - 200 - 160 + 512 \;=\; +2 \;>\; 0.$$

Two trial rates

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$$\mathrm{NPV}(0.25) \;=\; -150 - 200 - 160 + 512 \;=\; +2 \;>\; 0.$$

Only just positive, so try one percentage point higher. For $i_2 = 0.26$, $\mathrm{v} = 0.793651$:

$$\mathrm{NPV}(0.26) \;=\; -150 - 198.413 - 157.470 + 499.906 \;=\; -5.977 \;<\; 0.$$

The sign has changed between 25% and 26%, and the bracket is narrow.

Interpolating, and how good it was

$$i_0 \;\approx\; 0.25 + \frac{2}{2-(-5.977)}\,(0.01) \;=\; 0.25 + \frac{2}{7.977} \times 0.01 \;=\; 0.2525072,$$

about 25.25% per annum. The estimate sits only a quarter of the way from 25% to 26%, because NPV was much nearer zero at i_1.

bracket	NPV(i_1)	NPV(i_2)	estimate
[0.20, 0.30]	46.759	−35.071	0.2571421
[0.24, 0.28]	10.283	−21.063	0.2531221
[0.25, 0.26]	2.000	−5.977	0.2525072

Against the true root 0.2524721, every estimate is too high, as predicted, and narrowing the bracket cures it.

Lecture 10 on one slide

HOLD THE CASHFLOW, VARY THE RATE

$$\mathrm{NPV}(i) \;=\; \sum_{t} \mathrm{v}^{t} c_t + \int_0^T \mathrm{v}^{t}\rho(t)\,\mathrm{d}t \qquad (i > -1,\ \text{continuous})$$

$$\mathrm{NPV}(i_0) \;=\; 0 \text{ with } i_0 \text{ unique} \;\Longrightarrow\; i_0 \text{ is the } \textit{yield}$$

$$i_0 \;\approx\; i_1 + \frac{\mathrm{NPV}(i_1)}{\mathrm{NPV}(i_1) - \mathrm{NPV}(i_2)}\,(i_2 - i_1).$$

A unique root is guaranteed when all the outgo precedes all the income. Bracket it with a sign change, then take the chord, and repeat.

Exercises

EXERCISE 10.7

A project costs €5.00 *now and returns* €2,000.00 *at the end of each of three years. Find* NPV(0.09), NPV(0.10), *and the yield.*

EXERCISE 10.8

An investor pays €1,000.00 *now for* €600.00 *at the end of each of two years. Show that the yield satisfies* $3v^2 + 3v - 5 = 0$*; solve it exactly, then by interpolation on* [0.13, 0.14].

EXERCISE 10.9

Income is paid annually in arrears for 22 *years, the first payment being* 1.05 *and each later one* 5% *larger. Value it at* $i = 5\%$.

Solutions are in the backup slides.

END OF LECTURE 10

A yield equation need not be a polynomial in v that anyone wants to meet. Next: a substitution that tames one, and what to do when the cashflows might not arrive at all.

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Solution 1

Here $\mathrm{NPV}(i) = -5000 + 2000\, a_{\overline{3}|}$, and all the outgo precedes all the income, so the yield exists. At $i = 0.09$, $a_{\overline{3}|} = 2.531295$, so $\mathrm{NPV} = -5000 + 5062.59 = +62.59$; at $i = 0.10$, $a_{\overline{3}|} = 2.486852$, so $\mathrm{NPV} = -26.30$.

The sign changes, so

$$i_0 \;\approx\; 0.09 + \frac{62.59}{62.59 + 26.30} \times 0.01 \;=\; 0.09704.$$

The true root is 0.09701; the estimate is high by 3×10^{-5}, the overshoot predicted by the convexity of NPV.

Solution 2

The equation of value is $-1000 + 600\mathrm{v} + 600\mathrm{v}^2 = 0$. Dividing by 200 gives $3\mathrm{v}^2 + 3\mathrm{v} - 5 = 0$, a quadratic, so

$$\mathrm{v} \;=\; \frac{-3+\sqrt{9+60}}{6} \;=\; \frac{-3+\sqrt{69}}{6} \;=\; 0.8844373, \qquad i_0 \;=\; \frac{1}{\mathrm{v}} - 1 \;=\; 0.1306624,$$

taking the positive root, since $\mathrm{v} > 0$.

Interpolating instead, NPV(0.13) $= 0.8615$ and NPV(0.14) $= -12.0037$, giving 0.130670. It is overshooting by 8×10^{-6}. Two payments is the largest case in which the exact route is available; three would need a cubic.

Solution 3

The payment at time j is $(1.05)^j$ for $j = 1, \ldots, 22$, and the discount factor is $\mathrm{v} = 1/1.05$. So the j-th term of the present value is

$$(1.05)^j \, \mathrm{v}^j \;=\; \left(1.05 \times \tfrac{1}{1.05}\right)^j \;=\; 1,$$

the same for every j. The present value is therefore 22.

> KEY IDEA
>
> Income growing at exactly the rate at which it is discounted contributes the *same* present value every year as growth and discounting cancel term by term. This can simplify a long escalating cashflow.

If the growth rate g and the interest rate differ, the terms are $\left(\frac{1+g}{1+i}\right)^j$ and the sum is geometric with that ratio, which is the general case.