LECTURE 9

Deferred and varying annuities

Payments that wait, grow and shrink

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Where the symbols got to

$$\text{PV} = \frac{1 - \text{v}^n}{\text{rate}}, \qquad \text{AV} = \frac{(1+i)^n - 1}{\text{rate}},$$



Every payment the same size

$$\text{rate} \in \left\{ d,\ d^{(m)},\ \delta,\ i^{(m)},\ i \right\}$$

Two things have been true of every annuity so far. The payments all have the same size, and the first of them arrives at once. Today we drop both asumptions.

Today

1. Waiting before the payments begin
2. Payments that grow by one each period
3. Decreasing payments, and everything between
4. Growth in continuous time

Warm-up

EXERCISE 9.1

Payments of $1, 2, 3$ *and* 4 *are made at times* $1, 2, 3$ *and* 4*, the effective rate of interest being* 6% *per annum. Find the value of this cashflow at time* 0*.*

Warm-up

EXERCISE 9.1

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Nothing new is needed. Drag each payment back by its own discount factor and add.

Time k	1	2	3	4	total
v^k	0.943396	0.889996	0.839619	0.792094	
$k\,\mathrm{v}^k$	0.943396	1.779993	2.518858	3.168375	8.410622

The four payments total 10, and that total is worth 8.410622 today. By the end of Section 2 it will have a name.

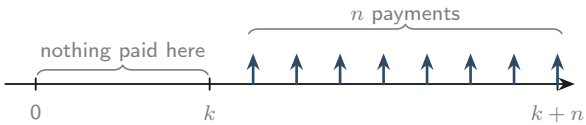
SECTION 1

Waiting before the payments begin

Deferred annuities

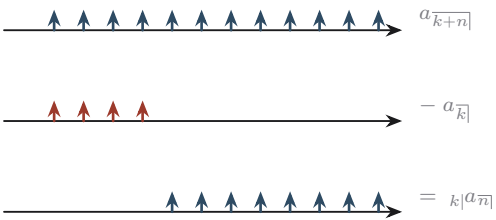
DEFINITION 9.2 · Deferred annuity

Payment of an annuity is **deferred** by k units of time if nothing is paid until time k, after which the annuity runs as usual. Its present value is written ${}_{k|}a_{\overline{n}|}$.



We note that the payments fall at times $k+1, \ldots, k+n$. The deferment is k, and the first payment comes one *further* period after that, because the annuity itself is in arrears.

Two routes to the same number



Subtract the years you did not get.
A deferred annuity is a long annuity with its early payments removed:

$$ {}_{k|}a_{\overline{n}|} \;=\; a_{\overline{k+n}|} - a_{\overline{k}|}. $$

Or **value it late, then discount.**
From time k the payments form an ordinary n-period annuity in arrears, so bring that one number back,

$$ {}_{k|}a_{\overline{n}|} \;=\; \mathsf{v}^k \, a_{\overline{n}|}. $$

The two must agree, and they do, $a_{\overline{k+n}|} - a_{\overline{k}|} = \frac{\mathsf{v}^k - \mathsf{v}^{k+n}}{i} = \mathsf{v}^k \, \frac{1-\mathsf{v}^n}{i} = \mathsf{v}^k \, a_{\overline{n}|}$.

Every level annuity defers the same way

Neither route noted anythig specific about annuities, only that the payments sit k units later than usual. So the rule applies across the whole family.

Payable	deferred	difference	discounted
in arrears	${}_{k\|}a_{\overline{n}\|}$	$a_{\overline{k+n}\|} - a_{\overline{k}\|}$	$\mathsf{v}^k a_{\overline{n}\|}$
in advance	${}_{k\|}\ddot{a}_{\overline{n}\|}$	$\ddot{a}_{\overline{k+n}\|} - \ddot{a}_{\overline{k}\|}$	$\mathsf{v}^k \ddot{a}_{\overline{n}\|}$
m-thly, in arrears	${}_{k\|}a^{(m)}_{\overline{n}\|}$	$a^{(m)}_{\overline{k+n}\|} - a^{(m)}_{\overline{k}\|}$	$\mathsf{v}^k a^{(m)}_{\overline{n}\|}$
m-thly, in advance	${}_{k\|}\ddot{a}^{(m)}_{\overline{n}\|}$	$\ddot{a}^{(m)}_{\overline{k+n}\|} - \ddot{a}^{(m)}_{\overline{k}\|}$	$\mathsf{v}^k \ddot{a}^{(m)}_{\overline{n}\|}$
continuously	${}_{k\|}\bar{a}_{\overline{n}\|}$	$\bar{a}_{\overline{k+n}\|} - \bar{a}_{\overline{k}\|}$	$\mathsf{v}^k \bar{a}_{\overline{n}\|}$

In practice the last column is usually preferred.

A deferred pension

EXAMPLE 9.3

A pension of €10,000.00 per annum is payable annually in arrears for twenty years, the first payment six years from now. At $i = 5\%$, find the present value.

A deferred pension

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A pension of €10,000.00 per annum is payable annually in arrears for twenty years, the first payment six years from now. At $i = 5\%$, find the present value.

The first payment is at time 6, so $k = 5$ and $n = 20$. The deferment is counted to the period *before* the first payment. With $v^5 = 0.783526$ and $a_{\overline{20}|} = 12.462210$, we have

$$10{,}000\ {}_{5|}a_{\overline{20}|} \;=\; 10{,}000 \times 0.783526 \times 12.462210 \;=\; €97{,}644.68.$$

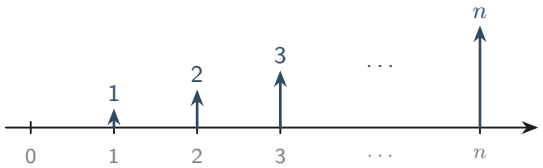
By subtraction, $a_{\overline{25}|} - a_{\overline{5}|} = 14.093945 - 4.329477 = 9.764468$, we get the same factor.

SECTION 2

Payments that grow by one each period

The set-up

Consider an annuity of n payments in arrears, where the payment at time k is k rather than 1. The total paid is $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.



DEFINITION 9.4 · Increasing annuity

The present value of payments of $1, 2, \ldots, n$ made at times $1, 2, \ldots, n$ is written $(Ia)_{\overline{n}|}$, read "I-a-n". That is,

$$(Ia)_{\overline{n}|} = \mathrm{v} + 2\mathrm{v}^2 + 3\mathrm{v}^3 + \cdots + n\mathrm{v}^n.$$

The derivation

This is not a geometric progression, so the closed form of Lecture 5 does not apply directly. The trick that produced it does. Multiply by $1+i$, which lowers every exponent by one, and write the two series above one another.

$$\begin{aligned}(1+i)\,(Ia)_{\overline{n}|} &= 1 + 2\mathsf{v} + 3\mathsf{v}^2 + \cdots + n\mathsf{v}^{n-1},\\ (Ia)_{\overline{n}|} &= \mathsf{v} + 2\mathsf{v}^2 + \cdots + (n-1)\mathsf{v}^{n-1} + n\mathsf{v}^{n}.\end{aligned}$$

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$$\begin{aligned}
(1+i)\,(Ia)_{\overline{n}|} &= 1 + 2\mathsf{v} + 3\mathsf{v}^2 + \cdots + n\mathsf{v}^{n-1}, \\
-\quad (Ia)_{\overline{n}|} &= \mathsf{v} + 2\mathsf{v}^2 + \cdots + (n-1)\mathsf{v}^{n-1} + n\mathsf{v}^n \\
\hline
(Ia)_{\overline{n}|} &= \mathsf{v} + (3-2)\mathsf{v}^2 + \cdots + (n-(n-1))\mathsf{v}^{n-1} + n\mathsf{v}^n
\end{aligned}$$

Subtracting, the coefficient of every power of v falls by exactly one, and a single term $n\mathsf{v}^n$ is left over at the end,

$$i\,(Ia)_{\overline{n}|} = 1 + \mathsf{v} + \mathsf{v}^2 + \cdots + \mathsf{v}^{n-1} - n\mathsf{v}^n = \ddot{a}_{\overline{n}|} - n\mathsf{v}^n.$$

Hence $$\boxed{(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\mathsf{v}^n}{i}.}$$

Comparision with the geometric progression

SHIFT AND SUBTRACT, FOR THE SECOND TIME

In Lecture 5, subtracting a shifted copy of the series collapsed a geometric progression. Here, it flattens a staircase into a level annuity. The $n+1$ terms of $\ddot{a}_{\overline{n}|}$ are the level part, and $n\mathrm{v}^n$ removes the payment the shift invented at the far end.

At $i = 6\%$ and $n = 4$, with $\ddot{a}_{\overline{4}|} = 3.673012$ and $4\mathrm{v}^4 = 3.168375$,

$$(Ia)_{\overline{4}|} \;=\; \frac{3.673012 - 3.168375}{0.06} \;=\; 8.410622,$$

which is the warm-up's number.

Carry the subtraction to more places than the six shown. Dividing by $i = 0.06$ multiplies whatever rounding survives it by nearly seventeen, so the last figure of any annuity symbol computed this way is never quite safe.

In advance, and at the far end

Use same argument that showed $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$ in Lecture 5: Regard the payments made at times $0, 1, \ldots, n-1$ from position $t = -1$ (meaning, regard advance payments as arrears by stepping back one step). But this discounts it one step too much and thus, we need to accumulate one more step. This same argument gives us analogous formulas:

$$(I\ddot{a})_{\overline{n}|} = (1+i)\,(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\mathrm{v}^n}{d}, \qquad (Is)_{\overline{n}|} = (1+i)^n\,(Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}.$$

KEY IDEA

The rule of Lectures 5 to 8 survives intact. Advance replaces i by d, and moving the valuation date to n replaces the numerator $\ddot{a}_{\overline{n}|} - n\mathrm{v}^n$ by $\ddot{s}_{\overline{n}|} - n$. This is just the transfer principle in action again.

A growing annuity

EXAMPLE 9.5

An annuity pays €1,000.00 at the end of the first year, €2,000.00 at the end of the second, and so on, the last payment of €10,000.00 falling at the end of the tenth year. At $i = 6\%$, find the present value.

A growing annuity

EXAMPLE 9.5

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With $\ddot{a}_{\overline{10|}} = 7.801692$ and $10v^{10} = 5.583948$,

$$(Ia)_{\overline{10|}} = \frac{7.801692 - 5.583948}{0.06} = 36.962408, \qquad 1000\,(Ia)_{\overline{10|}} = €36{,}962.41.$$

ASIDE

The same €55,000.00 paid level, at €5,500.00 a year, would be worth $5500\,a_{\overline{10|}} =$ €40,480.48. The growing annuity is worth less, because larger sums arrive later.

SECTION 3

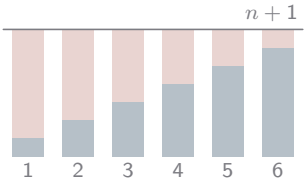
Decreasing payments, and everything between

Payments that shrink

DEFINITION 9.6 · Decreasing annuity

The present value of payments of $n, n-1, \ldots, 1$ made at times $1, 2, \ldots, n$ is written

$$(Da)_{\overline{n}|} \;=\; n\mathsf{v} + (n-1)\mathsf{v}^2 + \cdots + \mathsf{v}^n.$$



Lower block k, upper block $n+1-k$.

Pay both annuities at once. At time k they hand over k and $n+1-k$, which is $n+1$ whatever k is. Together they are a level annuity of size $n+1$,

$$(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} \;=\; (n+1)\, a_{\overline{n}|}.$$

Thus, we need not derive a formula for decreasing annuities separately. They are simply the difference between a level annuity and an increasing one.

The closed form

$$(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} \;=\; (n+1)\, a_{\overline{n}|}.$$

The closed form

$$(Da)_{\overline{n}|} = (n+1)\, a_{\overline{n}|} - (Ia)_{\overline{n}|}.$$

Move $(Ia)_n$ to the other side

The closed form

$$(Da)_{\overline{n}|} \;=\; (n+1)\, a_{\overline{n}|} \;-\; \frac{\ddot{a}_{\overline{n}|} - n\, \mathsf{v}^n}{i}.$$

Closed form of $(Ia)_n$

The closed form

$$(Da)_{\overline{n}|} = (n+1)\, a_{\overline{n}|} - \frac{(1+i)a_{\overline{n}|} - n\, v^n}{i}.$$

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$$

The closed form

$$(Da)_{\overline{n}|} = (n+1)\, a_{\overline{n}|} - \frac{(1+i)a_{\overline{n}|} - n\left(1 - i\, a_{\overline{n}|}\right)}{i}.$$

$v^n = 1 - i\, a_{\overline{n}|}$
(Lecture 5)

The closed form

$$(Da)_{\overline{n}|} = \frac{i(n+1)\, a_{\overline{n}|} - (1+i)a_{\overline{n}|} + n - ni\, a_{\overline{n}|}}{i}.$$

Unify the denominators, add and expand the brackets.

The closed form

$$(Da)_{\overline{n}|} = \frac{a_{\overline{n}|}\left[i(n+1)-(1+i)-ni\right]+n}{i}.$$

Group the $a_{\overline{n}|}$ terms.

The closed form

$$(Da)_{\overline{n}|} = \frac{a_{\overline{n}|}\left[\cancel{in} + \cancel{i} - 1 - \cancel{i} - \cancel{ni}\right] + n}{i}.$$

in, ni cancel, so do the two i's, leaving -1

The closed form

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}.$$

The closed form

$$(Da)_{\overline{n}|} \;=\; \frac{n - a_{\overline{n}|}}{i} \qquad \text{and, one period earlier,} \qquad (D\ddot{a})_{\overline{n}|} \;=\; \frac{n - a_{\overline{n}|}}{d}.$$

EXAMPLE 9.7

At $i = 6\%$ and $n = 10$, where $a_{\overline{10}|} = 7.360087$, this gives $(Da)_{\overline{10}|} = \frac{10 - 7.360087}{0.06} = 43.99855$, and the identity checks,

$$(Ia)_{\overline{10}|} + (Da)_{\overline{10}|} \;=\; 36.96241 + 43.99855 \;=\; 80.96096 \;=\; 11\, a_{\overline{10}|}.$$

Payments in arithmetic progression

Most varying annuities in practice de-/increase by a fixed amount rather than by 1. Let the payment at time t be $P + (t-1)Q$, the general arithmetic progression, and rewrite it as

$$P + (t-1)Q \;=\; \underbrace{(P-Q)}_{\text{Constant every year}} \;+\; \underbrace{tQ}_{\text{Changes by } Q}\,.$$

THE DECOMPOSITION

A cashflow in arithmetic progression is a level annuity plus a multiple of an increasing annuity, so it needs no third symbol,

$$\text{PV} \;=\; (P-Q)\,a_{\overline{n}|} \;+\; Q\,(Ia)_{\overline{n}|}.$$

$P - Q$ may well be negative, for instance in a falling schedule, where the level part is subtracted rather than added.

A rising rent

EXAMPLE 9.8

A tenant pays €1,500.00 at the end of the first year, and each year after that €100.00 more than the year before, for ten years. At $i = 6\%$, find the present value of the rent.

A rising rent

EXAMPLE 9.8

A tenant pays €1,500.00 at the end of the first year, and each year after that €100.00 more than the year before, for ten years. At $i = 6\%$, find the present value of the rent.

Here $P = 1500$, $Q = 100$ and $n = 10$, so $P - Q = 1400$. We get

$$1400\, a_{\overline{10}|} + 100\, (Ia)_{\overline{10}|} \;=\; 1400 \times 7.360087 + 100 \times 36.962408$$

$$= \;€10{,}304.12 + €3{,}696.24 \;=\; €14{,}000.36.$$

The tenant hands over €19,500.00 in all. Paid level, at €1,950.00 a year, that total would be worth $1950\, a_{\overline{10}|} = €14{,}352.17$. Thus, the rising schedule is cheaper by €351.81 for the tenant.

Deferring a varying annuity

Discounting works directly. Payments of $1, 2, \ldots, n$ at times $k+1, \ldots, k+n$ are worth $(Ia)_{\overline{n}|}$ at time k, so

$$ {}_{k|}(Ia)_{\overline{n}|} \;=\; \mathrm{v}^k\,(Ia)_{\overline{n}|}. $$

SUBTRACTION DOES NOT

$(Ia)_{\overline{k+n}|} - (Ia)_{\overline{k}|}$ is a different cashflow. It pays $k+1, k+2, \ldots, k+n$ at those dates, because the long annuity does not restart its count at the deferment. In fact

$$ (Ia)_{\overline{k+n}|} - (Ia)_{\overline{k}|} \;=\; \mathrm{v}^k\left((Ia)_{\overline{n}|} + k\,a_{\overline{n}|}\right), $$

the extra term being the level k that every surviving payment carries.

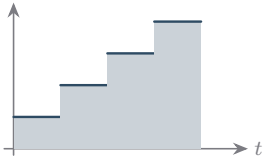
At $i = 6\%$, $k = 5$ and $n = 10$, deferment gives $0.747258 \times 36.962408 = 27.620462$, where subtraction gives $67.266800 - 12.146912 = 55.119888$. The two are not close.

SECTION 4

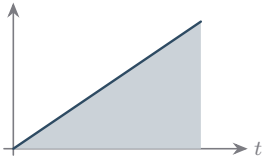
Growth in continuous time

Two different things can increase

Continuity for annuities is no longer a clear term. A continuous annuity that grows can mean either of two, distinct, cashflows:



$(I\bar{a})_{\overline{n}|}$, a step rate



$(\bar{I}\bar{a})_{\overline{n}|}$, a smooth ramp

- $(I\bar{a})_{\overline{n}|}$ pays continuously at the *constant* rate k throughout the k-th period. Each year's total is still $1, 2, \ldots, n$.
- $(\bar{I}\bar{a})_{\overline{n}|}$ pays continuously at the rate t at time t. The rate itself never stops rising.

A bar over the a says how the money is paid. A bar over the I says how the amount grows.

The two closed forms

For the step rate, the yearly totals are unchanged and only their timing within each year moves, which is the situation of Lectures 7 and 8. The numerator survives and the denominator becomes the force,

$$(I\bar{a})_{\overline{n}|} \;=\; \frac{\ddot{a}_{\overline{n}|} - n\mathsf{v}^n}{\delta} \;=\; \frac{i}{\delta}\,(Ia)_{\overline{n}|}.$$

For the smooth ramp, integrate by parts, taking $\mathsf{v}(t) = \mathsf{e}^{-\delta t}$,

The two closed forms

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For the smooth ramp, integrate by parts, taking $\mathsf{v}(t) = \mathsf{e}^{-\delta t}$, Let $u = t$, $v = -\int e^{-\delta t dt}$. Then $du = dt$, $dv = e^{-\delta t}dt$. Recall also integration by parts:

$$\int udv \;=\; uv - \int vdv.$$

The two closed forms

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$$(I\bar{a})_{\overline{n}|} \;=\; \frac{\ddot{a}_{\overline{n}|} - n\mathrm{v}^n}{\delta} \;=\; \frac{i}{\delta}\,(Ia)_{\overline{n}|}.$$

For the smooth ramp, integrate by parts, taking $\mathrm{v}(t) = \mathrm{e}^{-\delta t}$,

$$(\bar{I}\bar{a})_{\overline{n}|} \;=\; \int_0^n t\,\mathrm{v}^t\,\mathrm{d}t \;=\; \left[\frac{-t\,\mathrm{e}^{-\delta t}}{\delta}\right]_0^n + \frac{1}{\delta}\int_0^n \mathrm{e}^{-\delta t}\,\mathrm{d}t \;=\; \frac{\bar{a}_{\overline{n}|} - n\mathrm{v}^n}{\delta}.$$

KEY IDEA

The denominator says how often the money is paid. The numerator says how the amount grows, $\ddot{a}_{\overline{n}|}$ for a yearly step and $\bar{a}_{\overline{n}|}$ for a continuous one.

The three of them, side by side

At $i = 5\%$ and $n = 8$, where $\delta = 0.0487902$, $\ddot{a}_{\overline{8}|} = 6.7863734$, $\bar{a}_{\overline{8}|} = 6.6234792$ and $8v^8 = 5.4147149$,

Symbol	Payment of the k-th period	Denominator	Value
$(Ia)_{\overline{8}\|}$	k, all at the year end	i	27.4332
$(I\bar{a})_{\overline{8}\|}$	k, spread through the year	δ	28.1134
$(\bar{I}\bar{a})_{\overline{8}\|}$	on average $k - \frac{1}{2}$, spread likewise	δ	24.7748

- $(I\bar{a})_{\overline{8}|} > (Ia)_{\overline{8}|}$, the same yearly totals arriving earlier.
- $(\bar{I}\bar{a})_{\overline{8}|} < (Ia)_{\overline{8}|}$, and not because of discounting. The ramp simply hands over less money.

Lecture 9 on one slide

WAITING, GROWING, SHRINKING

$$ {}_{k|}a_{\overline{n}|} = \mathsf{v}^k a_{\overline{n}|} = a_{\overline{k+n}|} - a_{\overline{k}|}, \qquad (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\mathsf{v}^n}{i}, \qquad (Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}, $$

$$ (Ia)_{\overline{n}|} + (Da)_{\overline{n}|} = (n+1)a_{\overline{n}|}, \qquad P + (t-1)Q \longrightarrow (P-Q)a_{\overline{n}|} + Q\,(Ia)_{\overline{n}|}. $$

In advance divide by d, continuously by δ, and at time n replace $\ddot{a}_{\overline{n}|} - n\mathsf{v}^n$ by $\ddot{s}_{\overline{n}|} - n$. Deferment multiplies by v^k, but only when the payments themselves restart at 1.

Exercises

EXERCISE 9.9

An annuity of €5,000.00 per month in arrears runs for ten years, the first payment falling in the month after time 4. At $i = 6\%$, find its value today.

EXERCISE 9.10

Prove $(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} = (n+1)a_{\overline{n}|}$ from the definitions, and deduce the closed form of $(Da)_{\overline{n}|}$. Evaluate $(Da)_{\overline{10}|}$ at $i = 6\%$.

EXERCISE 9.11

An annuity pays €800.00 at the end of the first year and €50.00 more in each year after that, for twelve years, at $i = 7\%$. Find its present value.

EXERCISE 9.12

Payments are made continuously for ten years at the rate of €1,000.00 t per annum at time t. At $i = 4\%$, find the present value.

Solutions are in the backup slides.

END OF LECTURE 9

The symbols are finished. From here the rate stops being given to us. Next, the value of a cashflow as a function of i, and the rate at which a project is worth nothing at all.

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Solution 1

The total is €60,000.00 per annum with $m = 12$ and $n = 10$, deferred by $k = 4$. With $i^{(12)} = 12\left(1.06^{1/12} - 1\right) = 0.058411$ and $v^{10} = 0.558395$,

$$a^{(12)}_{\overline{10}|} \;=\; \frac{1 - 0.558395}{0.058411} \;=\; 7.560360,$$

and $v^4 = 1.06^{-4} = 0.792094$, so the present value is

$$60{,}000 \times 0.792094 \times 7.560360 \;=\; €359{,}310.80.$$

By the other route, $a^{(12)}_{\overline{14}|} - a^{(12)}_{\overline{4}|} = 5.988513$, which is 0.792094×7.560360 again.

Solution 2

The payment at time k is k in the first annuity and $n+1-k$ in the second, so the two together pay $n+1$ at every one of the n dates,

$$(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} \;=\; \sum_{k=1}^{n} \big(k + (n+1-k)\big)\, \mathsf{v}^k \;=\; (n+1) \sum_{k=1}^{n} \mathsf{v}^k \;=\; (n+1) a_{\overline{n}|}.$$

Therefore, using $\ddot{a}_{\overline{n}|} = (1+i) a_{\overline{n}|}$ and $\mathsf{v}^n = 1 - i\, a_{\overline{n}|}$,

$$(Da)_{\overline{n}|} \;=\; (n+1) a_{\overline{n}|} - \frac{\ddot{a}_{\overline{n}|} - n\mathsf{v}^n}{i} \;=\; \frac{i(n+1) a_{\overline{n}|} - (1+i) a_{\overline{n}|} + n - n\, i\, a_{\overline{n}|}}{i} \;=\; \frac{n - a_{\overline{n}|}}{i}.$$

At $i = 6\%$, $(Da)_{\overline{10}|} = (10 - 7.360087)/0.06 = 43.99855$. It values payments of $10, 9, \ldots, 1$ at times $1, 2, \ldots, 10$, a total of 55.

Solution 3

Here $P = 800$, $Q = 50$ and $n = 12$, so the decomposition gives $750\, a_{\overline{12}|} + 50\, (Ia)_{\overline{12}|}$. With $1.07^{12} = 2.252192$ we have $v^{12} = 0.444012$, so

$$a_{\overline{12}|} \;=\; \frac{1 - 0.444012}{0.07} \;=\; 7.942686, \qquad \ddot{a}_{\overline{12}|} \;=\; 1.07 \times 7.942686 \;=\; 8.498674,$$

$$(Ia)_{\overline{12}|} \;=\; \frac{8.498674 - 12 \times 0.444012}{0.07} \;=\; 45.293298.$$

Hence $750 \times 7.942686 + 50 \times 45.293298 = €\,5{,}957.01 + €\,2{,}264.66 = €\,8{,}221.68$, of which the level part accounts for €\,5,957.01.

Solution 4

The rate of payment at time t is $1000\,t$, which is the smooth ramp, so the factor wanted is $(\bar{I}\bar{a})_{\overline{10}|}$. With $\delta = \ln 1.04 = 0.0392207$, $v^{10} = 1.04^{-10} = 0.6755642$ and

$$\bar{a}_{\overline{10}|} \;=\; \frac{1 - 0.6755642}{0.0392207} \;=\; 8.2720533,$$

we get

$$(\bar{I}\bar{a})_{\overline{10}|} \;=\; \frac{8.2720533 - 6.7556417}{0.0392207} \;=\; \frac{1.5164116}{0.0392207} \;=\; 38.663540,$$

so the present value is $1000\,(\bar{I}\bar{a})_{\overline{10}|} = €\,38{,}663.54$.

The undiscounted total is $\int_0^{10} 1000\,t\,\mathrm{d}t = €\,50{,}000.00$, and a level stream of the same total, €5,000.00 per annum payable continuously, would be worth $5000\,\bar{a}_{\overline{10}|} = €\,41{,}360.27$. Both comparisons point the same way, the ramp's money arrives late.