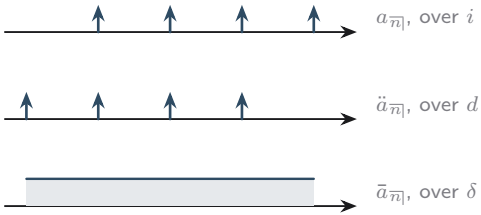
LECTURE 8

# Annuities payable $m$ times a period

Two nominal rates, and the whole family they complete

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
22 September 2026 · Winter 2026

# Three annuities, three denominators




$$\text{The present value } = \frac{1-\mathrm{v}^n}{\square}$$

Regular vs continuous payments.
Today, the case in between.

# Today

1. Payments in arrears
2. Payments in advance
3. The two nominal rates, and the whole family
4. Choosing the unit of time

## Warm-up

*EXERCISE 8.1*

*The effective rate of interest is* 6% *per annum. Find the nominal rate of interest and the nominal rate of discount, both convertible quarterly.*

## Warm-up

*EXERCISE 8.1*

*The effective rate of interest is* 6% *per annum. Find the nominal rate of interest and the nominal rate of discount, both convertible quarterly.*

Use the "brdige" from Lecture 1 and passes through $1+i$. Recall

$$1+i \;=\; \left(1+\frac{i^{(m)}}{m}\right)^{m}.$$

*EXERCISE 8.1*

*The effective rate of interest is* 6% *per annum. Find the nominal rate of interest and the nominal rate of discount, both convertible quarterly.*

Note, we are looking for the *nominal* rate, menaing $i^{(m)}$, so we can not simply take $0.06/4 = 0.015$. Using our formula, $i^{(m)} = m\left((1+i)^{1/m} - 1\right)$, we get

$$i^{(4)} = 4\left(1.06^{1/4} - 1\right) = 0.058695, \qquad d^{(4)} = 4\left(1 - 1.06^{-1/4}\right) = 0.057847.$$

*EXERCISE 8.1*

*The effective rate of interest is* 6% *per annum. Find the nominal rate of interest and the nominal rate of discount, both convertible quarterly.*

$$i^{(4)} = 4\left(1.06^{1/4} - 1\right) = 0.058695, \qquad d^{(4)} = 4\left(1 - 1.06^{-1/4}\right) = 0.057847.$$

ASIDE

As always $d^{(4)} < i^{(4)} < i$. Since the same interest earns more the more often we compound, to get the same growth, we must reduce the interest rate if we compound more often.

SECTION 1

# Payments in arrears

## The set-up

Consider an annuity payable over $n$ units of time, where payments of $\frac{1}{m}$ are made *in arrears* $m$ times per unit time. The total payment is again 1 per unit time. For example, an annual salary paid monthly.




There are $nm$ payments in all, at times $\frac{k}{m}$ for $k = 1, 2, \ldots, nm$. The last one falls exactly at time $n$.

## Two shorthand-notations used throughout this lecture

As the payment sits at a multiple of $\dfrac{1}{m}$, the factor we meet are powers of one of these:

$$q \;:=\; (1+i)^{\frac{1}{m}} \qquad\qquad w \;:=\; \mathsf{v}^{\frac{1}{m}} \;=\; q^{-1}$$

Accumulate one $m$-th of a period — Discount one $m$-th of a period

With these, the warm-up's two rates read

$$i^{(m)} \;=\; m\,(q-1)\,, \qquad\qquad d^{(m)} \;=\; m\,(1-w)\,.$$

KEY IDEA

Both sums below are geometric with ratio $q$ or $w$, $nm$ terms each. The only work is to recognise $m(q-1)$ and $m(1-w)$ at the end.

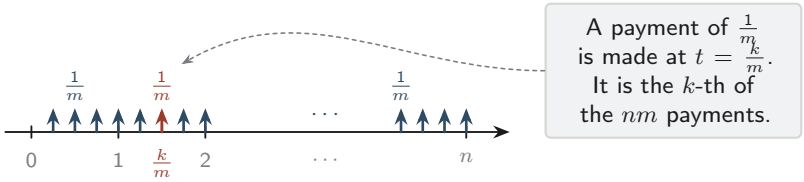
## Accumulated value $s^{(m)}_{\overline{n}|}$

Payments of $\frac{1}{m}$ are made in arrears $m$ times per unit time, over $n$ units of time.



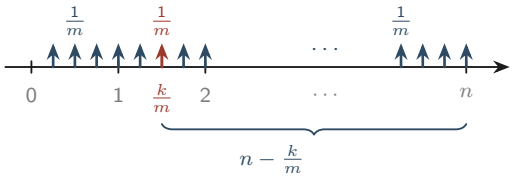
# Accumulated value $s_{\overline{n}|}^{(m)}$

Payments of $\frac{1}{m}$ are made in arrears $m$ times per unit time, over $n$ units of time.



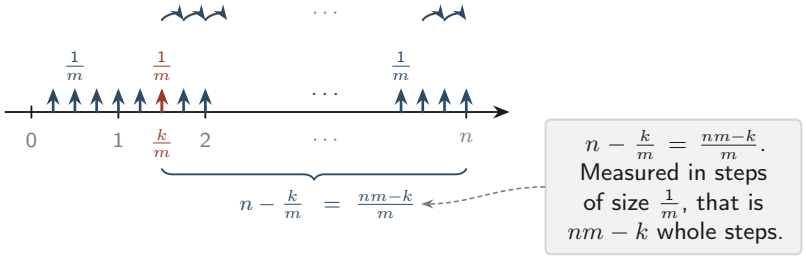
# Accumulated value $s_{\overline{n}|}^{(m)}$

Payments of $\frac{1}{m}$ are made in arrears $m$ times per unit time, over $n$ units of time.



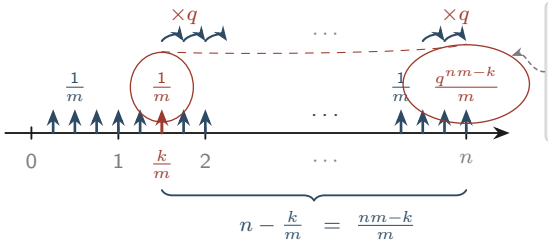
# Accumulated value $s^{(m)}_{\overline{n}|}$

Payments of $\frac{1}{m}$ are made in arrears $m$ times per unit time, over $n$ units of time.



# Accumulated value $s_{\overline{n}|}^{(m)}$

Payments of $\frac{1}{m}$ are made in arrears $m$ times per unit time, over $n$ units of time. The one made at $t = \frac{k}{m}$ arrives at time $n$ as $\frac{q^{nm-k}}{m}$.




Each of the $nm - k$ steps multiplies by $q$, so the payment lands as $\frac{q^{nm-k}}{m}$.

# Accumulated value $s^{(m)}_{\overline{n}|}$

Payments of $\frac{1}{m}$ are made in arrears $m$ times per unit time, over $n$ units of time. The one made at $t = \frac{k}{m}$ arrives at time $n$ as $\frac{q^{nm-k}}{m}$.



# Accumulated value $s^{(m)}_{\overline{n}|}$

Payments of $\frac{1}{m}$ are made in arrears $m$ times per unit time, over $n$ units of time. The one made at $t = \frac{k}{m}$ arrives at time $n$ as $\frac{q^{nm-k}}{m}$.

$$s^{(m)}_{\overline{n}|} = \sum_{k=1}^{nm} \frac{q^{nm-k}}{m}$$

Now, just expand the $\sum$ (notation only).

## Accumulated value $s^{(m)}_{\overline{n}|}$

Payments of $\frac{1}{m}$ are made in arrears $m$ times per unit time, over $n$ units of time. The one made at $t = \frac{k}{m}$ arrives at time $n$ as $\frac{q^{nm-k}}{m}$.

$$s^{(m)}_{\overline{n}|} = \frac{1}{m}\left(q^{nm-1} + q^{nm-2} + \cdots + q + 1\right)$$

This is just the geometric series with $a = 1$ and $r = q$. We reorder to make it more familiar.

# Accumulated value $s^{(m)}_{\overline{n}|}$

Payments of $\frac{1}{m}$ are made in arrears $m$ times per unit time, over $n$ units of time. The one made at $t = \frac{k}{m}$ arrives at time $n$ as $\frac{q^{nm-k}}{m}$.

$$s^{(m)}_{\overline{n}|} = \frac{1}{m}\left(1 + q + q^2 + \cdots + q^{nm-1}\right)$$

Recall the formula for a geometric series.

## Accumulated value $s^{(m)}_{\overline{n}|}$

Payments of $\frac{1}{m}$ are made in arrears $m$ times per unit time, over $n$ units of time. The one made at $t = \frac{k}{m}$ arrives at time $n$ as $\frac{q^{nm-k}}{m}$.

$$s^{(m)}_{\overline{n}|} = \frac{1}{m} \cdot \frac{q^{nm} - 1}{q - 1}$$

# Accumulated value $s^{(m)}_{\overline{n}|}$

Summing all $nm$ payments gave a geometric series. Two substitutions now turn $q$ back into $i$.

$$s^{(m)}_{\overline{n}|} = \frac{q^{nm} - 1}{m(q-1)}$$

As $q = (1+i)^{1/m}$,

$$q^{nm} = \left((1+i)^{1/m}\right)^{nm} = (1+i)^n.$$

# Accumulated value $s^{(m)}_{\overline{n}|}$

Summing all $nm$ payments gave a geometric series. Two substitutions now turn $q$ back into $i$.

$$s^{(m)}_{\overline{n}|} = \frac{(1+i)^n - 1}{m(q-1)}$$

Likewise, as $q = (1+i)^{1/m}$,

$$i^{(m)} = m\big((1+i)^{1/m} - 1\big) = m(q-1).$$

# Accumulated value $s^{(m)}_{\overline{n}|}$

Summing all $nm$ payments gave a geometric series. Two substitutions now turn $q$ back into $i$.



## Accumulated value $s^{(m)}_{\overline{n}|}$

Summing all $nm$ payments gave a geometric series. Two substitutions now turn $q$ back into $i$.

$$s^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}}\, s_{\overline{n}|}$$

KEY IDEA

The nominal rate $i^{(m)}$ was *defined* in Lecture 1 as the rate quoted when interest is credited $m$ times per period. It is, of course, no accident that it turns up here, where *payments* are made $m$ times per period. We are essentially still doing the same thing.

# Present value $a_{\overline{n}|}^{(m)}$

Discount each payment to time 0 instead. The payment at $t = \dfrac{k}{m}$ is worth $\dfrac{w^k}{m}$, so

$$a_{\overline{n}|}^{(m)} = \frac{1}{m} \sum_{k=1}^{nm} w^k$$

$$\sum_{k=1}^{N} w^k = \frac{w\left(1 - w^N\right)}{1 - w}$$

is the geometric series for any $N$, including $N = nm$.

This is a finite geometric series with first term $w$ and ratio $w$; recall its closed form.

## Present value $a_{\overline{n}|}^{(m)}$

Discount each payment to time 0 instead. The payment at $t = \dfrac{k}{m}$ is worth $\dfrac{w^k}{m}$, so

$$a_{\overline{n}|}^{(m)} = \frac{w\left(1 - w^{nm}\right)}{m(1-w)}$$

$= (w^m)^n$

To rewrite $w^{nm}$ in terms of the annual discount factor v, pplit the exponent with the power rule: $w^{nm} = (w^m)^n$.

## Present value $a_{\overline{n}|}^{(m)}$

Discount each payment to time 0 instead. The payment at $t = \dfrac{k}{m}$ is worth $\dfrac{w^k}{m}$, so

$$a_{\overline{n}|}^{(m)} = \frac{w\left(1 - w^{nm}\right)}{m(1-w)}$$

$= (w^m)^n = \mathsf{v}^n$ since $w = \mathsf{v}^{\frac{1}{m}}$ by definition

Since $w^m = \mathsf{v}$ (the $m$-thly factor compounded $m$ times over a year gives the annual one), $(w^m)^n = \mathsf{v}^n$.

## Present value $a^{(m)}_{\overline{n}|}$

Discount each payment to time 0 instead. The payment at $t = \dfrac{k}{m}$ is worth $\dfrac{w^k}{m}$, so

$$a^{(m)}_{\overline{n}|} = \frac{w\,(1 - \mathsf{v}^n)}{m(1 - w)}$$

Substituting gives the closed form.

## Present value $a_{\overline{n}|}^{(m)}$

Discount each payment to time 0 instead. The payment at $t = \dfrac{k}{m}$ is worth $\dfrac{w^k}{m}$, so

$$a_{\overline{n}|}^{(m)} = \frac{w\,(1 - \mathrm{v}^n)}{m(1 - w)}$$

Divide top and bottom by $w$

Dividing top and bottom by $w$ simplifies the denominator.

## Present value $a_{\overline{n}|}^{(m)}$

Discount each payment to time 0 instead. The payment at $t = \dfrac{k}{m}$ is worth $\dfrac{w^k}{m}$, so

$$a_{\overline{n}|}^{(m)} = \frac{1 - \mathrm{v}^n}{m\left(w^{-1} - 1\right)}$$

Recall $w^{-1} = q$

$w^{-1}$ is exactly $q$.

## Present value $a^{(m)}_{\overline{n}|}$

Discount each payment to time 0 instead. The payment at $t = \dfrac{k}{m}$ is worth $\dfrac{w^k}{m}$, so

$$a^{(m)}_{\overline{n}|} = \frac{1 - \mathsf{v}^n}{m(q-1)}$$

$$i^{(m)} := m(q-1)$$

$i^{(m)}$ is defined as $m$ times that quantity: the nominal rate compounded $m$ times a year.

## Present value $a_{\overline{n}|}^{(m)}$

Discount each payment to time 0 instead. The payment at $t = \dfrac{k}{m}$ is worth $\dfrac{w^k}{m}$, so

$$a_{\overline{n}|}^{(m)} = \frac{1 - \mathsf{v}^n}{i^{(m)}}$$

$$a_{\overline{n}|} = \frac{1 - \mathsf{v}^n}{i} \;\Rightarrow\; 1 - \mathsf{v}^n = i\, a_{\overline{n}|}$$

The numerator is exactly $i$ times the familiar annual annuity factor.

## Present value $a_{\overline{n}|}^{(m)}$

Discount each payment to time 0 instead. The payment at $t = \dfrac{k}{m}$ is worth $\dfrac{w^k}{m}$, so

$$a_{\overline{n}|}^{(m)} = \boxed{\frac{1 - \mathsf{v}^n}{i^{(m)}}} = \frac{i}{i^{(m)}}\, a_{\overline{n}|}$$

Substituting gives $a_{\overline{n}|}^{(m)}$ as a multiple of $a_{\overline{n}|}$.

# A quarterly annuity

EXAMPLE 8.2

An annuity pays €1,000.00 at the end of each quarter for 20 years. At $i = 6\%$ effective per annum, find its present value.

The total is €4,000.00 per annum, so the value is $4000\, a^{(4)}_{\overline{20}|}$. The warm-up gave $i^{(4)} = 0.058695$, and $v^{20} = 0.311805$. Hence

$$a^{(4)}_{\overline{20}|} = \frac{1 - 0.311805}{0.058695} = 11.724862, \qquad 4000\, a^{(4)}_{\overline{20}|} = €46{,}899.45.$$

As €4,000.00 once a year in arrears it would be worth $4000\, a_{\overline{20}|} = €45{,}879.68$, about €1,020.00 less. Three quarters of each year's money would arrive later.

SECTION 2

# Payments in advance

## The set-up

The same annuity, but the payments of $\frac{1}{m}$ are made *in advance*: at times $\frac{k}{m}$ for $k \in \{0, 1, \ldots, nm-1\}$. We again have $nm$ payments, each one $\frac{1}{m}$ of a period earlier than before. Examples are rent paid monthly, or a mortgage.




The first falls at time 0, the last one $\frac{1}{m}$ of a period before $n$.

# Present value $\ddot{a}^{(m)}_{\overline{n}|}$

Discount each payment to time 0 again. The payment at $t = \dfrac{k}{m}$ is still worth $\dfrac{w^k}{m}$, we just start at $t = 0$, meaning $k$ now starts at 0, so the sum starts at $w^0 = 1$.

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1}{m} \sum_{k=0}^{nm-1} w^k$$

$$\sum_{k=0}^{N-1} w^k = \frac{1 - w^N}{1 - w}$$

is the geometric series for any $N$, including $N = nm$.

A finite geometric series with first term 1 and ratio $w$ but starting at $k = 0$.

## Present value $\ddot{a}_{\overline{n}|}^{(m)}$

Discount each payment to time 0 again. The payment at $t = \dfrac{k}{m}$ is still worth $\dfrac{w^k}{m}$, we just start at $t = 0$, meaning $k$ now starts at 0, so the sum starts at $w^0 = 1$.

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - w^{nm}}{m(1-w)}$$

$= (w^m)^n$

Split the exponent with the power rule, $w^{nm} = (w^m)^n$.

## Present value $\ddot{a}_{\overline{n}|}^{(m)}$

Discount each payment to time 0 again. The payment at $t = \frac{k}{m}$ is still worth $\frac{w^k}{m}$, we just start at $t = 0$, meaning $k$ now starts at 0, so the sum starts at $w^0 = 1$.

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - w^{nm}}{m(1-w)}$$

$= (w^m)^n = \mathsf{v}^n$ as $w^m = \mathsf{v}$ by definition

Since $w = \mathsf{v}^{\frac{1}{m}}$ by our definition.

## Present value $\ddot{a}_{\overline{n}|}^{(m)}$

Discount each payment to time 0 again. The payment at $t = \frac{k}{m}$ is still worth $\frac{w^k}{m}$, we just start at $t = 0$, meaning $k$ now starts at 0, so the sum starts at $w^0 = 1$.

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - \mathsf{v}^n}{m(1-w)}$$

$d^{(m)} := m(1-w)$

This is again page 5.

## Present value $\ddot{a}^{(m)}_{\overline{n}|}$

Discount each payment to time 0 again. The payment at $t = \dfrac{k}{m}$ is still worth $\dfrac{w^k}{m}$, we just start at $t = 0$, meaning $k$ now starts at 0, so the sum starts at $w^0 = 1$.

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1 - \mathrm{v}^n}{d^{(m)}}$$

$$a_{\overline{n}|} = \frac{1 - \mathrm{v}^n}{i} \;\Rightarrow\; 1 - \mathrm{v}^n = i\, a_{\overline{n}|}$$

The numerator is again exactly $i$ times the familiar annual annuity factor.

## Present value $\ddot{a}^{(m)}_{\overline{n}|}$

Discount each payment to time 0 again. The payment at $t = \dfrac{k}{m}$ is still worth $\dfrac{w^k}{m}$, we just start at $t = 0$, meaning $k$ now starts at 0, so the sum starts at $w^0 = 1$.

$$\ddot{a}^{(m)}_{\overline{n}|} = \boxed{\frac{1 - \mathsf{v}^n}{d^{(m)}}} = \frac{i}{d^{(m)}}\, a_{\overline{n}|}$$

As so often before: Same nominator, but different denominator, in this case its $d^{(m)}$ instead of $i^{(m)}$.

> **KEY IDEA**
>
> Arrears bring in $i^{(m)}$ and advance brings in $d^{(m)}$, exactly as $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$ did in Lecture 5. Interest goes hand in hand with arrears, discount with advance, at every frequency.

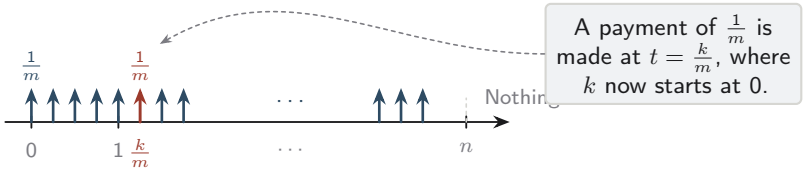
## Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm - 1$.



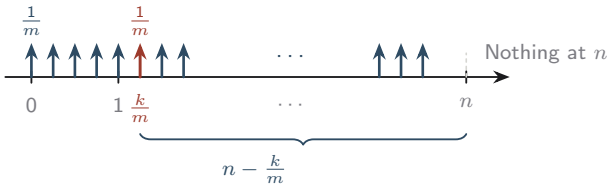
# Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm - 1$.



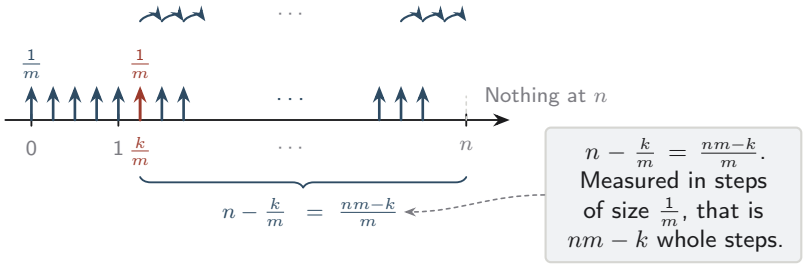
## Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm - 1$.



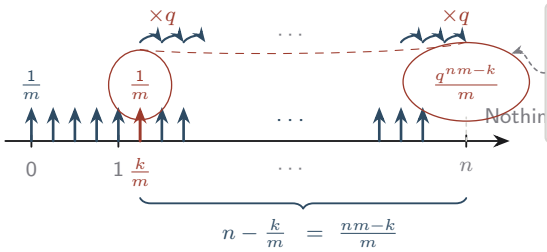
# Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm - 1$.



# Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

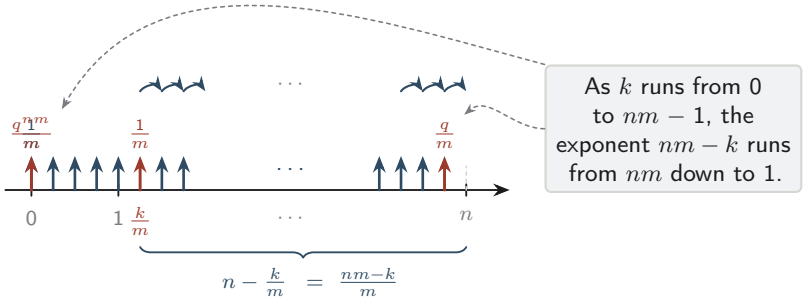
The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm-1$.




Each of the $nm-k$ steps multiplies by $q$, so the payment lands as $\frac{q^{nm-k}}{m}$.

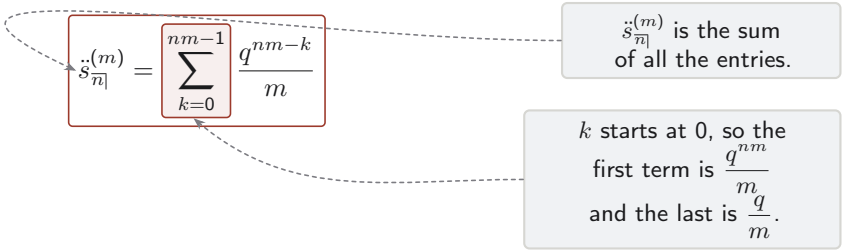
# Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm-1$. Each one is $\frac{1}{m}$ of a period earlier than in arrears, so each carries one more factor of $q$.



# Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm-1$. Each one is $\frac{1}{m}$ of a period earlier than in arrears, so each carries one more factor of $q$.



## Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm-1$. Each one is $\frac{1}{m}$ of a period earlier than in arrears, so each carries one more factor of $q$.

$$\ddot{s}^{(m)}_{\overline{n}|} = \frac{1}{m}\left(q^{nm} + q^{nm-1} + \cdots + q\right)$$

The geometric series again, now with $a = q$ and $r = q$.

## Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm-1$. Each one is $\frac{1}{m}$ of a period earlier than in arrears, so each carries one more factor of $q$.

$$\ddot{s}^{(m)}_{\overline{n}|} = \frac{q\,(q^{nm}-1)}{m(q-1)}$$

Now divide top and bottom by $q$.

## Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm-1$. Each one is $\frac{1}{m}$ of a period earlier than in arrears, so each carries one more factor of $q$.

$$\ddot{s}^{(m)}_{\overline{n}|} = \frac{q^{nm} - 1}{m\left(1 - q^{-1}\right)}$$

Recall $q^{-1} = w$, the discount factor for one $m$-th of a period.

## Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm-1$. Each one is $\frac{1}{m}$ of a period earlier than in arrears, so each carries one more factor of $q$.

$$\ddot{s}^{(m)}_{\overline{n}|} = \frac{q^{nm} - 1}{m(1-w)}$$

By definition, $d^{(m)} = m(1-w)$. Arrears gave $i^{(m)}$, advance gives $d^{(m)}$.

## Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

The same payments of $\frac{1}{m}$, made in advance at times $\frac{k}{m}$ for $k = 0, 1, \ldots, nm-1$. Each one is $\frac{1}{m}$ of a period earlier than in arrears, so each carries one more factor of $q$.

$$\ddot{s}^{(m)}_{\overline{n}|} = \frac{q^{nm} - 1}{d^{(m)}}$$

As $q = (1+i)^{1/m}$,

$$q^{nm} = \left((1+i)^{1/m}\right)^{nm} = (1+i)^n.$$

# Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

Summing all $nm$ payments gave a geometric series again. The same two substitutions turn $q$ back into $i$.

$$\ddot{s}^{(m)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(m)}}$$

$s_{\overline{n}|} = \dfrac{(1+i)^n - 1}{i}$, so this numerator is $i\, s_{\overline{n}|}$.

## Accumulated value $\ddot{s}^{(m)}_{\overline{n}|}$

Summing all $nm$ payments gave a geometric series again. The same two substitutions turn $q$ back into $i$.

$$\ddot{s}^{(m)}_{\overline{n}|} = \boxed{\frac{(1+i)^n - 1}{d^{(m)}}} = \frac{i}{d^{(m)}}\, s_{\overline{n}|}$$

KEY IDEA

All four $m$-thly symbols share the two numerators of Lecture 5. Only the denominator ever changes, and only between $i^{(m)}$ and $d^{(m)}$.

SECTION 3

# The two nominal rates, and the whole family

## The ratio $i^{(m)}/d^{(m)}$

Much as we compared $i$ and $d$, let us now also compare $i^{(m)}$ and $d^{(m)}$. In the notation of this lecture, $i^{(m)} = m(q-1)$ and $d^{(m)} = m(1-w)$ with $w = q^{-1}$, so

$$\frac{i^{(m)}}{d^{(m)}} \;=\; \frac{m(q-1)}{m\,(1-q^{-1})} \;=\; \frac{q-1}{\dfrac{q-1}{q}} \;=\; \boxed{\;q \;=\; (1+i)^{\frac{1}{m}}.\;}$$

EXAMPLE 8.3

At $i = 6\%$, $m = 4$: $\dfrac{0.058695}{0.057847} = 1.014674 = 1.06^{1/4}$, as the warm-up's two numbers promised.

At $m = 1$ this reads $i/d = 1 + i$, which is Lecture 1's relation $d = i/(1+i)$.

## Use of the ratio

All four $m$-thly symbols are multiples of the corresponding annual one, so the annual symbol cancels when two of them are compared:

$$\frac{\ddot{a}^{(m)}_{\overline{n}|}}{a^{(m)}_{\overline{n}|}} \;=\; \frac{\frac{i}{d^{(m)}}\,a_{\overline{n}|}}{\frac{i}{i^{(m)}}\,a_{\overline{n}|}} \;=\; \frac{i^{(m)}}{d^{(m)}} \;=\; (1+i)^{\frac{1}{m}},$$

and likewise for the accumulated pair, so

$$\ddot{a}^{(m)}_{\overline{n}|} \;=\; (1+i)^{\frac{1}{m}}\, a^{(m)}_{\overline{n}|}, \qquad \ddot{s}^{(m)}_{\overline{n}|} \;=\; (1+i)^{\frac{1}{m}}\, s^{(m)}_{\overline{n}|}.$$

KEY IDEA

Each payment "in advance" falls $\frac{1}{m}$ of a period earlier, so is worth one factor of $(1+i)^{1/m}$ more. At $m=1$, this is $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$.

## Everything thus far on one page

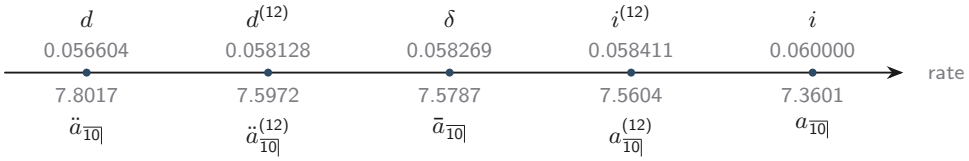
Every symbol proved in Lectures 5 to 8 has the same two numerators. Only the rate underneath changes, and it is decided by *how often* and *when* the money moves.

| Payments | Present value | Accumulated value | Rate |
|---|---|---|---|
| Yearly, in arrears | $a_{\overline{n}\vert}$ | $s_{\overline{n}\vert}$ | $i$ |
| $m$-thly, in arrears | $a^{(m)}_{\overline{n}\vert}$ | $s^{(m)}_{\overline{n}\vert}$ | $i^{(m)}$ |
| Continuously | $\bar{a}_{\overline{n}\vert}$ | $\bar{s}_{\overline{n}\vert}$ | $\delta$ |
| $m$-thly, in advance | $\ddot{a}^{(m)}_{\overline{n}\vert}$ | $\ddot{s}^{(m)}_{\overline{n}\vert}$ | $d^{(m)}$ |
| Yearly, in advance | $\ddot{a}_{\overline{n}\vert}$ | $\ddot{s}_{\overline{n}\vert}$ | $d$ |

$$\text{Present value} = \frac{1-\mathrm{v}^n}{\text{Rate}}, \qquad \text{Accumulated value} = \frac{(1+i)^n-1}{\text{Rate}}.$$

Arranged on a line. As before, continuity sits in the middle

Dividing by a *smaller* rate gives a *bigger* value, so the annuities run inversely to the rates.




KEY IDEA

All five annuities pay a total of 10 and differ only in timing. From Lecture 2, $i^{(m)} \to \delta$ and $d^{(m)} \to \delta$ as $m \to \infty$, and only the denominator depends on $m$. So $a^{(m)}_{\overline{n|}}$ and $\ddot{a}^{(m)}_{\overline{n|}}$ squeeze onto $\bar{a}_{\overline{n|}}$ from either side. Arrears and advance stop mattering once the payments are infinitely frequent.

## Perpetuities at every frequency

The term $n$ appears only inside $v^n$, which tends to 0 whenever $i > 0$. So every present value in the table survives the limit with its numerator replaced by 1 and its denominator untouched.

| Payments | Perpetuity value | at $i = 6\%$, $m = 4$ |
|---|---|---|
| Yearly, in arrears | $1/i$ | 16.6667 |
| $m$-thly, in arrears | $1/i^{(m)}$ | 17.0371 |
| Continuously | $1/\delta$ | 17.1618 |
| $m$-thly, in advance | $1/d^{(m)}$ | 17.2871 |
| Yearly, in advance | $1/d$ | 17.6667 |

CAUTION

There is still no $s_{\overline{\infty}|}$, at any frequency. An accumulated value must be accumulated *to* some date, and a perpetuity has none.

## A charitable trust

EXAMPLE 8.4

A trust is to pay €300.00 at the end of each quarter for ever. At $i = 6\%$ per annum effective, what capital is needed?

# A charitable trust

EXAMPLE 8.4

A trust is to pay €300.00 at the end of each quarter for ever. At $i = 6\%$ per annum effective, what capital is needed?

The total is €1,200.00 per annum, paid quarterly in arrears, so the capital is $1200\, a^{(4)}_{\overline{\infty}|} = 1200/i^{(4)}$. From the warm-up, $i^{(4)} = 0.058695$,

$$\frac{1200}{0.058695} \;=\; €20{,}444.54.$$

Paid once a year in arrears instead, the capital needed would be $1200/0.06 = €20{,}000.00$. Paying the same total earlier costs about €445.00 more capital, for ever.

SECTION 4

# Choosing the unit of time

## A question that looks harder than it is

EXAMPLE 8.5

An investor wishes to purchase a level annuity of €120.00 per annum payable quarterly in arrears for five years. Find the purchase price, calculated on the basis of an interest rate of 12% per annum convertible half-yearly.




Three periods are mentioned, a year, a quarter and a half-year. Our formulas need an *effective* rate per unit time, and we are handed a nominal rate convertible half-yearly.

# The trick, let the given rate be effective

**KEY IDEA**

12% per annum convertible half-yearly means $\frac{i^{(2)}}{2} = 6\%$ per half-year. Take the half-year as the unit and 6% is *effective*. This way, we need not convert anything.

| | in years | in half-years |
|---|---|---|
| Term | 5 years | $n = 10$ |
| Payment frequency | 4 per year | $m = 2$ per unit |
| Payment rate | €120.00 per year | €60.00 per unit |
| Rate of interest | not effective | $i = 6\%$ effective |

Everything the formula needs now sits in the right-hand column.

## Solution

Two times every half-year, for ten half-years, at €60.00 per half-year gives us

$$60\, a^{(2)}_{\overline{10}|} \;=\; 60\,\frac{1-\mathrm{v}^{10}}{i^{(2)}} \;=\; 60\,\frac{1-0.558395}{0.059126} \;=\; €448.13,$$

where $i^{(2)} = 2\left(1.06^{1/2}-1\right) = 0.059126$ with the half-year as the unit.

## Solution

Two times every half-year, for ten half-years, at €60.00 per half-year gives us

$$60\, a_{\overline{10}|}^{(2)} \;=\; 60\,\frac{1-\mathrm{v}^{10}}{i^{(2)}} \;=\; 60\,\frac{1-0.558395}{0.059126} \;=\; €448.13,$$

where $i^{(2)} = 2\left(1.06^{1/2}-1\right) = 0.059126$ with the half-year as the unit. In *years* the effective rate is $i = 1.06^2 - 1 = 0.1236$, the payments are quarterly so $m = 4$, the term is $n = 5$, and the rate of payment is €120.00 per annum,

$$i^{(4)} \;=\; 4\left(1.1236^{1/4}-1\right) \;=\; 0.118252, \qquad 120\, a_{\overline{5}|}^{(4)} \;=\; €448.13.$$

## Which unit to choose

KEY IDEA

Identical to the cent, of course, since we are valuing the same payments at the same date. Which unit we measure in matters as little as which ruler we measure with. We are still measuring the same thing, whether we use kg or lb.

So choose the unit that is, or seems, most convenient. It is usually not a bad idea to

- make the *given rate effective*, so that a rate quoted convertible half-yearly is worked in half-years, and
- keep the payment frequency $m$ a whole number, and the term $n$ with it.

Here the half-year satisfies both, which is why it was a clear choice.

# Lecture 8 on a slide

**$m$ PAYMENTS OF $\frac{1}{m}$ PER PERIOD**

$$a^{(m)}_{\overline{n}|} = \frac{1-\mathsf{v}^n}{i^{(m)}}, \qquad s^{(m)}_{\overline{n}|} = \frac{(1+i)^n-1}{i^{(m)}},$$

$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1-\mathsf{v}^n}{d^{(m)}}, \qquad \ddot{s}^{(m)}_{\overline{n}|} = \frac{(1+i)^n-1}{d^{(m)}}, \qquad \frac{i^{(m)}}{d^{(m)}} = (1+i)^{\frac{1}{m}}.$$

Arrears and advance are the same nominators divided by $i^{(m)}$ and $d^{(m)}$, respectively. This is Lecture 5's rule with "time running $m$ times faster" (smaller increments, but faster). With $d < d^{(m)} < \delta < i^{(m)} < i$, the five rates order the five annuities in reverse, and a perpetuity is worth one over its own rate.

# Exercises

### *EXERCISE 8.6*

*An annuity pays €3,000.00 at the end of each month for fifteen years at $i = 7\%$. Find its present value, and the value if the payments fell at the start of each month instead. Check the ratio of the two.*

### *EXERCISE 8.7*

*A mortgage of €200,000.00 is repaid by level monthly instalments in arrears over twenty-five years at $i = 5\%$. Find the monthly instalment.*

### *EXERCISE 8.8*

*A perpetuity pays €500.00 at the end of each quarter, at $i = 8\%$. Find the capital required, and again if the payments fell at the start of each quarter.*

### *EXERCISE 8.9*

*Show from the definitions that $d^{(m)} < i^{(m)}$ whenever $i > 0$, and say what that means for the two present values.*

Solutions are in the backup slides.

# END OF LECTURE 8

Every annuity so far has paid the same amount every time, and has started at once. Next, payments that wait, payments that grow and payments that shrink.

ilia.pirashvili@universityofgalway.ac.ie

## Solution 1

The total is €36,000.00 per annum, $m = 12$, $n = 15$, and $\mathrm{v}^{15} = 1.07^{-15} = 0.362446$. The two nominal rates are

$$i^{(12)} \;=\; 12\left(1.07^{1/12} - 1\right) \;=\; 0.067850, \qquad d^{(12)} \;=\; 12\left(1 - 1.07^{-1/12}\right) \;=\; 0.067468.$$

In arrears, $a^{(12)}_{\overline{15}|} = \dfrac{1 - 0.362446}{0.067850} = 9.396557$, giving $36{,}000 \times 9.396557 = €338{,}276.05$. In advance, $\ddot{a}^{(12)}_{\overline{15}|} = \dfrac{1 - 0.362446}{0.067468} = 9.449686$, giving €340,188.71.

The ratio is $340{,}188.71/338{,}276.05 = 1.005654 = 1.07^{1/12}$. It is one month's accumulation, as expected.

## Solution 2

Let $X$ be the *total* repaid per annum, so that each instalment is $X/12$. With $i = 0.05$, $n = 25$ and $m = 12$,

$$i^{(12)} \;=\; 12\left(1.05^{1/12} - 1\right) \;=\; 0.048889, \qquad v^{25} \;=\; 1.05^{-25} \;=\; 0.295303,$$

$$200{,}000 \;=\; X\, a_{\overline{25}|}^{(12)} \;=\; X\,\frac{0.704697}{0.048889} \;=\; X \times 14.414086,$$

so $X = €\,13{,}875.32$ per annum and the monthly instalment is $X/12 = €\,1{,}156.28$.

Worth comparing, yearly in arrears gives $200{,}000/a_{\overline{25}|} = €\,14{,}190.49$ per annum, which over twenty-five years totals €354,762.29 against €346,882.91. The monthly borrower pays out less in total but is no better off, the money simply leaves sooner, and at time 0 both schedules are worth exactly €200,000.00. The ratio of the two is $i/i^{(12)} = 1.022715$.

## Solution 3

The total is €2,000.00 per annum. In arrears the capital is $2000\, a^{(4)}_{\overline{\infty}|} = 2000/i^{(4)}$, and with

$$i^{(4)} \;=\; 4\left(1.08^{1/4} - 1\right) \;=\; 0.077706, \qquad \frac{2000}{0.077706} \;=\; €25{,}737.98.$$

In advance it is $2000/d^{(4)}$, and with $d^{(4)} = 4\left(1 - 1.08^{-1/4}\right) = 0.076225$,

$$\frac{2000}{0.076225} \;=\; €26{,}237.98.$$

The difference is exactly €500.00, one extra quarterly payment, made today and needing no discounting, exactly as for the annual perpetuities of Lecture 6.

## Solution 4

Write both rates with $q = (1+i)^{1/m} > 1$, so that $w = q^{-1} < 1$:

$$i^{(m)} \;=\; m(q-1), \qquad d^{(m)} \;=\; m(1-w) \;=\; m\,\frac{q-1}{q} \;=\; \frac{i^{(m)}}{q}.$$

Since $i > 0$ gives $q > 1$, dividing by $q$ strictly decreases a positive number, so $d^{(m)} < i^{(m)}$.

Both present values have the same numerator $1 - \mathrm{v}^n > 0$, so the one with the smaller denominator is the larger: $a^{(m)}_{\overline{n}|} < \ddot{a}^{(m)}_{\overline{n}|}$. This is as it should be, as the in-advance annuity pays the same money earlier.