LECTURE 7

Annuities payable continuously

Cashflow without the clock

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Recall

$$a_{\overline{n}|} = \frac{1 - \mathsf{v}^n}{i}, \qquad \ddot{a}_{\overline{n}|} = \frac{1 - \mathsf{v}^n}{d},$$



One equal payment a period

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}, \qquad \ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}.$$

Interest goes with arrears, discount with advance. In both cases, every payment so far has landed on a regular interval. Today the clock stops mattering.

Today

1. From lump sums to a flow
2. The two symbols
3. What continuity is worth

Warm-up

EXERCISE 7.1

Payments are made continuously at a rate of €1,000.00 *per annum for five years, the force of interest being a constant* 5%*. Find the present value.*

Recall lecture 3: Remember $\int_0^T \mathrm{v}(t)\rho(t)\,\mathrm{d}t$?

EXERCISE 7.1

Payments are made continuously at a rate of €1,000.00 *per annum for five years, the force of interest being a constant* 5%*. Find the present value.*

The present value of a continuous cashflow is $\int_0^T \mathrm{v}(t)\rho(t)\,\mathrm{d}t$.
Here $T = 5$, $\rho \equiv 1000$ and $\mathrm{v}(t) = \mathrm{e}^{-0.05t}$ since the force is constant. We get

$$\int_0^5 1000\,\mathrm{e}^{-0.05t}\,\mathrm{d}t \;=\; 1000\left[\frac{\mathrm{e}^{-0.05t}}{-0.05}\right]_0^5 \;=\; 1000\,\frac{1-\mathrm{e}^{-0.25}}{0.05} \;=\; 1000\times 4.423984,$$

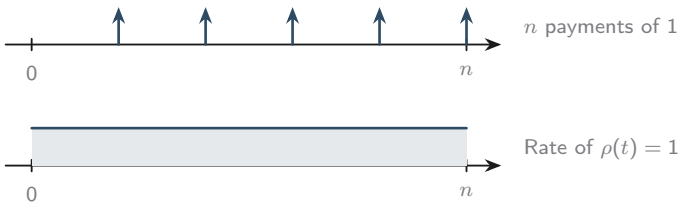
so the present value is €4,423.98.

SECTION 1

From lump sums to a flow

The set-up

Consider an annuity payable between time 0 and time n, with a continuous payment at the rate of 1 per unit time. Here n need not be an integer.



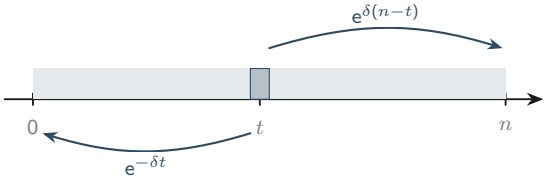
Both deliver a total of n over the total time n. The difference is *when*. The block hands the money over steadily, rather than in lumps at each year end.

Think of water from a tap, as opposed to arring in bottles.

Discounting and accumulating a single instant

Recall from Lecture 2 that, under a constant force of interest δ, a payment of 1 made at time $t \in [0, n]$

- discounts to $\mathrm{v}^t = \mathrm{e}^{-\delta t}$ at time 0;
- accumulates to $(1+i)^{n-t} = \mathrm{e}^{\delta(n-t)}$ at time n.



The shaded sliver is the payment $\rho(t)\,\mathrm{d}t = \mathrm{d}t$ received in the instant around t. Adding up the slivers *is* integrating. (That is why $\int$ looks like an S).

SECTION 2

The two symbols

Present value: The symbol $\bar{a}_{\overline{n}|}$

DEFINITION 7.2 · Continuously payable annuity

The present value of an annuity payable continuously at the rate of 1 per unit time for n units of time is written $\bar{a}_{\overline{n}|}$, read "a-bar-n". From Lecture 3 we know

$$\bar{a}_{\overline{n}|} = \int_0^n \mathrm{v}(t)\,\mathrm{d}t.$$

Note: This is a definition, not a formula. We made no assumptions on $\mathrm{v}(t)$. It is whatever it is and may vary with t in any way.

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This is $\int_0^T \mathrm{v}(t)\rho(t)\,\mathrm{d}t$ from Lecture 3 with $T = n$ and $\rho \equiv 1$.

Constant case: The closed form

Let the force of interest be a constant δ, so that $\mathrm{v}(t) = \mathrm{e}^{-\delta t}$. Then

$$\bar{a}_{\overline{n}|} \;=\; \int_0^n \mathrm{e}^{-\delta t}\,\mathrm{d}t \;=\; \left[\frac{\mathrm{e}^{-\delta t}}{-\delta}\right]_0^n \;=\; \frac{1-\mathrm{e}^{-\delta n}}{\delta} \;=\; \boxed{\frac{1-\mathrm{v}^n}{\delta}.}$$

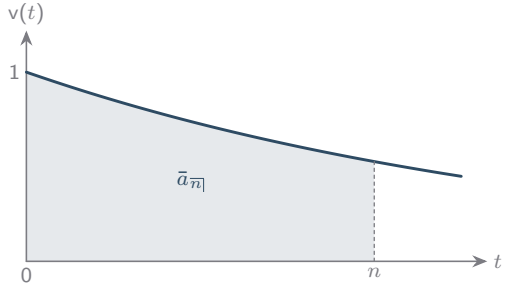
CAUTION

The denominator is δ, *not* i (which would be $\mathrm{e}^{\delta}-1$). Every annuity symbol in this module has the shape $\dfrac{1-\mathrm{v}^n}{\text{rate}}$. The rate, not interest, determines the payments.

And the warm-up at $\delta = 0.05$, gave us $\bar{a}_{\overline{5}|} = \frac{1-\mathrm{e}^{-0.25}}{0.05} = 4.423984$, the factor we computed by hand.

The present value, illustrated

Because the payment rate is 1, the present value *is* the area under the discount factor.



This illustration hows/hints at the following 1. The area lies inside the $1 \times n$ rectangle, so $\bar{a}_{\overline{n}|} < n$. 2. As n grows, the curve flattens onto the axis, so the area converges. This is perpetuity, discussed at the end of this lecture.

Accumulated value: The symbol $\bar{s}_{\overline{n}|}$

The accumulated value at the moment payments cease is $\bar{s}_{\overline{n}|}$. Each sliver $\mathrm{d}t$ at time t grows by $\mathrm{e}^{\delta(n-t)}$, so

$$\bar{s}_{\overline{n}|} = \int_0^n \mathrm{e}^{\delta(n-t)}\,\mathrm{d}t$$

Put $u = n - t$

Naming the exponent's remaining age $u := n - t$ will turn the integrand into a plain exponential.

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$$\bar{s}_{\overline{n}|} = \int_0^n \mathrm{e}^{\delta u}\, \mathrm{d}t$$

$$\mathrm{d}t = \frac{\mathrm{d}u}{\mathrm{d}t}$$

We need to know how u changes as t changes, so we differentiate the substitution.

Accumulated value: The symbol $\bar{s}_{\overline{n}|}$

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$$\bar{s}_{\overline{n}|} = \int_0^n \mathrm{e}^{\delta u}\, \mathrm{d}t$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}(n-t)}{\mathrm{d}t}$$

Apply the definition $u = n - t$.

Accumulated value: The symbol $\bar{s}_{\overline{n}|}$

The accumulated value at the moment payments cease is $\bar{s}_{\overline{n}|}$. Each sliver $\mathrm{d}t$ at time t grows by $\mathrm{e}^{\delta(n-t)}$, so

$$\bar{s}_{\overline{n}|} = \int_0^n \mathrm{e}^{\delta u}\,\mathrm{d}t$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}n - \mathrm{d}t}{\mathrm{d}t}$$

The differential of a difference is the difference of the differentials.

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$$\bar{s}_{\overline{n}|} = \int_0^n \mathrm{e}^{\delta u}\,\mathrm{d}t$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}n}{\mathrm{d}t} - \frac{\mathrm{d}t}{\mathrm{d}t}$$

Split the single fraction into two.

Accumulated value: The symbol $\bar{s}_{\overline{n}|}$

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$$\bar{s}_{\overline{n}|} = \int_0^n \mathrm{e}^{\delta u}\,\mathrm{d}t$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 0 - 1 = -1$$

n is a fixed constant – it does not change as t does – so $\mathrm{d}n/\,\mathrm{d}t = 0$; and t changes at exactly unit rate with respect to itself, so $\mathrm{d}t/\,\mathrm{d}t = 1$.

Accumulated value: The symbol $\bar{s}_{\overline{n}|}$

The accumulated value at the moment payments cease is $\bar{s}_{\overline{n}|}$. Each sliver $\mathrm{d}t$ at time t grows by $\mathrm{e}^{\delta(n-t)}$, so

$$\bar{s}_{\overline{n}|} = \int_0^n \mathrm{e}^{\delta u}\, \mathrm{d}t$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} \longleftarrow \mathrm{d}u = -\,\mathrm{d}t$$

So $\mathrm{d}u/\,\mathrm{d}t = -1$.Multiplying both sides by $\mathrm{d}t$: $\mathrm{d}u = -\,\mathrm{d}t$. This is what we substitute for $\mathrm{d}t$ above.

Accumulated value: The symbol $\bar{s}_{\overline{n}|}$

The accumulated value at the moment payments cease is $\bar{s}_{\overline{n}|}$. Each sliver $\mathrm{d}t$ at time t grows by $\mathrm{e}^{\delta(n-t)}$, so

$$\bar{s}_{\overline{n}|} = \int_{0}^{n} \mathrm{e}^{\delta u}\,(-\mathrm{d}u)$$

$t = 0 \Rightarrow u = n, \;\; t = n \Rightarrow u = 0$

The bounds are still written in t; as t runs from 0 to n, u runs from n down to 0.

Accumulated value: The symbol $\bar{s}_{\overline{n}|}$

The accumulated value at the moment payments cease is $\bar{s}_{\overline{n}|}$. Each sliver $\mathrm{d}t$ at time t grows by $\mathrm{e}^{\delta(n-t)}$, so

$$\bar{s}_{\overline{n}|} = -\int_{n}^{0} \mathrm{e}^{\delta u}\,\mathrm{d}u$$

Reversing the limits flips the sign

Substituting those u-bounds carries the minus sign from $\mathrm{d}t = -\,\mathrm{d}u$ out front.

Accumulated value: The symbol $\bar{s}_{\overline{n}|}$

The accumulated value at the moment payments cease is $\bar{s}_{\overline{n}|}$. Each sliver $\mathrm{d}t$ at time t grows by $\mathrm{e}^{\delta(n-t)}$, so

$$\bar{s}_{\overline{n}|} = \boxed{\int_0^n \mathrm{e}^{\delta u}\,\mathrm{d}u}$$

Reversing the limits of integration absorbs that minus sign, leaving a clean integral in u.

$$\bar{s}_{\overline{n}|} = \left[\frac{\mathrm{e}^{\delta u}}{\delta}\right]_0^n = \frac{\mathrm{e}^{\delta n}-1}{\delta} = \boxed{\frac{(1+i)^n-1}{\delta}}$$

Two dates, one annuity

Both symbols evaluate the same flow but at different times: $t = 0$ and $t = n$. The transfer principle relates them exactly as it did for $a_{\overline{n}|}$ and $s_{\overline{n}|}$:

$$\bar{a}_{\overline{n}|} \;=\; \mathrm{v}^n\, \bar{s}_{\overline{n}|}, \qquad \text{equivalently} \qquad \bar{s}_{\overline{n}|} \;=\; (1+i)^n\, \bar{a}_{\overline{n}|}.$$

Against the closed forms, $\mathrm{v}^n \dfrac{(1+i)^n - 1}{\delta} = \dfrac{1 - \mathrm{v}^n}{\delta}$, since $\mathrm{v}^n(1+i)^n = 1$.

> NOTHING HERE IS NEW
>
> Both symbols are Lecture 3's valuation formula with $\rho \equiv 1$. $\bar{s}_{\overline{n}|}$ is just $\bar{a}_{\overline{n}|}$ divided by $\mathrm{v}(n)$. It's the same cashflow, quoted at two dates.

A continuous flow for eight years

EXERCISE 7.3

An investor receives payments continuously at a rate of €2,000.00 *per annum for eight years, the force of interest being a constant* 4%. *Find the present value.*

A continuous flow for eight years

EXERCISE 7.3

An investor receives payments continuously at a rate of €2,000.00 *per annum for eight years, the force of interest being a constant* 4%. *Find the present value.*

The term is eight years at a constant force, so the factor is $\bar{a}_{\overline{8}|}$ at $\delta = 0.04$:

$$\bar{a}_{\overline{8}|} = \frac{1 - \mathrm{e}^{-0.32}}{0.04} = \frac{1 - 0.726149}{0.04} = 6.846274, \qquad 2000\,\bar{a}_{\overline{8}|} = €13{,}692.55.$$

ASIDE

Undiscounted, the flow would be worth €16,000.00, so discounting equated to about a seventh. This can be intuited by our illustrated on page 8: Area under v(t) opposed to the rectangle of height 1.

SECTION 3

What continuity is worth

Comparing with the annual annuity

As so often in our examples $a_{\overline{n}|} = \dfrac{1 - \mathsf{v}^n}{i}$ and $\bar{a}_{\overline{n}|} = \dfrac{1 - \mathsf{v}^n}{\delta}$ differ only by their denominator. We have

$$\bar{a}_{\overline{n}|} \;=\; \frac{i}{\delta}\, a_{\overline{n}|}, \qquad\qquad \bar{s}_{\overline{n}|} \;=\; \frac{i}{\delta}\, s_{\overline{n}|}.$$

KEY IDEA

From Lecture 2, $\delta < i$, so $i/\delta > 1$. Continuous payment is worth *more* than the same total paid in arrears (e.g., annually), as it arrives sooner (instantly).

At $i = 6\%$: $\delta = 0.058269$ and $i/\delta = 1.029709$, so continuous payment is worth about 3% more. The ratio does not depend on n, since the numerator cancels.

A salary

EXAMPLE 7.4

A firm pays a salary of €30,000.00 per annum, treated as payable continuously, for five years. At $i = 4\%$, find the present value of the salary and its accumulated value at the end.

Here $\delta = \ln 1.04 = 0.039221$ and $v^5 = 1.04^{-5} = 0.821927$, so

$$\bar{a}_{\overline{5}|} = \frac{1 - 0.821927}{0.039221} = 4.540277, \qquad \bar{s}_{\overline{5}|} = \frac{1.04^5 - 1}{0.039221} = 5.523941,$$

giving $30{,}000\,\bar{a}_{\overline{5}|} = €136{,}208.30$ and $30{,}000\,\bar{s}_{\overline{5}|} = €165{,}718.23$.

Paid annually in arrears instead, $a_{\overline{5}|} = 4.451822$ and the present value would be only €133,554.67. The same €150,000.00 of salary, worth €2,653.00 less for arriving later.

A flow that never stops

Let the payments run for ever. If $i > 0$ then $\delta > 0$, so $\mathrm{v}^n = \mathrm{e}^{-\delta n} \to 0$ and

$$\bar{a}_{\overline{\infty}|} = \lim_{n\to\infty} \frac{1-\mathrm{v}^n}{\delta} = \frac{1}{\delta}.$$

The numerator goes to 1 but the denominator stays constant. This is exactly as for $a_{\overline{\infty}|} = 1/i$ and $\ddot{a}_{\overline{\infty}|} = 1/d$.

EXAMPLE 7.5

At $i = 5\%$, so $\delta = 0.048790$ and $d = 0.047619$, a perpetuity of 1 per annum is worth

$$\underbrace{20.0000}_{1/i\text{, in arrears}} < \underbrace{20.4959}_{1/\delta\text{, continuous}} < \underbrace{21.0000}_{1/d\text{, in advance}}.$$

The ordering is the ordering of the rates

The three perpetuities with one income of 1 per annum are ordered as follows:

LECTURE 2'S CHAIN OF INEQUALITIES:

$$d < \delta < i \qquad \Longrightarrow \qquad \frac{1}{i} < \frac{1}{\delta} < \frac{1}{d}.$$

Dividing by a smaller rate gives a larger value, so the annuities run the opposite way from the rates. Continuous payment sits between the two. Think "on average the money arrives in the middle of each year", later than advance and earlier than arrears.

The same ordering holds over a a finite time, since the symbols only differ by the denominator.

Lecture 7 on one slide

CONTINUOUS ANNUITY: THE RATE OF THE FLOW

$$\bar{a}_{\overline{n}|} = \int_0^n \mathsf{v}(t)\,\mathrm{d}t \overset{\delta \text{ const.}}{=} \frac{1-\mathsf{v}^n}{\delta}, \qquad \bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}, \qquad \bar{a}_{\overline{\infty}|} = \frac{1}{\delta}.$$

Level payment in arrears divides by i, in advance by d, continuously by δ. Since $d < \delta < i$, the continuous annuity sits between the other two, and $\bar{a}_{\overline{n}|} = \frac{i}{\delta} a_{\overline{n}|}$ prices the difference.

Exercises

EXERCISE 7.6

At $i = 8\%$ and $n = 12$, compute $\bar{a}_{\overline{12}|}$ and $\bar{s}_{\overline{12}|}$, and verify that $\bar{a}_{\overline{12}|} = \frac{i}{\delta} a_{\overline{12}|}$.

EXERCISE 7.7

An investor receives €1,200.00 per annum, payable continuously, for twenty years at $i = 7\%$. Find the present value, and compare it with the value of the same total paid annually in arrears.

EXERCISE 7.8

Show that $\bar{s}_{\overline{n}|} = (1+i)^n \bar{a}_{\overline{n}|}$, both from the closed forms and by a one-sentence argument about valuation dates.

Solutions are in the backup slides.

END OF LECTURE 7

A salary paid every instant is a fiction; a salary paid every month is not. Next: payments m times a period, and the two nominal rates that price them.

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Solution 1

First the force, then everything else: $\delta = \ln 1.08 = 0.076961$, and $1.08^{12} = 2.518170$, so $v^{12} = 0.397114$. Hence

$$\bar{a}_{\overline{12}|} = \frac{1 - 0.397114}{0.076961} = 7.833655, \qquad \bar{s}_{\overline{12}|} = \frac{2.518170 - 1}{0.076961} = 19.726476.$$

For the check, the annual annuity in arrears has the same numerator over i:

$$a_{\overline{12}|} = \frac{1 - 0.397114}{0.08} = 7.536078, \qquad \frac{i}{\delta} = \frac{0.08}{0.076961} = 1.039487,$$

and $1.039487 \times 7.536078 = 7.833655$, as claimed. Continuous payment is worth about 4% more here, against 3% at $i = 6\%$: the gap between i and δ widens as the rate rises.

Solution 2

With $\delta = \ln 1.07 = 0.067659$ and $v^{20} = 1.07^{-20} = 0.258419$,

$$\bar{a}_{\overline{20}|} = \frac{1 - 0.258419}{0.067659} = 10.960624, \qquad 1200\,\bar{a}_{\overline{20}|} = €13{,}152.75.$$

Paid annually in arrears, the same €1,200.00 a year is worth

$$a_{\overline{20}|} = \frac{1 - 0.258419}{0.07} = 10.594014, \qquad 1200\,a_{\overline{20}|} = €12{,}712.82,$$

lower by €439.93. The ratio of the two is $i/\delta = 0.07/0.067659 = 1.034601$. The continuous payment is worth about 3.5% more, and that figure would be the same for any term.

Solution 3

From the closed forms, using $(1+i)^n \mathsf{v}^n = 1$:

$$(1+i)^n \bar{a}_{\overline{n}|} \;=\; (1+i)^n \,\frac{1-\mathsf{v}^n}{\delta} \;=\; \frac{(1+i)^n - (1+i)^n \mathsf{v}^n}{\delta} \;=\; \frac{(1+i)^n - 1}{\delta} \;=\; \bar{s}_{\overline{n}|}.$$

Expressed in words, $\bar{a}_{\overline{n}|}$ and $\bar{s}_{\overline{n}|}$ value the *same* payment flow, at time 0 and at time n respectively. By the transfer principle a single cashflow has one value, quoted at a date, so the two answers can differ only by the accumulation factor between those dates, which over n periods is $(1+i)^n$.