LECTURE 6

The four symbols at work

Edgecases, perpetuities and loan repayments

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
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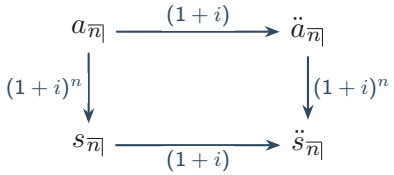
Four symbols, two multiplications



Horizontal: the payments move.
Vertical: the valuation date moves.

$$\frac{1-\mathsf{v}^n}{\square} \text{ at } 0, \qquad \frac{(1+i)^n-1}{\square} \text{ at } n,$$

The placeholder-square being i in arrears, and d in advance

Today we examine special cases of these formulas: No payments, no interest, one payment more, payments that never stop, and finaly examine it on an extended loan example.

Today

1. Two edge cases
2. One payment more
3. Payments that never stop
4. A loan repaid

Warm-up

EXERCISE 6.1

At $i = 6\%$, an annuity of 1 per year for ten years in arrears has present value $a_{\overline{10}|} = 7.360087$. Calculate $\ddot{a}_{\overline{10}|}$, $s_{\overline{10}|}$ and $\ddot{s}_{\overline{10}|}$.

EXERCISE 6.1

At $i = 6\%$, an annuity of 1 per year for ten years in arrears has present value $a_{\overline{10}|} = 7.360087$. Calculate $\ddot{a}_{\overline{10}|}$, $s_{\overline{10}|}$ and $\ddot{s}_{\overline{10}|}$.

Move the payments one period earlier, or move the valuation date ten periods later:

$$\ddot{a}_{\overline{10}|} = 1.06\, a_{\overline{10}|} = 7.801692, \qquad s_{\overline{10}|} = 1.06^{10}\, a_{\overline{10}|} = 13.180795,$$

$$\ddot{s}_{\overline{10}|} = 1.06\, s_{\overline{10}|} = 1.06^{10}\, \ddot{a}_{\overline{10}|} = 13.971643.$$

ASIDE

The last line is the square commuting: Across – then down, gives the same number as down – then across.

SECTION 1

Two edge cases

No payments, and no interest

- **No payments.** The annuity $s_{\overline{0}|} = a_{\overline{0}|} = \ddot{s}_{\overline{0}|} = \ddot{a}_{\overline{0}|} = 0$ with no payments is worth nothing. The closed forms agree, since $v^0 = 1$.
- **No interest.** If $i = 0$ then $s_{\overline{n}|} = a_{\overline{n}|} = \ddot{s}_{\overline{n}|} = \ddot{a}_{\overline{n}|} = n$, because every payment of 1 is worth 1 at every date. There are n of them, and it no longer matters when they fall.

The no-interest-case is the only non-trivial one in which the four symbols coincide, as interest is precisely what tells them apart.

Hold on: $i = 0$ needs a limit

At $i = 0$ the closed form reads $0/0$, which is not a number. The value n is a *limit*, and here is where it comes from.

BINOMIAL EXPANSION

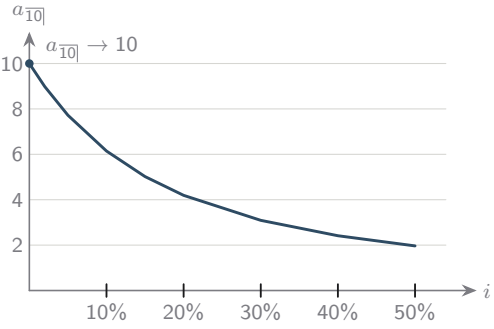
For small i, $(1+i)^{-n} = 1 - ni + \frac{n(n+1)}{2} i^2 - \cdots$.

$$a_{\overline{n}|} = \frac{\left(1 - ni + \frac{n(n+1)}{2} i^2 - \cdots\right) - 1}{i} = -n + \frac{n(n+1)}{2}\, i - \cdots \xrightarrow[i \to 0]{} -n.$$

CAUTION

Never evaluate a function where the denominator is zero, irrelevant if the nominator is also zero. Always use limits.

Sensible at both ends



Ten payments of 1, valued at time 0, as the rate varies.

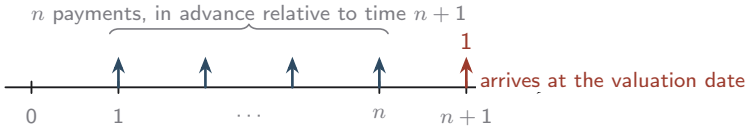
- As $i \to 0$ the curve rises to 10: nothing is discounted, so the annuity is worth its face total.
- As i grows, v $\to 0$ and the later payments become meaningless.

The value never leaves $(0, n]$.

SECTION 2

One payment more

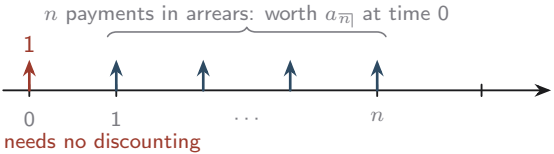
$s_{\overline{n+1}|} = 1 + \ddot{s}_{\overline{n}|}$



Value $n+1$ payments in arrears at time $n+1$. The last one earns nothing and contributes exactly 1. The other n sit at times $1, \ldots, n$. This is exactly an annuity of n payments in advance, valued n periods later. Hence

$$s_{\overline{n+1}|} \;=\; 1 + \ddot{s}_{\overline{n}|}.$$

$$\ddot{a}_{\overline{n+1}|} = 1 + a_{\overline{n}|}$$



The mirror image: $n+1$ payments in advance, valued at time 0. The first is paid immediately and contributes 1. The rest form an ordinary annuity in arrears. Both identities can be easily derived by their closed forms, but the picture above also gives a clear illustration why

$$\ddot{a}_{\overline{n+1}|} \;=\; 1 + a_{\overline{n}|}$$

holds.

SECTION 3

Payments that never stop

Infinitely many terms

By definition

$$\sum_{k=0}^{\infty} ar^k \;:=\; \lim_{n\to\infty} \sum_{k=0}^{n} ar^k,$$

and for $a \neq 0$ this exists *if and only if* $|r| < 1$, in which case $r^{n+1} \to 0$ giving us

$$\sum_{k=0}^{\infty} ar^k \;=\; \frac{a}{1-r}.$$



If $|r| \geq 1$ the terms do not shrink and the total is infinite, or oscillates ($r = -1$).

The main formual is that of a geometric series with infinitely many terms.

Perpetuities

An annuity where the payments never stop is called a **perpetuity**. If $i > 0$ then $\mathrm{v} < 1$, and so $\mathrm{v}^n \to 0$. Thus, the limits exist and is

$$a_{\overline{\infty}|} = \lim_{n\to\infty} \frac{1-\mathrm{v}^n}{i} = \frac{1}{i}, \qquad \ddot{a}_{\overline{\infty}|} = \lim_{n\to\infty} \frac{1-\mathrm{v}^n}{d} = \frac{1}{d}.$$

> CAUTION
>
> There is no $s_{\overline{\infty}|}$ or $\ddot{s}_{\overline{\infty}|}$. An accumulated value has to be accumulated *to* some date, and a perpetuity has none.

Note also $\frac{1}{d} = \frac{1+i}{i} = 1 + \frac{1}{i}$. That is, $\ddot{a}_{\overline{n+1}|} = 1 + a_{\overline{n}|}$ survives the limit.

Why $1/i$ is obvious in hindsight

KEY IDEA

Put $1/i$ in the bank at rate i. Each year it earns $i \times \frac{1}{i} = 1$ in interest. Withdraw exactly that 1 and the capital is untouched, ready to do the same next year, and for ever. So $1/i$ buys an income of 1 a year in perpetuity.



An infinite stream of payments has a finite value as eventually, each becomes almost worthless.

Mathematically: A constant stream of money is linear, a constant intrest rate is exponential.

Example

EXAMPLE 6.2

Determine the value of a perpetuity that pays €500.00 a year at $i = 5\%$ if the payments are made in (a) arrears, (b) advance.

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(a) $500\, a_{\overline{\infty}|} = \dfrac{500}{0.05} = €10{,}000.00.$

Example

EXAMPLE 6.2

Determine the value of a perpetuity that pays €500.00 a year at $i = 5\%$ if the payments are made in (a) arrears, (b) advance.

(b) Here $d = 0.05/1.05 = 0.047619$, so $1/d = 21$ and
$500\,\ddot{a}_{\overline{\infty}|} = 500 \times 21 = €10{,}500.00$.
Or simpler: $\ddot{a}_{\overline{\infty}|} = a_{\overline{\infty}|} + 1 \implies 500\ddot{a}_{\overline{\infty}|} = 500(a_{\overline{\infty}|} + 1)$.

Example

EXAMPLE 6.2

Determine the value of a perpetuity that pays €500.00 a year at $i = 5\%$ if the payments are made in (a) arrears, (b) advance.

(a): $a_{\overline{\infty}|} = 500 \cdot 1/i = €10{,}000.00$

(b): $\ddot{a}_{\overline{\infty}|} = 1/d = €10{,}500.00 = 1 + a_{\overline{\infty}|}$.

> ASIDE
>
> The difference is exactly €500.00, being one extra payment, made today and needing no discounting.

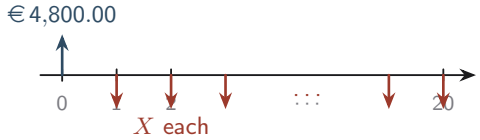
SECTION 4

A loan repaid

The question

EXAMPLE 6.3

A loan of €4,800.00 is to be repaid in 20 equal annual instalments at an effective rate of 10% per annum. Find the instalment if the payments are made in (a) arrears, (b) advance.



The basic idea is as follows: The lender hands over €4,800.00 at time 0 and receives the repayments, so at time 0 the two must have the same value.

(a) Payments in arrears

Let the annual repayment be X, with $i = 0.1$, so $\mathrm{v} = \dfrac{1}{1.1} = 0.909091$. Equating values at time 0 gives

$$4800 \;=\; X\, a_{\overline{20}|} \;=\; X\,\frac{1-\mathrm{v}^{20}}{i} \;=\; X\,\frac{1-0.148644}{0.1} \;=\; X \times 8.513564,$$

$$\Longrightarrow \quad X \;=\; \frac{4800}{8.513564} \;=\; €563.81.$$

> ASIDE
>
> Keep $\mathrm{v} = 0.909091$, not 0.9. Raised to the twentieth power, a rounding in the third decimal place moves the answer by about 18%: Here $0.9^{20} = 0.1216$ against $0.909^{20} = 0.1483$.

(b) Payments in advance, the short way

Let the repayment now be Y. The loan is the same, so both annuities must have the same value at time 0:

$$X\, a_{\overline{20}|} \;=\; 4800 \;=\; Y\, \ddot{a}_{\overline{20}|} \;=\; Y\,(1+i)\, a_{\overline{20}|}.$$

Cancelling $a_{\overline{20}|}$ yields

$$X \;=\; (1+i)\,Y \qquad \Longrightarrow \qquad Y \;=\; \mathrm{v}\,X \;=\; \frac{563.81}{1.1} \;=\; €512.55.$$

> KEY IDEA
>
> Paying in advance hands the lender every instalment a year earlier, so each instalment can be smaller by exactly one year's discount factor.

(b) The direct way, for comparison

We can also answer (b) by direct computation, but it is longer. With

$$d \;=\; \frac{i}{1+i} \;=\; \frac{0.1}{1.1} \;=\; 0.090909,$$

$$4800 \;=\; Y\,\ddot{a}_{\overline{20|}} \;=\; Y\,\frac{1-0.148644}{0.090909} \;=\; Y \times 9.364920 \quad\Longrightarrow\quad Y \;=\; €512.55.$$

By "direct" computation I mean using the formulat for $\ddot{a}_{\overline{20|}}$ without going through $a_{\overline{20|}}$, not by going back to first principle. The latter would be no fun.

What the borrower actually pays

	instalment	$\times 20$	interest paid
In arrears	563.81	11,276.20	6,476.20
In advance	512.55	10,251.00	5,451.00

- Over twenty years at 10% the borrower repays more than twice what was borrowed. Long loans are dominated by interest, not principal.
- The "in advance" schedule looks cheaper in total, but nothing has been gained (from a finance perspective). The money is simply handed over sooner. At time 0 both schedules are worth exactly €4,800.00.

Note, saying they are worth the same is only true if one "use" the money. This is, of course, not always the case, irl.

Lecture 6 on one slide

THE EDGE-CASES OF OUR SYMBOLS

$n = 0$: All four are 0; $i = 0$: All four are n; $n = \infty$: $a_{\overline{\infty}|} = \frac{1}{i}$; $\ddot{a}_{\overline{\infty}|} = \frac{1}{d}$.

$$s_{\overline{n+1}|} = 1 + \ddot{s}_{\overline{n}|}, \qquad \ddot{a}_{\overline{n+1}|} = 1 + a_{\overline{n}|}.$$

A payment landing on the valuation date is worth 1. At $i = 0$ the formulas read $0/0$ and must be replaced by limits. To price a loan, equate the advance with the repayments at one date.

Exercises

EXERCISE 6.4

A loan of €20,000.00 *is repaid by ten equal annual instalments at* $i = 7\%$*. Find the instalment if the payments are in arrears, and again if they are in advance.*

EXERCISE 6.5

Prove $s_{\overline{n+1}|} = 1 + \ddot{s}_{\overline{n}|}$ *from the closed forms, and say in one sentence what the* 1 *is.*

EXERCISE 6.6

A perpetuity pays €750.00 *a year at* $i = 4\%$*. Find its value in arrears and in advance, and explain why the two differ by exactly one payment.*

Solutions are in the backup slides.

END OF LECTURE 6

Every annuity so far has paid once a period. Salaries arrive monthly and dividends quarterly — next, what changes when the money moves more often than the clock ticks.

ilia.pirashvili@universityofgalway.ac.ie

Solution 1

Ten payments in arrears. With $1.07^{10} = 1.967151$ we have $\mathrm{v}^{10} = 0.508349$, so

$$a_{\overline{10}|} = \frac{1 - 0.508349}{0.07} = 7.023582, \qquad X = \frac{20{,}000}{7.023582} = €2{,}847.55.$$

In advance, the quickest route is the one from the lecture: Every instalment now falls a year earlier, so it may be smaller by one year's discount factor,

$$Y = \mathrm{v}\,X = \frac{2847.55}{1.07} = €2{,}661.26.$$

Directly, $\ddot{a}_{\overline{10}|} = 1.07 \times 7.023582 = 7.515232$ and $20{,}000/7.515232$ gives the same €2,661.26.

Solution 2

Write both symbols over the denominator i. Since $\ddot{s}_{\overline{n}|} = \dfrac{(1+i)^n - 1}{d}$ and $\dfrac{1}{d} = \dfrac{1+i}{i}$,

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)\left((1+i)^n - 1\right)}{i} = \frac{(1+i)^{n+1} - (1+i)}{i}.$$

Therefore

$$s_{\overline{n+1}|} - \ddot{s}_{\overline{n}|} = \frac{(1+i)^{n+1} - 1}{i} - \frac{(1+i)^{n+1} - (1+i)}{i} = \frac{(1+i) - 1}{i} = \frac{i}{i} = 1.$$

The 1 is the final payment. It lands on the valuation date, time $n+1$, and so earns no interest at all.

Solution 3

In arrears the value is

$$750\, a_{\overline{\infty}|} \;=\; \frac{750}{0.04} \;=\; €\,18{,}750.00.$$

In advance, $d = \dfrac{0.04}{1.04} = 0.038462$, so $\dfrac{1}{d} = 26$ and the value is $750 \times 26 = €\,19{,}500.00$.

The difference is €750.00, exactly one payment. The reason is that the two perpetuities are the same cashflow apart from a single payment made today: Shift the in-advance stream forward by one period and it becomes the in-arrears stream, having dropped only its time-0 payment, which needed no discounting and so contributed its full face value. In symbols, this is $\frac{1}{d} = 1 + \frac{1}{i}$.