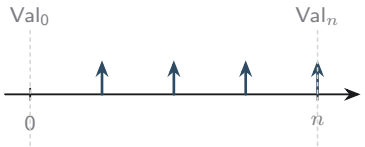
LECTURE 5

Level annuities

When every payment is the same size

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
15 September 2026 · Winter 2026

One value, quoted at a date



$$\mathrm{Val}_{t_1}(\mathcal{C})\,\mathrm{v}(t_1) \;=\; \mathrm{PV}(\mathcal{C})$$

One cashflow has one value;
the date only determines how its represented numerically.

We considered arbitrary payments at arbitrary times and aligned them all to the same to date to be able to compare.

Today we specialise: We consider payments of exactly the same size at regular intervals. This allows us to derive simpler formulas.

Today

1. Annuities: Constant and regular cashflows
2. The sum behind every annuity
3. Payments in arrears
4. Payments in advance

Warm-up

EXERCISE 5.1

Payments of 1 *are due at times* 1, 2 *and* 3, *and the effective rate of interest is* 6% *p. a. Find the value of this cashflow at times* 0 *and* 3.

Warm-up

EXERCISE 5.1

Payments of 1 *are due at times* 1, 2 *and* 3, *and the effective rate of interest is* 6% *p. a. Find the value of this cashflow at times* 0 *and* 3.

With $v = 1/1.06 = 0.943396$, the payments are dragged back by one, two and three years:

$$\mathrm{Val}_0(\mathcal{C}) \;=\; v + v^2 + v^3 \;=\; 0.943396 + 0.889996 + 0.839619 \;=\; 2.673012.$$

At $t = 3$, we push the first two payments forward by two and one years. Third stays:

$$\mathrm{Val}_3(\mathcal{C}) \;=\; 1.06^2 + 1.06 + 1 \;=\; 1.1236 + 1.06 + 1 \;=\; 3.1836.$$

EXERCISE 5.1

Payments of 1 *are due at times* 1, 2 *and* 3, *and the effective rate of interest is* 6% *p. a. Find the value of this cashflow at times* 0 *and* 3.

With $v = 1/1.06 = 0.943396$, the payments are dragged back by one, two and three years:

$$\mathrm{Val}_0(\mathcal{C}) \;=\; v + v^2 + v^3 \;=\; 0.943396 + 0.889996 + 0.839619 \;=\; 2.673012.$$

At $t = 3$, we push the first two payments forward by two and one years. Third stays:

$$\mathrm{Val}_3(\mathcal{C}) \;=\; 1.06^2 + 1.06 + 1 \;=\; 1.1236 + 1.06 + 1 \;=\; 3.1836.$$

> WE COULD HAVE ALSO PUSHED THE FIRST SOLUTION FORWARD
>
> The two agree, as they must: $2.673012 \times 1.06^3 = 2.673012 \times 1.191016 = 3.1836$.

Series like these are the content of our current lecture.

SECTION 1

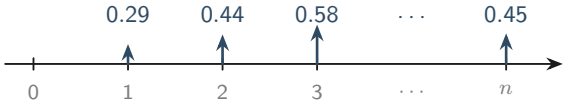
Annuities: Constant and regular cashflows

Annuities

DEFINITION 5.2 · Annuity

An **annuity** is a regular series of payments.

Insurance premiums, stock dividends, mortgage repayments, salaries, rents, pensions,...
Most cashflows people actually meet (irl) are annuities.



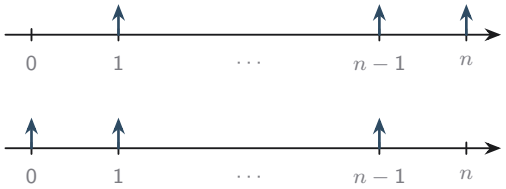
The vocabulary

An annuity is called

- **annuity certain**, if payable for a definite period of time;
- **level**, if each payment is of the same amount;
- **increasing** (**decreasing**), if the payments increase (decrease) by the same amount;
  - it's called **strictly** increasing (decreasing) if the inequalities are strict;
- paid **in arrears**, if paid at the end of each time period;
- paid **in advance**, if paid at the beginning;
  - also called **annuity due**. It means the first payment is made at time $t = 0$;
- **immediate**, if the first payment is made at time $t = 1$;
- **deferred**, if the first payment is made at time $t \geq 2$.

Everything today is a *level annuity certain*; either in *arrears* or in *advance*.

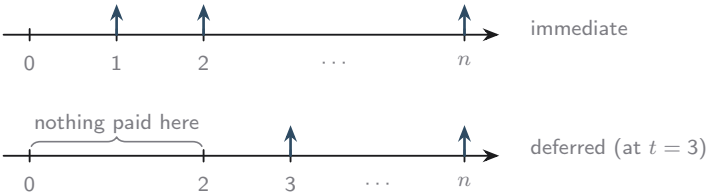
"In arrears" vs. "in advance"



Same number of payments, and of the same size, but shifted by one period. This changes their total values.

This is another perspective to what we did in previous lectures: Back then, the total value stayed constant, but numerical representations of the transactions differed. Now, the numerical transactions are constant, but the total values are different.

Immediate versus deferred



BEHIND THE JARGON

Immediate does *not* mean "paid now". An annuity-immediate in arrears pays nothing at time 0; its first payment is at time 1. The word promises only that payments start in the first period. The one that actually pays *now* is the annuity *due*.

SECTION 2

The sum behind every annuity

Why a geometric progression appears

A payment of 1 at time t is worth v^t today. So the present values of the successive payments of an annuity are

$$v,\ v^2,\ v^3,\ \ldots$$

each one v times the one before. Valuing an annuity therefore means summing a *geometric progression*, so it is worth doing that once, carefully.

DEFINITION 5.3 · Geometric progression

A **geometric progression** is a sequence a, ar, ar^2, ... for real numbers a and r. The number r is the **common ratio**. Throughout we assume $r \neq 1$.

An **arithmetic progression** is a sequence of the form $an + b = \{a, a + b, a + 2b, a + 3b, \ldots\}$. Among other things, they are realated to beautiful pieces of maths, such as *Dirichlet's theorem* on arithmetic sequences, which says that if $(a, b) = 1$ are coprime, there are infinitely many primes in such a sequence.

Shift and sum it

Write S for the sum we want to evaluate, and its one-instant shift rS.

$$\begin{aligned} S &= a + ar + ar^2 + \cdots + ar^n \\ rS &= \qquad ar + ar^2 + \cdots + ar^n + ar^{n+1} \end{aligned}$$

Shift and sum it

Write S for the sum we want to evaluate, and its one-instant shift rS.

$$\begin{aligned} S &= a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^n} \\ rS &= \quad \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^n} + ar^{n+1} \end{aligned}$$

Since each summand of S is a shift of its neighbour, shifting all of S causes every part to be identical, save for the *first* and *last*.

Shift and sum it

Write S for the sum we want to evaluate, and its one-instant shift rS.

$$\begin{aligned} S &= a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^n} \\ rS &= \quad \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^n} + ar^{n+1} \end{aligned}$$

Subtracting rS from S thus cancels everything except the two extremital terms.

$$S - rS = a - ar^{n+1} \qquad \Longrightarrow \qquad S(1-r) = a\left(1 - r^{n+1}\right).$$

The final formula

Dividing by $1 - r$ (we assume $r \neq 1$) gives us

$$S = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n = \frac{a\left(1 - r^{n+1}\right)}{1 - r}.$$

ALWAYS ASK: **HOW MANY PAYMENTS?**

That sum has $n + 1$ terms, not n. The exponents run $0, 1, \ldots, n$. Miscounting terms is the commonest error in annuity questions.

Started one term later instead, the same argument gives $\sum_{k=1}^{n} ar^k = \dfrac{ar\left(1 - r^n\right)}{1 - r}$, with n terms.

SECTION 3

Payments in arrears

Two dates, two symbols

Take n payments of 1 in arrears, at times $1, 2, \ldots, n$, at an effective rate $i \neq 0$ per period. There are two canonical valuation dates and we will use special symbols for each:

- The **accumulated value** $s_{\overline{n}|}$ is evaluated at the time of the *last* payment $t = n$;
- The **present value** $a_{\overline{n}|}$ is evaluated one period *before the first* payment $t = 0$.

> READING THE SYMBOLS
>
> The angle carries the number of payments, not the number of years to anything. "Dots" or an "umlaut" will be added on top later, to mean "in advance".

The "umlaut" is really just two dots and can be reffered to as such, as opposed to a bar, which we will introduce further down the line as well.

Accumulating to time n

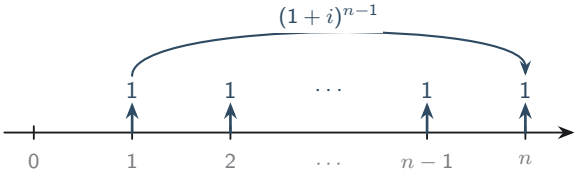
Each payment is pushed forward to $t = n$ by the number of whole periods it has left to run.



The payment at time 1 has $n - 1$ periods left, the one at time 2 has $n - 2$, and so on.

Accumulating to time n

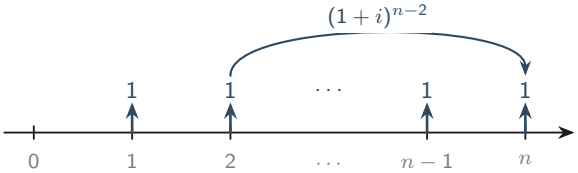
Each payment is pushed forward to $t = n$ by the number of whole periods it has left to run.



Each arrow multiplies its payment by an accumulation factor.

Accumulating to time n

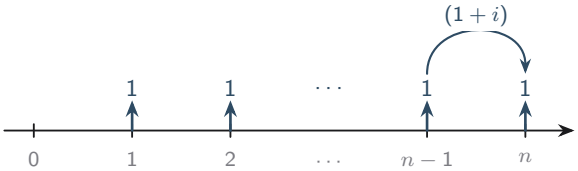
Each payment is pushed forward to $t = n$ by the number of whole periods it has left to run.



Each arrow multiplies its payment by an accumulation factor.

Accumulating to time n

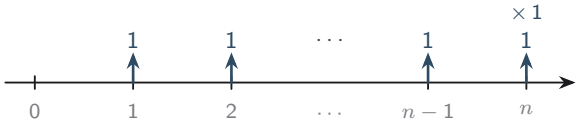
Each payment is pushed forward to $t = n$ by the number of whole periods it has left to run.



Each arrow multiplies its payment by an accumulation factor.

Accumulating to time n

Each payment is pushed forward to $t = n$ by the number of whole periods it has left to run.



The last payment arrives exactly at the valuation date, so it is multiplied by 1, earning no interest at all.

The closed form for $s_{\overline{n}|}$

Adding the accumulated payments, largest exponent first, we get

$$s_{\overline{n}|} \;=\; (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1.$$

Read backwards this is a geometric progression with $a = 1$, $r = 1+i$, and n terms. Thus

$$s_{\overline{n}|} \;=\; \frac{1-(1+i)^n}{1-(1+i)} \;=\; \frac{1-(1+i)^n}{-i} \;=\; \boxed{\frac{(1+i)^n - 1}{i}}$$

EXAMPLE 5.4

The warm-up example had $n = 3$ and $i = 6\%$, so $s_{\overline{3}|} = \dfrac{1.06^3 - 1}{0.06} = \dfrac{0.191016}{0.06} = 3.1836.$

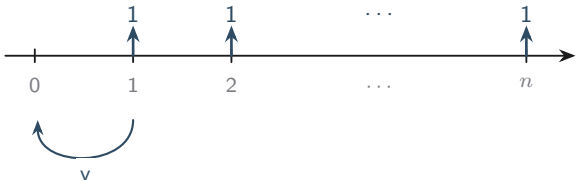
Much easier with a formula even for $n = 3$. Now think if $n = 300$.

Discounting to time 0



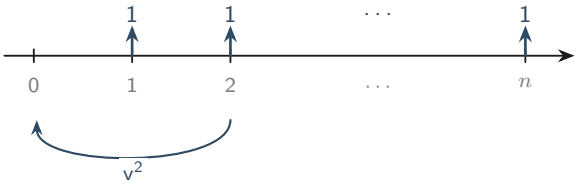
Now every payment is dragged back to time 0 instead, the payment at time t by its own factor v^t.

Discounting to time 0



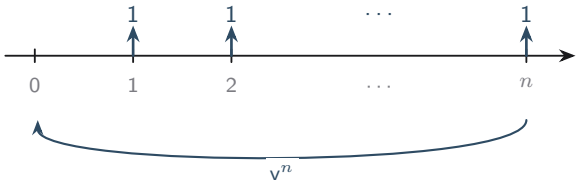
Now every payment is dragged back to time 0 instead, the payment at time t by its own factor v^t.

Discounting to time 0



Now every payment is dragged back to time 0 instead, the payment at time t by its own factor v^t.

Discounting to time 0



Summing the discounted payments gives us

$$a_{\overline{n}|} = \mathsf{v} + \mathsf{v}^2 + \cdots + \mathsf{v}^n.$$

The closed form for $a_{\overline{n}|}$

A geometric progression with first term v, factor v and n terms, we have

$$a_{\overline{n}|} = \frac{\mathsf{v}\,(1-\mathsf{v}^n)}{1-\mathsf{v}}.$$

The closed form for $a_{\overline{n}|}$

A geometric progression with first term v, factor v and n terms, we have

$$a_{\overline{n}|} = \frac{\mathsf{v}\left(1-\mathsf{v}^n\right)}{1-\mathsf{v}}.$$

Now recall two facts from Lecture 1: $1-\mathsf{v}=d$ and $d=i\mathsf{v}$. Hence $\dfrac{\mathsf{v}}{1-\mathsf{v}}=\dfrac{\mathsf{v}}{i\mathsf{v}}=\dfrac{1}{i}$, and

$$\boxed{a_{\overline{n}|} = \frac{1-\mathsf{v}^n}{i}}$$

The closed form for $a_{\overline{n}|}$

A geometric progression with first term v, factor v and n terms, we have

$$a_{\overline{n}|} = \frac{\mathsf{v}\left(1-\mathsf{v}^n\right)}{1-\mathsf{v}}.$$

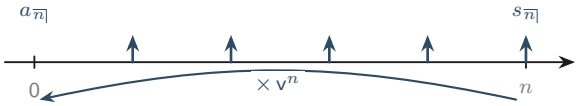
Now recall two facts from Lecture 1: $1-\mathsf{v}=d$ and $d=i\mathsf{v}$. Hence $\dfrac{\mathsf{v}}{1-\mathsf{v}}=\dfrac{\mathsf{v}}{i\mathsf{v}}=\dfrac{1}{i}$, and

$$\boxed{a_{\overline{n}|} = \frac{1-\mathsf{v}^n}{i}}$$

EXAMPLE 5.5

Again with $n=3$, $i=6\%$: $\mathsf{v}^3=0.839619$, so $a_{\overline{3}|}=\dfrac{1-0.839619}{0.06}=2.673012$.

One annuity, valued twice



The two symbols represent the same value at different dates, so the transfer principle relates them as

$$a_{\overline{n}|} \;=\; v^n\, s_{\overline{n}|}, \qquad \text{or equivalently} \qquad s_{\overline{n}|} \;=\; (1+i)^n\, a_{\overline{n}|}.$$

The same statement in Lecture 4's notation

Observe that these are just Lecture 4's formulas with every c_t set to 1.

KEY IDEA

With $t \in \{1, 2, \dots, n\}$, $c_t = 1$ and $\mathsf{v}(t) = \mathsf{v}^t = (1+i)^{-t}$,

$$a_{\overline{n}|} \;=\; \mathsf{Val}_0(\mathcal{C}) \;=\; \sum_t c_t\, \mathsf{v}(t), \qquad s_{\overline{n}|} \;=\; \mathsf{Val}_n(\mathcal{C}) \;=\; \sum_t c_t\, \frac{\mathsf{v}(t)}{\mathsf{v}(n)}.$$

So the transfer principle $\mathsf{Val}_n(\mathcal{C})\mathsf{v}(n) = \mathsf{Val}_0(\mathcal{C})$ is exactly $a_{\overline{n}|} = \mathsf{v}^n s_{\overline{n}|}$.

A twelve-year annuity

EXERCISE 5.6

An annuity pays €400.00 *at the end of each year for twelve years, at* $i = 5\%$. *Find its present value.*

A twelve-year annuity

EXERCISE 5.6

An annuity pays €400.00 at the end of each year for twelve years, at $i = 5\%$. Find its present value.

Twelve payments in arrears, so the symbol is $a_{\overline{12}|}$. Here $v^{12} = 1.05^{-12} = 0.556837$, and

$$a_{\overline{12}|} = \frac{1 - 0.556837}{0.05} = 8.863252, \qquad 400\, a_{\overline{12}|} = 400 \cdot 8.863252 = €3{,}545.30.$$

ASIDE

Undiscounted, the payments come to €4,800.00. $a_{\overline{12}|} = 8.863$ says that twelve payments of 1 at this rate i are worth 8.863 payments made today.

SECTION 4

Payments in advance

Two more symbols

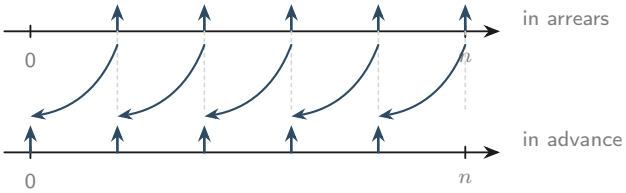
Now the n payments of 1 are made *in advance*, at times $0, 1, \ldots, n-1$. We use the same symbols, but with an umlaut/dots:

- $\ddot{s}_{\overline{n}|}$, the accumulated value one period *after the last* payment, meaning at time n;
- $\ddot{a}_{\overline{n}|}$, the present value *at the time of the first* payment, meaning at time 0.

KEY IDEA

All four symbols value n transactions of size 1, at $t = 0$ or $t = n$. The dots only says when the *payments* fall: ($\{1, 2, \ldots, n\}$ vs $\{0, 1, \ldots, n-1\}$), *not* the value is taken.

Everything one period earlier



Each payment happens one period sooner, so at any fixed valuation date, each is worth $(1+i)$ times as much:

$$\ddot{a}_{\overline{n}|} \;=\; (1+i)\, a_{\overline{n}|}, \qquad \ddot{s}_{\overline{n}|} \;=\; (1+i)\, s_{\overline{n}|}.$$

Closed forms

Substituting, and using $\dfrac{1+i}{i} = \dfrac{1}{d}$ because $d = \dfrac{i}{1+i}$,

$$\ddot{a}_{\overline{n}|} = (1+i)\frac{1-\mathsf{v}^n}{i} = \boxed{\frac{1-\mathsf{v}^n}{d}} \qquad \ddot{s}_{\overline{n}|} = (1+i)\frac{(1+i)^n-1}{i} = \boxed{\frac{(1+i)^n-1}{d}}$$

Closed forms

Substituting, and using $\dfrac{1+i}{i} = \dfrac{1}{d}$ because $d = \dfrac{i}{1+i}$,

$$\ddot{a}_{\overline{n}|} \;=\; (1+i)\frac{1-\mathsf{v}^n}{i} \;=\; \boxed{\frac{1-\mathsf{v}^n}{d}} \qquad \ddot{s}_{\overline{n}|} \;=\; (1+i)\frac{(1+i)^n-1}{i} \;=\; \boxed{\frac{(1+i)^n-1}{d}}$$

Or sum them directly, which is just as short. The payments now fall at $0, 1, \ldots, n-1$. Since $1 - \mathsf{v} = d$, we get

$$\ddot{a}_{\overline{n}|} \;=\; 1 + \mathsf{v} + \cdots + \mathsf{v}^{n-1} \;=\; \frac{1-\mathsf{v}^n}{1-\mathsf{v}} \;=\; \frac{1-\mathsf{v}^n}{d}.$$

The same n terms as before, each one period earlier. The only change is denominator going from i to d.

Six years of rent

EXERCISE 5.7

A tenant pays €900.00 *at the start of each year for six years, at* $i = 5\%$. *Find the present value of the rent.*

Six years of rent

EXERCISE 5.7

A tenant pays €900.00 at the start of each year for six years, at $i = 5\%$. Find the present value of the rent.

Six payments in advance, so the symbol is $\ddot{a}_{\overline{6}|}$. Here $v^6 = 0.746215$ and $d = 0.047619$, so

$$\ddot{a}_{\overline{6}|} = \frac{1 - 0.746215}{0.047619} = 5.329477, \qquad 900\,\ddot{a}_{\overline{6}|} = €4{,}796.53.$$

ASIDE

Paid in arrears the same rent is worth $900\,a_{\overline{6}|} = €4{,}568.12$. This is lower by the factor 1.05, one year's interest on the whole annuity.

All four, on one set of numbers

At $i = 6\%$ and $n = 10$, so that $v^{10} = 0.558395$ and $d = 0.056604$, we have

	payments at	valued at	value
$a_{\overline{10}\rceil}$	$1, \ldots, 10$	0	7.360087
$\ddot{a}_{\overline{10}\rceil}$	$0, \ldots, 9$	0	7.801692
$s_{\overline{10}\rceil}$	$1, \ldots, 10$	10	13.180795
$\ddot{s}_{\overline{10}\rceil}$	$0, \ldots, 9$	10	13.971643

Each "due" value is 1.06 times the one above it.

Lecture 5 on one slide

FOUR SYMBOLS, ONE GEOMETRIC SERIES

$$a_{\overline{n}|} = \frac{1-\mathsf{v}^n}{i}, \qquad s_{\overline{n}|} = \frac{(1+i)^n-1}{i},$$

$$\ddot{a}_{\overline{n}|} = \frac{1-\mathsf{v}^n}{d}, \qquad \ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d}.$$

Exercises

EXERCISE 5.8

An annuity pays €2,000.00 at the end of each year for fifteen years, at $i = 4\%$. Find its present value and its accumulated value at the end, and check the two against each other.

EXERCISE 5.9

Payments of 1 are made at times $0, 1, \ldots, n$. How many payments are there? Write the value at time 0 as a single "due" symbol, and again as 1 plus an in-arrears symbol; check they agree at $i = 6\%$, $n = 10$.

EXERCISE 5.10

Prove $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$ from the closed forms, and say what the factor $(1+i)$ is doing.

Solutions are in the backup slides.

END OF LECTURE 5

The formulas are built; next they are put to work — annuities that never stop, annuities that start late, and the loan whose repayment they fix.

ilia.pirashvili@universityofgalway.ac.ie

Solution 1

Fifteen payments in arrears, so the symbols are $a_{\overline{15}|}$ and $s_{\overline{15}|}$. With $1.04^{15} = 1.800944$ we have $v^{15} = 0.555265$, and

$$a_{\overline{15}|} = \frac{1 - 0.555265}{0.04} = 11.118387, \qquad s_{\overline{15}|} = \frac{1.800944 - 1}{0.04} = 20.023588.$$

Hence the present value is $2000 \times 11.118387 = €\,22{,}236.77$ and the accumulated value is $2000 \times 20.023588 = €\,40{,}047.18$.

The check is the transfer principle: the two answers value the same payments fifteen years apart, so they must differ by 1.04^{15}, and indeed $22{,}236.77 \times 1.800944 = €\,40{,}047.18$.

Solution 2

The times $0, 1, \ldots, n$ are $n+1$ payments, not n. Recall the "counting trap" of Section 2.

Read as an annuity-due, the payments fall at $0, 1, \ldots, (n+1)-1$, which is $\ddot{a}_{\overline{n+1}|}$. Read the other way, the payment at time 0 needs no discounting at all and contributes exactly 1, while the remaining payments at $1, \ldots, n$ are an ordinary annuity in arrears worth $a_{\overline{n}|}$. So

$$\ddot{a}_{\overline{n+1}|} \;=\; 1 + a_{\overline{n}|}.$$

At $i = 6\%$, $n = 10$: $v^{11} = 0.526788$ and $d = 0.056604$, so

$$\ddot{a}_{\overline{11}|} \;=\; \frac{1 - 0.526788}{0.056604} \;=\; 8.360087, \qquad 1 + a_{\overline{10}|} \;=\; 1 + 7.360087 \;=\; 8.360087.$$

Solution 3

From the closed forms the two symbols differ only in their denominator:

$$\frac{\ddot{a}_{\overline{n}|}}{a_{\overline{n}|}} = \frac{(1-\mathrm{v}^n)/d}{(1-\mathrm{v}^n)/i} = \frac{i}{d}.$$

Since $d = \dfrac{i}{1+i}$, the ratio $\dfrac{i}{d} = 1+i$, and therefore $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$.

The factor $(1+i)$ is one period's accumulation: the two annuities have the same n payments and the same valuation date, but every payment of the annuity-due falls one period earlier, and so has one extra period in which to earn interest.