LECTURE 4

Values at any date

One cashflow, one value, quoted where you like

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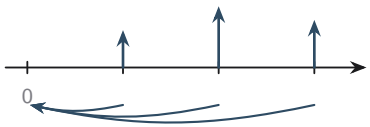
Everything, brought back to one date



each by its own $\mathsf{v}(t)$

$$\mathsf{PV}(\mathcal{C}) \;=\; \sum_t c_t\,\mathsf{v}(t) \;+\; \int_0^T \mathsf{v}(t)\,\rho(t)\,\mathrm{d}t$$

Lumps summed against $\mathsf{v}(t)$,
Flows integrated against it

"Today" (time 0) has been the preferred date throughout. This lecture considers other dates and we will observe that everything works as expected.

Today

1. The value of a payment at any date
2. The value of a cashflow at any date
3. A worked example

Warm-up

EXERCISE 4.1

An investor receives €1,000.00 *at time* 2 *and a continuous stream of* €300.00 *per annum between times* 0 *and* 4*. The force of interest is a constant* 4%*. Find the present value.*

EXERCISE 4.1

An investor receives €1,000.00 *at time* 2 *and a continuous stream of* €300.00 *per annum between times* 0 *and* 4. *The force of interest is a constant* 4%. *Find the present value.*

The force is constant, so $\mathrm{v}(t) = \mathrm{e}^{-0.04t}$. Evaluate the two parts separately and add:

$$\mathrm{PV}(\mathcal{C}) = \underbrace{1000\,\mathrm{e}^{-0.08}}_{\text{the lump}} + \underbrace{\int_0^4 300\,\mathrm{e}^{-0.04t}\,\mathrm{d}t}_{\text{the flow}} = 923.12 + 300 \cdot \frac{1-\mathrm{e}^{-0.16}}{0.04} = €2{,}032.04.$$

The stream pays €1,200.00 in total and the lump €1,000.00, so €2,200.00 of money is worth €2,032.04 today.

SECTION 1

The value of a payment at any date

Why time 0 is not always the interesting date

Present value answers "what is this worth now?". A very sensible question, but there are other interesting ones:

- What will this pension pot be worth at retirement?
- What will fixed the payments be worth in 5 years from now?
- Two offers, with payments on different dates, which is better?

So we need the value of a payment at a general time t_1, where the payment may fall *after* t_1 or *before* it.

Of course, we can always solve for $t = 0$ and pushforward or back, which is basically our plan.

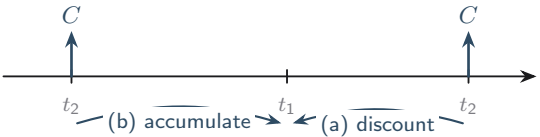
Value at time t_1 of a single payment

DEFINITION 4.2 · Value at time t_1

Consider times t_1 and t_2, (where $t_1 < t_2$, $t_1 > t_2$ and $t_1 = t_2$ are all allowed). The value at time t_1 of the sum C due at time t_2 is

(a) the discounted value at time t_1 of C due at time t_2, if $t_1 < t_2$;

(b) the accumulated value (interest) of C from time t_2 until time t_1, if $t_1 \geq t_2$.



One formula covers both cases

In both cases the value at time t_1 of C due at time t_2 is

$$\boxed{C\,\mathrm{e}^{-\int_{t_1}^{t_2}\delta(t)\,\mathrm{d}t}.}$$

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It works for $t_1 > t_2$ and $t_1 \leq t_2$ at the same time since reversing signs reverses limits:

$$\int_{t_1}^{t_2}\delta(t)\,\mathrm{d}t \;=\; -\int_{t_2}^{t_1}\delta(t)\,\mathrm{d}t \qquad\Longrightarrow\qquad C\,\mathrm{e}^{-\int_{t_1}^{t_2}\delta(t)\,\mathrm{d}t} \;=\; C\,\mathrm{e}^{\int_{t_2}^{t_1}\delta(t)\,\mathrm{d}t}.$$

EXAMPLE 4.3

For $\delta = 0.06$, the value of 100 at time 2, due at time 5, is $100\mathrm{e}^{0.06\times(2-5)} = €83.53$. The value of 100 at time 5, due at time 2, is $100\mathrm{e}^{0.18} = €119.72$.

Rewriting it with discount factors

Integrals over adjacent intervals add, so for any t_1, t_2

$$\int_{t_1}^{t_2} \delta(t)\,\mathrm{d}t \;=\; \int_0^{t_2} \delta(t)\,\mathrm{d}t - \int_0^{t_1} \delta(t)\,\mathrm{d}t,$$

and therefore, using $\mathrm{e}^{x-y} = \mathrm{e}^x/\mathrm{e}^y$, the value at t_1 of C due at t_2 is

$$C\,\mathrm{e}^{-\int_{t_1}^{t_2}\delta(t)\,\mathrm{d}t} \;=\; C \times \frac{\mathrm{e}^{-\int_0^{t_2}\delta(t)\,\mathrm{d}t}}{\mathrm{e}^{-\int_0^{t_1}\delta(t)\,\mathrm{d}t}} \;=\; C\,\frac{\mathsf{v}(t_2)}{\mathsf{v}(t_1)}.$$

This reflects our small comment on Slide 4. What we did is the same as calculating at time 0 and moving forward to time t_1, but done in one go.

A payment moved under a varying force

EXERCISE 4.4

Let $\delta(t) = 0.04 + 0.002t$ *p.a. be the force of interest. Find the value at time* 3 *of* €2,000.00*, due at time* 6*.*

A payment moved under a varying force

EXERCISE 4.4

Let $\delta(t) = 0.04 + 0.002t$ *p.a. be the force of interest. Find the value at time* 3 *of* €2,000.00*, due at time* 6*.*

Only the force between $t = 3$ and $t = 6$ matters, so we integrate in that range:

$$\int_3^6 (0.04 + 0.002t)\,\mathrm{d}t \;=\; \Big[0.04t + 0.001t^2\Big]_3^6 \;=\; 0.276 - 0.129 \;=\; 0.147,$$

so the value is $2000\,\mathrm{e}^{-0.147} = €1{,}726.59$.

The route through time 0 gives $2000\,\mathrm{v}(6)/\mathrm{v}(3)$, whose exponents are those same 0.276 and 0.129. Subtracting them is what the direct integral did in one step.

SECTION 2

The value of a cashflow at any date

Applying the factor to every payment

Values add, so we may apply the factor to the whole cashflow at once.

VALUE OF A CASHFLOW AT TIME t_1

Write $\mathrm{Val}_{t_1}(\mathcal{C})$ for the value at time t_1 of the cashflow $\mathcal{C}$. Then

$$\mathrm{Val}_{t_1}(\mathcal{C}) \;=\; \sum_{t} c_t \,\frac{\mathsf{v}(t)}{\mathsf{v}(t_1)} \;+\; \int_{-\infty}^{\infty} \rho(t)\,\frac{\mathsf{v}(t)}{\mathsf{v}(t_1)}\;\mathrm{d}t,$$

the sum being over those t for which $c_t \neq 0$.

The limits $\pm\infty$ are deliberate: From a valuation date t_1 there may be payments on *both* sides, and the formula handles it all at once.

The special case $t_1 = 0$

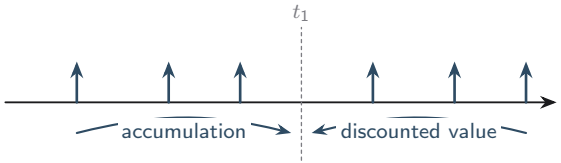
Since $\mathsf{v}(0) = 1$, taking $t_1 = 0$ returns the present value:

$$\mathsf{Val}_0(\mathcal{C}) \;=\; \sum_t c_t\, \mathsf{v}(t) \;+\; \int_{-\infty}^{\infty} \mathsf{v}(t)\, \rho(t)\, \mathrm{d}t.$$

This is Lecture 3's formula, now covering the past as well as the future.

> THE FORUMLA HANDLES SIGN BY ITSELF
>
> For $t < 0$ the same definition $\mathsf{v}(t) = \mathrm{e}^{-\int_0^t \delta(s)\,\mathrm{d}s}$ gives $\mathsf{v}(t) > 1$. The limits are the wrong way round, so the exponent changes sign and past payments are *accumulated* rather than discounted. No separate rule is needed.

Two words for two situations



- If all the payments are due *at or after* t_1, their value at t_1 is also called their **discounted value**.
- If all the payments are *at or before* t_1, their value is called their **accumulation**.
- In general $\mathrm{Val}_{t_1}(C)$ is the sum of the two, splitting the cashflow at t_1.

If $t_1 = 0$ and everything lies in the future, this is the **present value** we began with.

Moving a value from one date to another

Take the formula for $\mathsf{Val}_{t_2}(\mathcal{C})$ and multiply it by $\dfrac{\mathsf{v}(t_2)}{\mathsf{v}(t_1)}$. Every term $c_t \dfrac{\mathsf{v}(t)}{\mathsf{v}(t_2)}$ becomes $c_t \dfrac{\mathsf{v}(t)}{\mathsf{v}(t_1)}$, and likewise inside the integral, so

$$\boxed{\mathsf{Val}_{t_1}(\mathcal{C}) \;=\; \mathsf{Val}_{t_2}(\mathcal{C}) \times \frac{\mathsf{v}(t_2)}{\mathsf{v}(t_1)}}$$

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Multiplying up, this says

$$\mathsf{Val}_{t_1}(\mathcal{C})\,\mathsf{v}(t_1) \;=\; \mathsf{Val}_{t_2}(\mathcal{C})\,\mathsf{v}(t_2),$$

and each side is $\mathsf{PV}(\mathcal{C})$, the value at time 0.

The transfer principle

KEY IDEA

A cashflow has *one* value, quoted at a date. To move that quotation to another date, multiply by the appropriate factor. You never have to go back to the individual payments.

Taking $t_2 = 0$ and $t_1 = t$,

$$\mathrm{Val}_t(\mathcal{C}) \;=\; \mathrm{PV}(\mathcal{C}) \times \frac{1}{\mathrm{v}(t)} \;=\; \mathrm{PV}(\mathcal{C})\,\mathrm{A}(t).$$

So, value once, at whichever date is convenient, then slide the answer along. Choosing the origin and the unit time makes is often just arithmetically more conventient.

The warm-up, at a different date

EXERCISE 4.5

The warm-up cashflow was worth €2,032.04 *at time* 0*, at a constant force of* 4%*. What is it worth at time* 4*?*

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The warm-up cashflow was worth €2,032.04 *at time* 0*, at a constant force of* 4%*. What is it worth at time* 4*?*

Every payment falls at or before time 4, so this became an accumulation question. Using the transfer principle

$$\mathrm{Val}_4(\mathcal{C}) \;=\; \mathrm{PV}(\mathcal{C})\,\mathrm{A}(4) \;=\; 2032.04\,\mathrm{e}^{0.16} \;=\; 2032.04 \times 1.173511 \;=\; €2{,}384.62.$$

> ASIDE
>
> The lump and the integral need not be revisited. Remember, whatever the cashflow is, it has one value. Time 0, 1 or 4 are only a different places to quote it from.

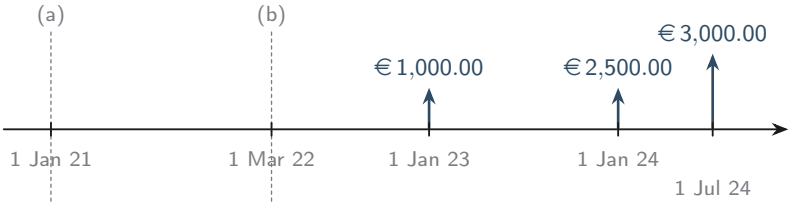
SECTION 3

A worked example

Alfred's payments

EXAMPLE 4.6

Alfred is owed €1,000.00 on 1st January 2023, €2,500.00 on 1st January 2024, and €3,000.00 on 1st July 2024. Assuming a constant force of interest of 0.06 per annum, find the value of these payments on (a) 1st January 2021, (b) 1st March 2022.



(a) Choosing the origin

Measure time in years from 1st January 2021, so that the valuation date in (a) is $t = 0$. The payment dates are then

Date	t	c_t
1 Jan 2023	2	1,000
1 Jan 2024	3	2,500
1 Jul 2024	3.5	3,000

The force is constant, so $\mathsf{v}(t) = \mathrm{e}^{-0.06t}$, and everything lies in the future. This is a plain present value question

$$\mathsf{Val}_0(\mathcal{C}) \;=\; 1000\,\mathsf{v}(2) + 2500\,\mathsf{v}(3) + 3000\,\mathsf{v}(3.5).$$

(a) The arithmetic

That is $1000\,\mathrm{e}^{-0.12} + 2500\,\mathrm{e}^{-0.18} + 3000\,\mathrm{e}^{-0.21}$.

t	c_t	$\mathsf{v}(t)$	$c_t\mathsf{v}(t)$
2	1,000	0.886920	886.92
3	2,500	0.835270	2,088.18
3.5	3,000	0.810584	2,431.75
	Value on 1 Jan 2021		5,406.85

€6,500.00 of promised money is worth €5,406.85 on 1st January 2021.

(b) Sliding the answer along

1st March 2022 is fourteen months after the origin, so $t_1 = \frac{14}{12} = \frac{7}{6}$. Rather than start again, use the transfer principle on the value we already have:

$$\mathrm{Val}_{7/6}(\mathcal{C}) \;=\; \mathrm{Val}_0(\mathcal{C}) \times \frac{1}{\mathrm{v}(7/6)} \;=\; 5406.85\,\mathrm{e}^{0.06\times\frac{7}{6}} \;=\; 5406.85\,\mathrm{e}^{0.07} \;=\; €5{,}798.89.$$

A QUIET APPROXIMATION

We treated fourteen months as exactly 14/12 of a year, and 1 July as exactly half a year. Real contracts specify a **day-count convention** (actual/365, 30/360, and so on) for this. We ignore it and measure time in exact fractions.

(b) The same answer the long way

Let us do it the other way once: Valuing each payment directly at $t_1 = \frac{7}{6}$, by $C\,\mathrm{e}^{-0.06(t-t_1)}$, we get

t	$t - \frac{7}{6}$	factor	value
2	$\frac{5}{6}$	$\mathrm{e}^{-0.05} = 0.951229$	951.23
3	$\frac{11}{6}$	$\mathrm{e}^{-0.11} = 0.895834$	2,239.59
3.5	$\frac{7}{3}$	$\mathrm{e}^{-0.14} = 0.869358$	2,608.07
		Value on 1 Mar 2022	5,798.89

Identical, as $\mathrm{Val}_{t_1}(\mathcal{C})\mathrm{v}(t_1) = \mathrm{Val}_0(\mathcal{C})$.

Lecture 4 on one slide

ONE VALUE, QUOTED AT A DATE

$$\mathrm{Val}_{t_1}(\mathcal{C}) \;=\; \sum_t c_t \,\frac{\mathsf{v}(t)}{\mathsf{v}(t_1)} \;+\; \int \frac{\mathsf{v}(t)}{\mathsf{v}(t_1)}\,\rho(t)\,\mathrm{d}t, \qquad \mathrm{Val}_{t_1}(\mathcal{C})\,\mathsf{v}(t_1) \;=\; \mathrm{PV}(\mathcal{C}).$$

Value once, at whichever date suits you, then slide the answer along. Payments before the date **accumulate** and payments after it **discount**, but a single formula captures both.

Exercises

EXERCISE 4.7

Payments of €500.00 *and* €700.00 *are due at times* 1 *and* 4*, and* $\delta = 0.05$ *is constant. Find the value of the cashflow at time* 3*, and check it against* $\mathrm{PV}(\mathcal{C})/\mathrm{v}(3)$*.*

EXERCISE 4.8

With $\delta = 0.05$ *constant, find the value at time* 0 *of a cashflow consisting of* €800.00 *paid at time* -2 *and* €500.00 *due at time* 3*. Which of the two is being accumulated?*

EXERCISE 4.9

Show directly from the definition that $\mathrm{Val}_{t_1}(\mathcal{C})\,\mathrm{v}(t_1)$ *does not depend on* t_1*.*

Solutions are in the backup slides.

END OF LECTURE 4

So far the payments could have been anything. Next, we consider a special case, when they all have the same size and derive a simpler formula.

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Solution 1

Time 3 lies between the two payments, so one is accumulated and one discounted. The single formula $C\,\mathrm{e}^{-0.05(t-3)}$ does both, the exponent changing sign of its own accord:

$$\mathrm{Val}_3(\mathcal{C}) \;=\; 500\,\mathrm{e}^{0.05\times 2} + 700\,\mathrm{e}^{-0.05\times 1} \;=\; 552.59 + 665.86 \;=\; €1{,}218.45.$$

As a check, value everything at time 0 first and then slide:

$$\mathrm{PV}(\mathcal{C}) \;=\; 500\,\mathrm{e}^{-0.05} + 700\,\mathrm{e}^{-0.2} \;=\; 475.61 + 573.12 \;=\; 1048.73,$$

and $1048.73\,\mathrm{e}^{0.15} = €1{,}218.45$, as the transfer principle requires.

Solution 2

The payment at time -2 is in the past, so its discount factor exceeds 1:

$$\mathsf{v}(-2) \;=\; \mathrm{e}^{-\int_0^{-2} 0.05\,\mathrm{d}s} \;=\; \mathrm{e}^{0.1} \;=\; 1.105171, \qquad \mathsf{v}(3) \;=\; \mathrm{e}^{-0.15} \;=\; 0.860708.$$

Hence

$$\mathsf{Val}_0(\mathcal{C}) \;=\; 800\,\mathrm{e}^{0.1} + 500\,\mathrm{e}^{-0.15} \;=\; 884.14 + 430.35 \;=\; €\,1{,}314.49.$$

It is the €800.00 that is being accumulated. It was paid two years ago and has been earning interest ever since. The €500.00 is still in the future and is discounted. No separate rule needs to be applied, only $\mathsf{v}(t)$. We just read it a negative t or at a positive one.

Solution 3

Multiply the definition through by $\mathsf{v}(t_1)$. Every denominator cancels:

$$\mathsf{Val}_{t_1}(\mathcal{C})\,\mathsf{v}(t_1) \;=\; \mathsf{v}(t_1)\left[\sum_t c_t\,\frac{\mathsf{v}(t)}{\mathsf{v}(t_1)} + \int_{-\infty}^{\infty}\frac{\mathsf{v}(t)}{\mathsf{v}(t_1)}\,\rho(t)\,\mathrm{d}t\right] \;=\; \sum_t c_t\,\mathsf{v}(t) + \int_{-\infty}^{\infty}\mathsf{v}(t)\,\rho(t)\,\mathrm{d}t,$$

and the right-hand side contains no t_1 at all. It is $\mathsf{PV}(\mathcal{C})$.

That is the whole content of the transfer principle: any two valuation dates give answers differing only by the factor $\mathsf{v}(t_2)/\mathsf{v}(t_1)$, because both are the same number $\mathsf{PV}(\mathcal{C})$ divided by different discount factors.