LECTURE 3

Cashflows

Many payments, one number

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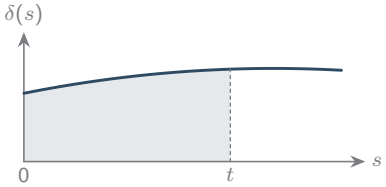
The canonical discount factor



$$\mathsf{v}(t) \;=\; \mathrm{e}^{-\int_0^t \delta(s)\,\mathrm{d}s}$$

The present value of 1 due at time t
is the shaded area in the exponent, negated.

Constant force: $\mathsf{v}(t) = \mathrm{e}^{-\delta t} = (1+i)^{-t}$.

This is all we will need today, as we stop moving one payment, and start moving many.

Today

1. Payments at many dates
2. Payments made continuously
3. Money in, money out

Warm-up

EXERCISE 3.1

Let the force of interest be

$$\delta(t) = \begin{cases} 0.04 & \text{for } t < 10, \\ 0.03 & \text{for } t \geq 10. \end{cases}$$

Find $\mathsf{v}(t)$ *for every* $t \geq 0$.

The integrand changes at $t = 10$, so the integral does too.

Warm-up: solution

For $t < 10$ the force is constant, so $\mathsf{v}(t) = \mathrm{e}^{-0.04t}$. For $t \geq 10$ split the integral where the integrand changes:

$$\int_0^{10} 0.04\,\mathrm{d}s + \int_{10}^{t} 0.03\,\mathrm{d}s \;=\; 0.4 + \Big[0.03s\Big]_{10}^{t} \;=\; 0.1 + 0.03t,$$

$$\mathsf{v}(t) \;=\; \begin{cases} \mathrm{e}^{-0.04t} & t < 10, \\ \mathrm{e}^{-(0.1+0.03t)} & t \geq 10. \end{cases}$$

A SMALL NOTE

The lines agree at $t = 10$, both giving $\mathrm{e}^{-0.4}$. This means continuity and is important since interest/discount cannot jump (must be continuous). For $t \geq 10$ the factor is $\mathrm{e}^{-0.1}\mathrm{e}^{-0.03t}$, where the first ten years leave a fixed discount behind, after which the interest slows down.

SECTION 1

Payments at many dates

Many payments, one number

Suppose sums $c_{t_1}, c_{t_2}, \ldots, c_{t_n}$ are due at times $t_1 < t_2 < \cdots < t_n$. How do we associate one value to this sequence? We may *not* simply add values at *different* times, but we may add them at *the same* time, e.g., **today**. The present value of each is $c_{t_j}\mathsf{v}(t_j)$ and thus:

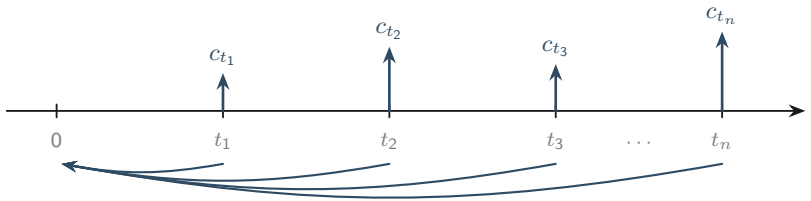
PRESENT VALUE OF A DISCRETE CASHFLOW

$$\sum_{j=1}^{n} c_{t_j}\mathsf{v}(t_j) \;=\; c_{t_1}\mathsf{v}(t_1) + c_{t_2}\mathsf{v}(t_2) + \cdots + c_{t_n}\mathsf{v}(t_n).$$

We can use the same formula if the payments never stop: $\sum_{j=1}^{\infty} c_{t_j}\mathsf{v}(t_j)$.

Provided the series converges, which, for $\delta > 0$, it usually does, since $\mathsf{v}(t)$ decays exponentially.

The same thing, illustrated



Each dragged back to time 0 by its own v(t_j)

The payments and their dates are independent, we simply moved them to a common date.

Example

EXAMPLE 3.2

The force of interest is a constant 5% per annum. Find the present value of: €1,000.00 due at time 1, €2,000.00 at time 3 and €1,500.00 at time 5. Find the value of the total cashflow.

We assumed a constant force and thus, $\mathsf{v}(t) = \mathrm{e}^{-0.05t}$.

t_j	c_{t_j}	$\mathsf{v}(t_j) = \mathrm{e}^{-0.05t_j}$	$c_{t_j}\mathsf{v}(t_j)$
1	1,000	0.951229	951.23
3	2,000	0.860708	1,721.42
5	1,500	0.778801	1,168.20
		Present value	3,840.85

The €4,500.00 of promised money is worth €3,840.85 today.

Two payments, one number

EXERCISE 3.3

The force of interest is a constant 4% *per annum. Find the present value of* €600.00 *due at time* 2 *together with* €900.00 *due at time* 5.

$$600\,\mathrm{e}^{-0.04\cdot 2} + 900\,\mathrm{e}^{-0.04\cdot 5} \;=\; 600\,\mathrm{e}^{-0.08} + 900\,\mathrm{e}^{-0.20} \;=\; 553.87 + 736.86 \;=\; €1{,}290.73.$$

> ASIDE
>
> The €1,500.00 of total payments is worth €1,290.73 today. The later we have to pay, we less a euro is worth, since v(t) falls with t. In this case, it fall at the same proportional rate every year, δ.

SECTION 2

Payments made continuously

Rate of payments as a flow

Salaries, rents and dividends (really everything) arrive in lumps, but when the lumps are small and frequent it is easier, and accurate enough, to model them as a flow.

So suppose that between times 0 and T money is paid *continuously*, at a **rate of payment** $\rho(t)$ at time t.

> CAUTION
>
> $\rho(t)$ is a rate, in euro per unit time, not an amount. Saying $\rho(2) = 1000$ does **not** mean that €1,000.00 changed hands; indeed, at time 2, no money whatsoever changes hands!
>
> Think of a car moving. Its speed being $100\,km/h$ at time 2 does *not* mean that at time 2 it traveld $100\,km$.

The money changing hand is the **integral** of the rate, meaning the area that $\rho(t)$ defines.

Total paid, and rate of payment

Let $M(t)$ be the total paid between time 0 and time t. By definition

$$\rho(t) \;=\; M'(t),$$

so, for $0 \leq \alpha < \beta$, the total paid between α and β is

$$M(\beta) - M(\alpha) \;=\; \int_{\alpha}^{\beta} M'(t)\,\mathrm{d}t \;=\; \int_{\alpha}^{\beta} \rho(t)\,\mathrm{d}t$$

by the fundamental theorem of calculus. The rate of payment is the derivative of the total paid and the total paid is the area under the rate.

The present value of a flow

Between times t and $t + \mathrm{d}t$ the amount paid is $M(t + \mathrm{d}t) - M(t)$, which for small $\mathrm{d}t$ is approximately $\rho(t)\,\mathrm{d}t$. It all arrives at roughly time t, so it is worth roughly

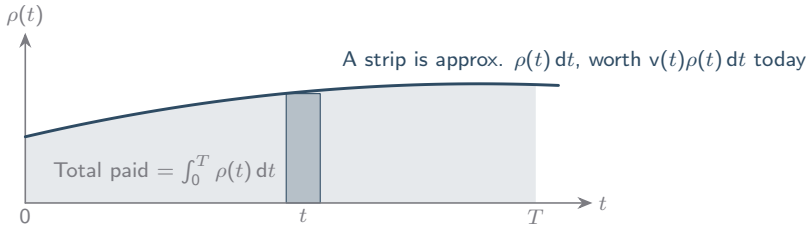
$$\mathsf{v}(t)\,\rho(t)\,\mathrm{d}t$$

today. Adding up the strips, that is integrating, we get

PRESENT VALUE OF A CONTINUOUS CASHFLOW

$$\int_0^T \mathsf{v}(t)\,\rho(t)\,\mathrm{d}t, \qquad \text{with} \qquad \mathsf{v}(t) \;=\; \mathsf{e}^{-\int_0^t \delta(s)\,\mathrm{d}s}.$$

The strip, drawn



The whole area is the total money received. Each strip is the euro that physically change hands around time t. They cannot simply be added, because money changes value with time. Each strip must thus first be discounted from its own date. That is why $\mathrm{v}(t)$ sits *inside* the integral, and why the present value is the area under $\mathrm{v}(t)\rho(t)$, which lies below the area under $\rho(t)$.

A level stream for five years

EXERCISE 3.4

Payments are made continuously at a rate of €1,000.00 *per annum for five years and the force of interest is a constant* 5%*. Find the present value of these transactions.*

RECALL 3.5

$$\int_0^T \mathrm{e}^{-\delta t}\,\mathrm{d}t = \left[\frac{\mathrm{e}^{-\delta t}}{-\delta}\right]_0^T = \frac{1-\mathrm{e}^{-\delta T}}{\delta}.$$

A level stream for five years

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$$\int_0^5 1000\,\mathrm{e}^{-0.05t}\,\mathrm{d}t \;=\; 1000\cdot\frac{1-\mathrm{e}^{-0.25}}{0.05} \;=\; 1000\times 4.423984 \;=\; €4{,}423.98.$$

Worth comparing

Five annual payments of €1,000.00 at the *ends* of years $1, .., 5$ are worth, at the same constant 5% force of interest,

$$1000 \sum_{k=1}^{5} \mathrm{e}^{-0.05k} \;=\; 1000\,(0.951229 + \cdots + 0.778801) \;=\; €4{,}314.31.$$

KEY IDEA

The same €5,000.00 over the same five years payed discretely are worth €109.68 less than the continous case, as in that case, the money arrives earlier. On average, halfway through each year, rather than at its end.

Example: A growing stream

EXAMPLE 3.6

An investor is paid continuously for ten years at the rate $\rho(t) = 1000\,\mathrm{e}^{0.03t}$ p. a., the force of interest being a constant 5%. Find the present value.

$$\int_0^{10} \mathrm{e}^{-0.05t}\,1000\,\mathrm{e}^{0.03t}\,\mathrm{d}t \;=\; 1000\int_0^{10}\mathrm{e}^{-0.02t}\,\mathrm{d}t \;=\; 1000\cdot\frac{1-\mathrm{e}^{-0.2}}{0.02} \;=\; €9{,}063.46.$$

Always simply *before* integrating (in this case, combine the integrals).

KEY IDEA

A stream growing at force 0.03 and discounted at force 0.05 behaves exactly like a stream discounted at force 0.02.

SECTION 3

Money in, money out

Cashflow

Both halves now live in one object.

DEFINITION 3.7 · Cashflow

A **cashflow** $\mathcal{C}$ consists of discrete payments c_t at times t (only countably many non-zero), together with a continuous payment stream at rate $\rho(t)$ per unit time. Its **present value** is

$$\mathrm{PV}(\mathcal{C}) \;=\; \sum_{t} c_t \, \mathsf{v}(t) + \int_0^T \rho(t) \mathsf{v}(t) \, \mathrm{d}t,$$

the sum being over those t for which $c_t \neq 0$ (called *support*).

A stream that never stops

We can take $T = \infty$, provided the integral converges.

EXAMPLE 3.8

With a constant force $\delta = 0.05$ and $\rho(t) \equiv 1$ constantly, we have

$$\int_0^\infty \mathrm{e}^{-0.05t}\,\mathrm{d}t \;=\; \left[\frac{\mathrm{e}^{-0.05t}}{-0.05}\right]_0^\infty \;=\; \frac{1}{0.05} \;=\; 20.$$

An income of €1.00 a year, paid for all eternity, is worth €20.00 today. Discounting makes the far future almost worthless: On these terms, all the payments after year 60 are together worth under €1.00.

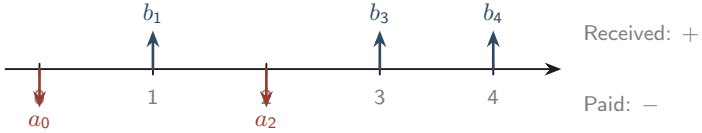
This is the idea behind **perpetual bonds**, possibly the earliest examples are by the Dutch, such as the Dutch Water Board perpetual bonds (1648), or the Yale University's 1648 Dutch bond. In 1751, the Bank of England issued this on a major scale with the consols.

Money in and money out

So far every payment has been positive. In practice, there are many examples where this is not the case.

DEFINITION 3.9 · Net cashflow

Let b_t be the payment *received* at time t and a_t the payment *made* at time t. The net discrete cashflow at time t is $c_t = b_t - a_t$. Similarly, if ρ_2 and ρ_1 are the rates of receiving and of paying, the net rate is $\rho = \rho_2 - \rho_1$.



Net present value

The **net present value** $\mathrm{NPV}(\mathcal{C})$ is the PV applied to the net payments.

> KEY IDEA
>
> Netting works precisely because present values add. Two payments on one date may be replaced by their difference, so one formula with signed payments is enough.

EXAMPLE 3.10

An investor pays €5,000.00 now and receives €2,000.00 per annum continuously for four years (starting immeditaly), at a constant force of 5%.

$$\mathrm{NPV}(\mathcal{C}) \;=\; -5000 + \int_0^4 2000\,\mathrm{e}^{-0.05t}\,\mathrm{d}t \;=\; -5000 + 7250.77 \;=\; €2{,}250.77,$$

positive, so at this force the project is worth doing.

Back to the warm-up

EXAMPLE 3.11

Under a force of 4% for ten years, then 3% (see warm-up exercise) an investment of €10.00 buys an income of €1.00 per annum, paid continuously for fifteen years. Is it worth making?

The income is a flow with $\rho \equiv 1$ on $[0, 15]$, and $\mathrm{v}(t)$ is given by two formulas, so the integral splits at $t = 10$:

$$\int_0^{15} \mathrm{v}(t)\,\mathrm{d}t \;=\; \int_0^{10} \mathrm{e}^{-0.04t}\,\mathrm{d}t + \int_{10}^{15} \mathrm{e}^{-(0.1+0.03t)}\,\mathrm{d}t.$$

In the second integral, the constant comes out, leaving $\mathrm{e}^{-0.1}\int_{10}^{15} \mathrm{e}^{-0.03t}\,\mathrm{d}t$. After year ten a fixed discount has already been suffered, and only the new rate is still running.

Back to the warm-up: Continued

REMINDER

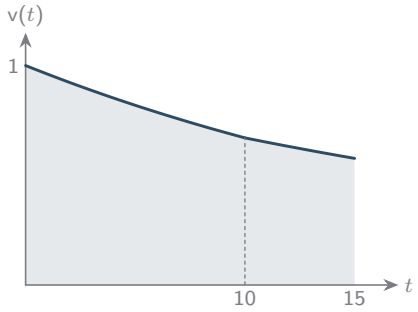
$$\int_{\alpha}^{\beta} \mathrm{e}^{-kt}\,\mathrm{d}t = \frac{\mathrm{e}^{-k\alpha} - \mathrm{e}^{-k\beta}}{k}.$$

$$\underbrace{\frac{1-\mathrm{e}^{-0.4}}{0.04}}_{\text{first ten years}} + \underbrace{\frac{\mathrm{e}^{-0.1}}{0.03}\left(\mathrm{e}^{-0.3} - \mathrm{e}^{-0.45}\right)}_{\text{last five years}} = 8.2420 + 3.1123 = 11.35.$$

So the income is worth €11.35, against €10.00 paid for it:

$$\mathrm{NPV}(\mathcal{C}) = -10 + 11.35 = €1.35 > 0.$$

Was that answer sensible?



As $\rho \equiv 1$, the present value *is* the shaded area under $\mathrm{v}(t)$.

- Fifteen years of €1.00 undiscounted would be €15.00. We got €11.35, so discounting cost about a quarter.
- Had the force stayed at 0.04 throughout, we'd get $\frac{1-\mathrm{e}^{-0.6}}{0.04} = €11.28$. The fall to 0.03 at year ten raises the value, as it must.

Lecture 3 on one slide

ONE DISCOUNT FACTOR, TWO WAYS TO USE IT

$$\mathrm{NPV}(\mathcal{C}) \;=\; \sum_t c_t\, \mathsf{v}(t) \;+\; \int_0^T \mathsf{v}(t)\, \rho(t)\, \mathrm{d}t, \qquad c_t \;=\; \mathsf{in} \;-\; \mathsf{out}\,.$$

Lumps are summed against $\mathsf{v}(t)$ and flows are integrated against it. When evaluating numerous cashflows, move everything to a commmon date (e.g., today) and sum up.

Exercises

EXERCISE 3.12

The force of interest is $\delta(t) = 0.05 - 0.002t$ *per annum. Find the present value of* €1,000.00 *due at time* 3 *together with* €2,000.00 *due at time* 8*.*

EXERCISE 3.13

An investor is paid continuously for ten years at the rate $\rho(t) = 800\,\mathrm{e}^{0.02t}$ *per annum, the force of interest being a constant* 5%*. Find the present value.*

EXERCISE 3.14

An investor pays €10,000.00 *now and receives* €3,000.00 *at each of times* 2*,* 4 *and* 6*. With* $\delta = 0.06$ *constant, find* $\mathrm{NPV}(\mathcal{C})$ *and say whether the investment is worth making.*

Solutions are in the backup slides.

END OF LECTURE 3

Every value here was quoted at time 0. Next: quoting the same cashflow at any date — and why the date you choose never changes what you own.

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Solution 1

Find the discount factor once, for all t, and only then put the dates in:

$$\int_0^t (0.05 - 0.002s)\,\mathrm{d}s \;=\; 0.05t - 0.001t^2, \qquad \mathsf{v}(t) \;=\; \mathrm{e}^{-(0.05t - 0.001t^2)}.$$

At the two payment dates the exponents are $0.15 - 0.009 = 0.141$ and $0.4 - 0.064 = 0.336$, so

$$\mathsf{v}(3) \;=\; \mathrm{e}^{-0.141} \;=\; 0.868489, \qquad \mathsf{v}(8) \;=\; \mathrm{e}^{-0.336} \;=\; 0.714623,$$

and the present value is

$$1000(0.868489) + 2000(0.714623) \;=\; 868.49 + 1429.25 \;=\; €2{,}297.74.$$

The force is falling but still positive, so $\mathsf{v}(t)$ keeps decreasing: later money is still worth less, just less so with each passing year.

Solution 2

Discounting at force 0.05 and growing at force 0.02, the two exponentials combine before anything is integrated:

$$\int_0^{10} \mathrm{e}^{-0.05t}\, 800\, \mathrm{e}^{0.02t}\, \mathrm{d}t \;=\; 800 \int_0^{10} \mathrm{e}^{-0.03t}\, \mathrm{d}t \;=\; 800 \cdot \frac{1 - \mathrm{e}^{-0.3}}{0.03} \;=\; €6{,}911.51.$$

As a check on the size: a level stream of €800.00 for ten years at force 0.05 would be worth $800(1 - \mathrm{e}^{-0.5})/0.05 = €6{,}293.57$. The growth in the rate of payment is worth about €618.00 of that total, and it arrives mostly at the end, where discounting bites hardest.

Solution 3

The outgo is a single payment at time 0, so $c_0 = -€10{,}000.00$; the income is three lumps. With $\delta = 0.06$ constant, $\mathrm{v}(t) = \mathrm{e}^{-0.06t}$ and

$$3000\left(\mathrm{e}^{-0.12} + \mathrm{e}^{-0.24} + \mathrm{e}^{-0.36}\right) \;=\; 3000\,(0.886920 + 0.786628 + 0.697676) \;=\; 3000 \times 2.371225,$$

which is €7,113.67. Hence

$$\mathrm{NPV}(\mathcal{C}) \;=\; -10000 + 7113.67 \;=\; -€2{,}886.33 \;<\; 0.$$

The investment is not worth making at this force of interest: €9,000.00 of income spread over six years does not repay €10,000.00 today, and the same €10,000.00 simply left to accumulate would do better.