LECTURE 2

The force of interest

A rate for an instant, and what happens when it moves

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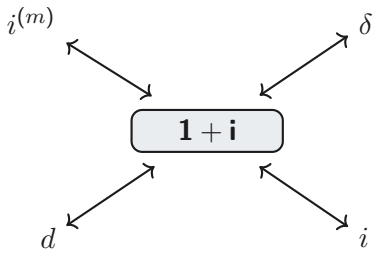
One language for all rates: the accumulation factor



Convert *to* $1+i$, then convert *from* $1+i$.

$$
\begin{aligned}
1+i &= \left(1+\frac{i^{(m)}}{m}\right)^{m} \\
1+i &= (1-d)^{-1} \\
1+i &= \left(1-\frac{d^{(m)}}{m}\right)^{-m} \\
1+i &= \mathsf{v}^{-1} \\
1+i &= \mathsf{e}^{\delta} \qquad \text{soon to be defined} \\
(1+i)^{n} &= \mathsf{A}(n) \qquad \text{for acc. of compound rate.}
\end{aligned}
$$

Today

1. The limit of the nominal rates
2. Working with a constant force
3. A force that varies with time

Warm-up

EXERCISE 2.1

The nominal rate of discount convertible monthly is 6% *per annum. Find the effective annual rate of discount, and the effective annual rate of interest.*

Only one route is needed: Get to $1 + i$, then leave from it.

Warm-up: solution

A rate of discount of $d^{(12)}/12 = 0.5\%$ is applied in each month, so

$$1 - d = \left(1 - \frac{0.06}{12}\right)^{12} = 0.995^{12} = 0.941623, \qquad d = 5.8377\%,$$

$$1 + i = (1 - d)^{-1} = \frac{1}{0.941623} = 1.061996, \qquad i = 6.1996\%.$$

> SANITY CHECK
>
> $d = 5.8377\% < d^{(12)} = 6\% < i = 6.1996\%$, as it must be. By the end of the lecture it will have one more number attached to it.

SECTION 1

The limit of the nominal rates

What happens as m grows

Fix $i = 8\%$ p.a., and pay interest (compound) ever more frequently:

m	1	2	4	12	52	365
$i^{(m)}$	0.080000	0.078461	0.077706	0.077208	0.077018	0.076969

The numbers get smaller, but do not approach zero. Instead, they settle near (converge towards) 0.0769610 . . . (where did we get this number? wait a few slides), and they converge fast. Daily and continuous compounding differ in the sixth decimal place. That limit is the most canonical number of the lot.

The force of interest

DEFINITION 2.2 · Force of interest

The **force of interest** δ corresponding to an effective rate i is the limiting nominal rate $\delta = \lim_{m\to\infty} i^{(m)}$.

Substituting $i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]$ and putting $h = \frac{1}{m}$, so that $h \to 0$, we get

$$\delta = \lim_{m\to\infty} m\left[(1+i)^{\frac{1}{m}} - 1\right] = \lim_{m\to\infty} \frac{(1+i)^{\frac{1}{m}} - 1}{\frac{1}{m}} = \lim_{h\to 0} \frac{(1+i)^h - 1}{h}.$$

$\delta = \ln(1+i)$

RECALL 2.3

For f differentiable at a, $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

Since $(1+i)^0 = 1$, the limit above is the derivative of $f(a) = (1+i)^a$ at $a = 0$:

$$\delta \;=\; \lim_{h\to 0} \frac{(1+i)^h - (1+i)^0}{h} \;=\; \frac{\mathrm{d}}{\mathrm{d}x}(1+i)^x\Big|_{x=0} \;=\; (1+i)^x \ln(1+i)\big|_{x=0} \;=\; \ln(1+i).$$

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So δ is the nominal rate of interest per unit time convertible **continuously**. One also says the rate is **continuously compounded**. Exponentiating,

$$\mathrm{e}^{\delta} \;=\; 1+i, \qquad \mathsf{A}(t) \;=\; (1+i)^t \;=\; \mathrm{e}^{\delta t}, \qquad \mathsf{v}(t) \;=\; \mathrm{e}^{-\delta t}.$$

The discount side gives the same number

The same argument on $d^{(m)} = m\left[1 - (1-d)^{1/m}\right]$ gives

$$\lim_{m\to\infty} d^{(m)} \;=\; -\ln(1-d) \;=\; -\ln v \;=\; \ln(1+i) \;=\; \delta.$$

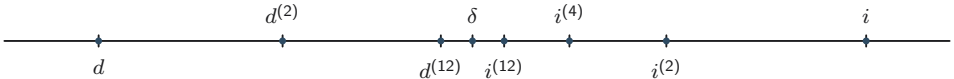
Paid infinitely often, interest and discount stop being different things.

ONE NUMBER FOR THE WHOLE COLLECTION

$$\delta \;=\; \ln(1+i) \;=\; -\ln(1-d) \;=\; -\ln v.$$

The whole family on one line

At $i = 8\%$ p.a., we can see that as $m \to \infty$, $d = d^{(0)} \to d^{(m)} \to \delta \leftarrow i^{(m)} \leftarrow i^{(0)}$.



KEY IDEA

For $i > 0$: $d < d^{(m)} < \delta < i^{(m)} < i$, with $d^{(m)}$ increasing and $i^{(m)}$ decreasing in m, squeezing onto δ.

SECTION 2

Working with a constant force

The warm-up account, finished

EXERCISE 2.4

The nominal rate of discount convertible monthly is 6% *per annum. Find the force of interest.*

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Twelve monthly discounts of 0.5% give $v = 0.995^{12}$, so

$$\delta \;=\; -\ln v \;=\; -\ln(0.995^{12}) = \; -12\ln(0.995) \;=\; 0.060151.$$

DIFFERENT NUMBERS REPRESENTING THE **SAME** CHANGE OF MONEY

$$\underbrace{5.8377\%}_{d} < \underbrace{6\%}_{d^{(12)}} < \underbrace{6.0151\%}_{\delta} < \underbrace{6.0301\%}_{i^{(12)}} < \underbrace{6.1996\%}_{i}.$$

Everything from the force

EXAMPLE 2.5

A bank credits interest continuously at a force of 5% per annum. Find i, d, and the accumulated value of €1,000.00 after three years.

Everything from the force

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Using our "bridge": $1 + i = e^{0.05} = 1.051271$.

$$i = (1 + i) - 1$$

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using $1 + i = 1.051271$

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As $v = (1+i)^{-1}$.

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Using $(1 + i) = 1.051271.$

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$$\boxed{d = 4.8771\%}$$

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Everything from the force

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$$1000\,A(3) = 1000(1+i)^3$$

$i = 5.1271\%$ $d = 4.8771\%$

Everything from the force

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Again $1 + i = 1.051271$.

$i = 5.1271\%$ $\qquad$ $d = 4.8771\%$

Everything from the force

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$$1000\,A(3) = €1{,}161.83$$

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$i = 5.1271\%$ $\quad$ $d = 4.8771\%$ $\quad$ $1000\,\mathrm{A}(3) = €1{,}161.83$

With a constant force, $(1+i)^t$ and $\mathrm{e}^{\delta t}$ are the same computation; the second needs no intermediate rounding.

Why anyone bothers with δ

- Forces **add**, rather than **multiply**: $\mathrm{e}^{\delta s}\mathrm{e}^{\delta(t-s)} = \mathrm{e}^{\delta t}$.
- **No period is attached.** i needs "per annum", $i^{(12)}$ needs "convertible monthly". δ is instantaneous, so it suits payments arriving at arbitrary times.

  Mathatically speaking, its the only **canonical choice**. With time, one will apprecaite the differnce.
- It **differentiates and integrates cleanly**, which is what lets the rate itself move, the subject of the rest of the lecture.

The r in the Black-Scholes formula is a force of interest, not an effective rate.

SECTION 3

A force that varies with time

$\delta(t)$: The general definition

So far the rate has been one number (constant) for all time. Nothing in $\mathsf{A}(t)$ required that.

DEFINITION 2.6 · $\delta(t)$: Force of interest as a function of time t

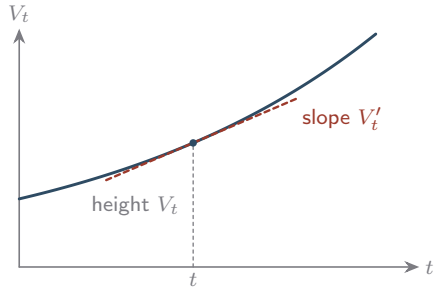
The force of interest is the instantaneous change in the fund value, as an annualised proportion of the current value. With V_t the fund value at time t,

$$\delta(t) \;=\; \frac{V_t'}{V_t} \;=\; \frac{\mathsf{d}}{\mathsf{d}t} \ln V_t.$$

REMINDERS: CHAINRULE AND DERIVATIVE OF $\ln(x)$

The second equality is the chain rule: For f, g, we have $(f(g(t)))' = f'(g(t)) \cdot g'(t)$. In particular, let $f = \ln$ and $g = V_t$. We have $(\ln(V_t))' = \ln'(V_t) \cdot V_t'$. Recall also that $\frac{\mathsf{d}}{\mathsf{d}x} \ln(x) = \frac{1}{x}$ and thus, we get $(\ln(V_t))' = \ln'(V_t) \cdot V_t' = \frac{1}{V_t} \cdot V_t' = \frac{V_t'}{V_t}$.

Why that is the right definition



The slope alone is not a rate: €1,000,000.00 growing by €50,000.00 a year and €1,000.00 growing by €50.00 have very different slopes (€50,000.00 and €1,000.00 respectively) and identical performance. Dividing slope by height removes the size of the fund and leaves the rate. Over a short interval the fund grows by about $\delta(t)V_t\ \mathrm{d}t$.

A fund whose growth rate falls

EXERCISE 2.7

A fund has value $V_t = 1000(1 + 0.02t)^2$ *at time* t*. Find* $\delta(t)$*.*

A fund whose growth rate falls

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A fund has value $V_t = 1000(1 + 0.02t)^2$ *at time* t*. Find* $\delta(t)$*.*

By the chain rule $V_t' = 1000 \cdot 2(1 + 0.02t)(0.02) = 40(1 + 0.02t)$, so

$$\delta(t) \;=\; \frac{V_t'}{V_t} \;=\; \frac{40(1 + 0.02t)}{1000(1 + 0.02t)^2} \;=\; \frac{0.04}{1 + 0.02t}.$$

ASIDE

$\delta(0) = 4\%$ and $\delta(5) = 0.04/1.1 = 3.6364\%$. The fund grows, but its growth *rate* falls. A quadratic is not an exponential, and as exponential has a constant force, polynomials — being always less powerful — must always be decreasing over time.

Integrating the force

Integrate $\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t} \ln V_t$ from t_1 to t_2. The right-hand side is a derivative, so the fundamental theorem of calculus applies:

$$\int_{t_1}^{t_2} \delta(t)\,\mathrm{d}t \;=\; \Big[\ln V_t\Big]_{t_1}^{t_2} \;=\; \ln V_{t_2} - \ln V_{t_1} \;=\; \ln\left(\frac{V_{t_2}}{V_{t_1}}\right).$$

Exponentiating, $V_{t_2}/V_{t_1} = \mathrm{e}^{\int_{t_1}^{t_2} \delta(t)\,\mathrm{d}t}$.

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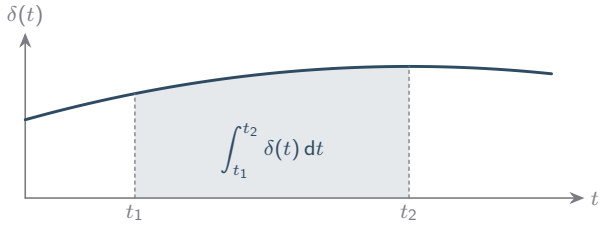
Exponentiating, $V_{t_2}/V_{t_1} = \mathrm{e}^{\int_{t_1}^{t_2} \delta(t)\,\mathrm{d}t}$. But V_{t_2}/V_{t_1} is exactly the factor by which money in the fund grows between t_1 and t_2, so

$$\boxed{\mathsf{A}(t_1, t_2) \;=\; \mathrm{e}^{\int_{t_1}^{t_2} \delta(t)\,\mathrm{d}t}}$$

Reading the formula off a picture



EQUIVALENCE, GEOMETRICALLY

The accumulation factor is e raised to the *shaded area*. Two different-looking forces enclosing the same area accumulate money identically.

Discount factors, and the constant case

Recall $\mathsf{v}(t) = 1/\mathsf{A}(0,t)$. Hence

$$\mathsf{v}(t) \;=\; \mathsf{e}^{-\int_0^t \delta(s)\,\mathrm{d}s},$$

the minus sign in the exponent doing all the discounting.

Discount factors, and the constant case

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$$\mathsf{v}(t) \;=\; \mathrm{e}^{-\int_0^t \delta(s)\,\mathrm{d}s},$$

the minus sign in the exponent doing all the discounting. If the force is a *constant* δ, then $\int_0^t \delta \,\mathrm{d}s = \delta t$ and

$$\mathsf{A}(0,t) \;=\; \mathrm{e}^{\delta t}, \qquad \mathsf{v}(t) \;=\; \mathrm{e}^{-\delta t},$$

which against $\mathsf{A}(t) = (1+i)^t$ gives $1 + i = \mathrm{e}^{\delta}$ once more.

> KEY IDEA
>
> The constant-force case *is* the compound-interest case. Everything in Sections 1 and 2 is the special case $\delta(t) \equiv \delta$ of what follows.

Example: a force that drifts upwards

EXAMPLE 2.8

The force of interest is $\delta(t) = 0.04 + 0.01t$ per annum. Find the accumulated value at time 5 of €1,000.00 invested at time 0.

$$\int_0^5 (0.04 + 0.01t)\,\mathrm{d}t \;=\; \Big[0.04t + 0.005t^2\Big]_0^5 \;=\; 0.2 + 0.125 \;=\; 0.325,$$

so $A(5) = e^{0.325} = 1.384031$ and the fund is worth €1,384.03.

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EXAMPLE 2.9 · Follow-up

Which constant effective annual rate does the same? From $(1+i)^5 = e^{0.325}$, follows $i = e^{0.065} - 1 = 6.7159\%$; the average force, exponentiated.

Example: a force that changes once

EXAMPLE 2.10

The force of interest is 5% per annum for three years and 4% thereafter. Find the present value of €10,000.00 due at time 5.

Split the integral where the integrand changes:

$$\int_0^5 \delta(s)\,\mathrm{d}s \;=\; \int_0^3 0.05\,\mathrm{d}s + \int_3^5 0.04\,\mathrm{d}s \;=\; 0.15 + 0.08 \;=\; 0.23,$$

so $v(5) = e^{-0.23} = 0.794534$ and the present value is €7,945.34.

Splitting the integral is consistency written in exponents: $e^{0.15}e^{0.08} = e^{0.23}$, that is, $A(0,3)A(3,5) = A(5)$.

Revisiting the caution from Lecture 1

Lecture 1 left this open: A rate quoted per year but valid for an instant.

EXAMPLE 2.11

The force is 6% for the first half-year and 4% for the second. Then $A(1) = e^{0.03}e^{0.02} = e^{0.05} = 1.051271$, a year's growth of 5.1271%: not 6, not 4, not 5.

KEY IDEA

An instantaneous rate is *integrated*, not multiplied:

$$A(t, t+h) \;=\; e^{\int_t^{t+h} \delta(s)\,ds} \;(\approx\; 1 + \delta(t)h \quad \text{for small } h),$$

and the familiar $1 + rh$ is only its first term.

Lecture 2 on one slide

THE FORCE OF INTEREST: LIMIT OF INTEREST AND DISCOUNT

$$\delta = \ln(1+i) = -\ln(1-d) = \lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} d^{(m)}$$

$$\mathsf{A}(t_1,t_2) = \mathrm{e}^{\int_{t_1}^{t_2}\delta(t)\,\mathrm{d}t}, \qquad \mathsf{v}(t) = \mathrm{e}^{-\int_0^t \delta(s)\,\mathrm{d}s}$$

Rates multiply and forces add as consistency is the law of exponents, and only the area under δ matters.

Exercises

EXERCISE 2.12

The force of interest is $\delta(t) = 0.03 + 0.005t$ *per annum. Find* $\mathsf{A}(4)$*, and the constant effective annual rate of interest that would give the same accumulation over those four years.*

EXERCISE 2.13

The force of interest is $\delta(t) = 0.06 - 0.002t$ *per annum. Find the present value of* €500.00 *due at time* 2 *together with* €800.00 *due at time* 6*.*

EXERCISE 2.14

Show, from $\mathsf{A}(t_1, t_2) = \mathrm{e}^{\int_{t_1}^{t_2} \delta(t)\,\mathrm{d}t}$*, that* $\mathsf{A}(t_1, t_2)\mathsf{A}(t_2, t_3) = \mathsf{A}(t_1, t_3)$ *for* $t_1 \leq t_2 \leq t_3$*.*

Solutions are in the backup slides.

END OF LECTURE 2

One force behind every quotation, and an area under it behind every accumulation. Next: many payments at many dates, priced with the same $\mathsf{v}(t)$.

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Solution 1

The exponent is the area under the force:

$$\int_0^4 (0.03 + 0.005t)\,\mathrm{d}t \;=\; \Big[0.03t + 0.0025t^2\Big]_0^4 \;=\; 0.12 + 0.04 \;=\; 0.16,$$

so $\mathrm{A}(4) = \mathrm{e}^{0.16} = 1.173511$.

For the equivalent constant rate we need the same accumulation over the same four years, $(1+i)^4 = \mathrm{e}^{0.16}$, so

$$1 + i \;=\; \mathrm{e}^{0.16/4} \;=\; \mathrm{e}^{0.04}, \qquad i \;=\; 4.0811\%.$$

Equivalently: the constant force with the same area is $0.16/4 = 0.04$, the average of δ over $[0, 4]$, and $i = \mathrm{e}^{0.04} - 1$.

Solution 2

First the discount factor, once, for all t:

$$\int_0^t (0.06 - 0.002s)\,\mathrm{d}s \;=\; 0.06t - 0.001t^2, \qquad \mathrm{v}(t) \;=\; \mathrm{e}^{-(0.06t - 0.001t^2)}.$$

Then evaluate it at the two payment dates:

$$\mathrm{v}(2) \;=\; \mathrm{e}^{-0.116} \;=\; 0.890475, \qquad \mathrm{v}(6) \;=\; \mathrm{e}^{-0.324} \;=\; 0.723250,$$

and add the two discounted payments:

$$500(0.890475) + 800(0.723250) \;=\; 445.24 + 578.60 \;=\; €1{,}023.84.$$

Note that the force is falling but still positive, so $\mathrm{v}(t)$ keeps decreasing — later money is still worth less, just less so each year.

Solution 3

Integrals over adjacent intervals add: for $t_1 \le t_2 \le t_3$,

$$\int_{t_1}^{t_2} \delta(t)\,\mathrm{d}t + \int_{t_2}^{t_3} \delta(t)\,\mathrm{d}t \;=\; \int_{t_1}^{t_3} \delta(t)\,\mathrm{d}t.$$

Since $\mathrm{e}^{x}\mathrm{e}^{y} = \mathrm{e}^{x+y}$,

$$\mathsf{A}(t_1,t_2)\mathsf{A}(t_2,t_3) \;=\; \mathrm{e}^{\int_{t_1}^{t_2}\delta}\,\mathrm{e}^{\int_{t_2}^{t_3}\delta} \;=\; \mathrm{e}^{\int_{t_1}^{t_2}\delta + \int_{t_2}^{t_3}\delta} \;=\; \mathrm{e}^{\int_{t_1}^{t_3}\delta} \;=\; \mathsf{A}(t_1,t_3).$$

So any model built from a force of interest satisfies the principle of consistency automatically: additivity of the integral *is* the no-arbitrage condition of Lecture 1.