LECTURE 1

Interest and discount

What a rate of interest actually says

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Today

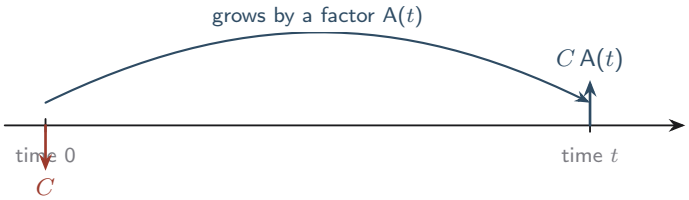
1. The price of time
2. Interest and discount: Simple, compound, continuous
3. Accumulation factors and consistency
4. Effective rates
5. Nominal rates

SECTION 1

The price of time

Money now is not money later

When you lend €100.00 you expect to recieve more. The extra is the price of *time*.



PRINCIPAL C:	Amount lent at start.
INTEREST:	Payment received for lending it.
ACCUMULATED VALUE:	Principal + interest.

What influences interest

The rate of interest is a *price*. It depends on numerous factors, the major ones being:

- **Default risk.** The higher the risk that the capital or interest might not be repaid, the higher the interest.
- **Inflation.** Money repaid in three years buys less, and this is more true in a high-inflation periods where this can dominate all other factors.
- **Term and liquidity.** Tying money up for longer, or in something you cannot sell on demand, costs more.

> A TERM OFTEN USED
>
> **Risk-free rate**: What is charged when none of the above applies.

Standing assumptions

THROUGHOUT WE WILL ASSUME:

- No expenses, charges, commissions or tax.
- Payments are *certain*, in full and on the date promised.
- The rates are known in advance.
- Everything is proportional to the size of money (scalability).

GOAL OF THIS LECTURE

Introduce the notion of accumulation $\mathsf{A}(t)$. Introduce four rates i, d, $i^{(m)}$, $d^{(m)}$ as different ways of reporting the same accumulation and learn how to move between them. A fifth, the *force of interest* will close the circle next lecture.

SECTION 2

Interest and discount: Simple, compound, continuous

Simple interest

DEFINITION 1.1 · Simple interest

Interest is called **simple** if it only depends on the orignal captial C. At rate r per unit time (e.g., day, month, year) it returns $C(1+rt)$ at time t.

EXAMPLE 1.2

€1,000.00 at 6% simple interest for five years pays €60.00 *each* year. Over the five years it comes to $1000(1+0.06\times 5) = €1{,}300.00$.

The €60.00 interest per year earns nothing. In practice simple interest is used for short-dated money-market instruments, where $t < 1$ and the difference to the compound interest is negligible.

Compound interest

DEFINITION 1.3 · Compound interest

Under **compound interest** interest itself yields money. At rate i per unit time C is worth $C(1+i)^t$ at time t.

EXAMPLE 1.4

€1,000.00 at 6% compound for five years gives $1000 \times 1.06^5 = €1{,}338.23$.

End of year	1	2	3	4	5
Simple	1,060	1,120	1,180	1,240	1,300.00
Compound	1,060	1,123.60	1,191.02	1,262.48	1,338.23

The gap of €38.23 is interest earned *on interest*. Unless told otherwise, interest from here on means compound or continuous.

Your turn

EXERCISE 1.5 · three minutes

€2,000.00 is invested for three years at 4.5% per unit time. How much is it worth at the end under (a) simple, (b) compound interest?

Your turn

EXERCISE 1.5 · three minutes

€2,000.00 is invested for three years at 4.5% per unit time. How much is it worth at the end under (a) simple, (b) compound interest?

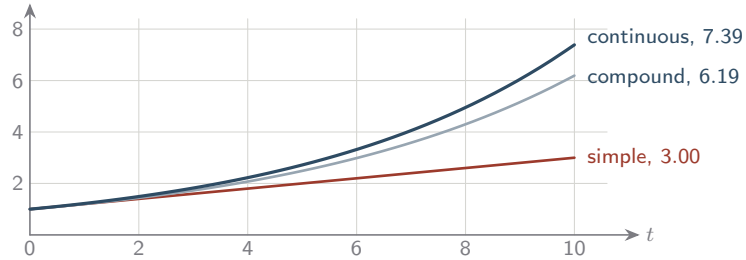
(a) $2000(1 + 0.045 \times 3) = 2000 \times 1.135 = €2{,}270.00$.

(b) $2000 \times 1.045^4 = €2{,}282.33$.

The €12.33 difference is interest on the two interest payments of €90.00 already credited. Over three years at a modest rate it is small. The next slide shows what it does over ten years at a large rate.

The three growth laws, side by side

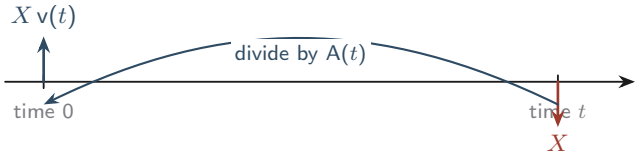
Starting from 1, at 20% per annum, over ten years:



Simple interest grows along a *line*, compound interest along an *exponential*. The third curve is what compound interest becomes when the interest is added ever more often. We make "ever more often" precise next lecture.

Discounting: The same game in reverse

Accumulating/interest asks "C now, how much later?". **Discounting** asks "X later, how much now?".



EXAMPLE 1.6

If you want €10,000.00 in four years, and rates are 5%, you have to invest $10{,}000/1.05^4 = €8{,}227.02$ today. Meaning, its **value today** or **present value** is €8,227.02.

"The present value, discounted at 5%" just means: Divide by 1.05^4.

A rate of discount is not equal to a rate of interest

Some short-term financial contracts are often quoted the other way round: A **rate of discount** d is deducted from the *final* amount, so a bill of face F maturing at time t costs $F(1 - dt)$.

EXAMPLE 1.7

A bill for €5,000.00 due in three months, quoted at a simple discount rate of 4% p.a., costs $5000(1 - 0.04 \times \frac{1}{4}) = €4{,}950.00$.

A SUBTLETY IN MINIATURE

You paid 4,950 and received 5,000, giving a return of $50/4950 = 1.0101\%$ for the quarter, not $1\% = \frac{4\%}{4}$! Interest is measured against what you *put in*, discount against what you *take out*.

Or simpler: $90 \longrightarrow 100$ is an increase by $\frac{100}{90} - 1 = 11.\overline{1}\%$, but $90 \longleftarrow 100$ is a decrease by $1 - \frac{90}{100} = 10\%$.

SECTION 3

Accumulation factors and consistency

Accumulation factors

Rather than commit to a single formula, we build a tool for studying them in a general setting.

DEFINITION 1.8 · Accumulation factor

For $t_1 \leq t_2$, let $\mathsf{A}(t_1, t_2)$ be the accumulation of 1 from time t_1 to t_2.

- Our scalability assumption says that C invested at t_1 accumulates to $C\,\mathsf{A}(t_1, t_2)$.

  Hence the word *factor*.
- Write $\mathsf{A}(t) := \mathsf{A}(0, t)$. Note also that $\mathsf{A}(t, t) = 1$.
- Nothing yet says what A looks like (linear, exponential, continuous, or even increasing). Simple interest, compound interest and a static balance in a bank are all covered by the same notation.

Discount factors

DEFINITION 1.9 · Discount factor

Let $\mathsf{v}(t)$ be the present value at time 0 of a payment of 1 due at time t.

Investing $1/\mathsf{A}(t)$ now produces exactly 1 at time t. The present value of a certain payment is the cost of creating exactly that payment. So

$$\mathsf{v}(t) \;=\; \frac{1}{\mathsf{A}(t)}.$$

EXAMPLE 1.10

Under compound interest i we have $\mathsf{A}(t) = (1+i)^t$ and $\mathsf{v}(t) = (1+i)^{-t}$. We write $\mathsf{v} := \mathsf{v}(1) = (1+i)^{-1}$, which will appear constantly.

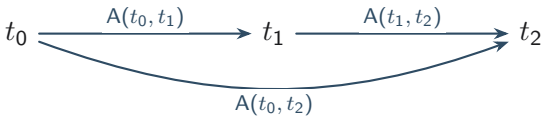
The principle of consistency

Invest 1 at t_0 and take it out at t_2. You could leave it alone, or withdraw at some t_1 in between and immediately reinvest.

CONSISTENCY

In a **consistent** market both routes pay the same: for $t_0 \leq t_1 \leq t_2$

$$\mathsf{A}(t_0, t_2) \;=\; \mathsf{A}(t_0, t_1)\,\mathsf{A}(t_1, t_2).$$



This is true under no-arbitrage assumptions. One might ask "Why must markets be consistent?" Because otherwise there is an arbitrage opportunity ("No free lunch").

Compound interest is consistent

Under compound interest at rate i, an investment of 1 at t_1 grows for $t_2 - t_1$ periods, so $\mathsf{A}(t_1, t_2) = (1+i)^{t_2 - t_1}$ and

$$\mathsf{A}(t_0, t_1)\,\mathsf{A}(t_1, t_2) \;=\; (1+i)^{t_1 - t_0}(1+i)^{t_2 - t_1} \;=\; (1+i)^{t_2 - t_0} \;=\; \mathsf{A}(t_0, t_2).$$

The law of exponents is exactly consistency here.

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The law of exponents is exactly consistency here. Applying it for $t_1 \le t_2 \le \cdots \le t_n$ gives

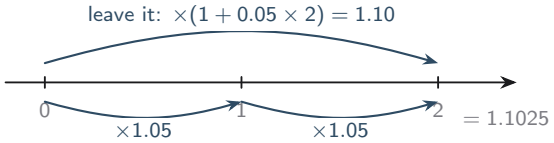
$$\boxed{\mathsf{A}(t_1, t_n) \;=\; \mathsf{A}(t_1, t_2)\mathsf{A}(t_2, t_3)\cdots\mathsf{A}(t_{n-1}, t_n).}$$

EXAMPLE 1.11

A fund returns 5% in year one and 4% in year two. So $\mathsf{A}(2) = 1.05 \times 1.04 = 1.092$ must hold for it to be consistent and €1,000.00 becomes €1,092.00. Note its 9.2% *not* 9%.

Simple interest is not consistent

A bank offers 5% simple interest to anyone, whenever they invest. Two routes from time 0 to time 2:



CAUTION

The routes disagree, so this A is not consistent and the gap is free money: Borrow on the two-year deal, lend it out a year at a time, keep 0.0025 per unit borrowed. No capital and no risk. For $t < 1$, though, the mismatch is tiny and hence, can be regularly found in real life.

SECTION 4

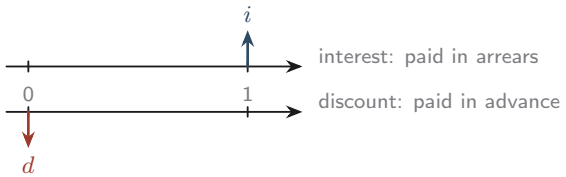
Effective rates

Effective versus nominal

DEFINITION 1.12 · Effective rate

An **effective rate** is the actual rate of change over one complete unit of time — of interest if measured from the beginning to the end of the period, and of discount if measured from the end to the beginning.

A **nominal rate** is a quotation over a fixed time period, whose interest or discount is applied over fixed subperiods. Usually, a rated quoted annually but applied monthly or quarterly. Thus, its effective annual rate depends on the frequency.



The effective rate expressed through A

Let i_n be the effective rate of interest in period n (meaning over $[n-1, n]$), i.e., the interest earned, as a fraction of what you had at the *start*. Let d_n be the same interest as a fraction of what you hold at the *end* (meaning discount). In formulas, we have

$$i_n = \frac{\mathsf{A}(n) - \mathsf{A}(n-1)}{\mathsf{A}(n-1)}, \qquad d_n = \frac{\mathsf{A}(n) - \mathsf{A}(n-1)}{\mathsf{A}(n)}.$$

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BEHIND THE JARGON

Numerator = interest. Denominator = "compared with what?". That's all the choice between i and d. Since the end-of-period amount is $(1 + i_n)$ times the start, this is where the relation $d = i/(1+i)$ in compound interest, proved shortly, comes from.

Example

EXAMPLE 1.13 · Simple interest has a drifting effective rate

Take $\mathsf{A}(t) = 1 + 0.05t$. (Convince yourself that this means exactly 5% *simple interest*). Then

$$i_1 = \frac{1.05 - 1.00}{1.00} = 5\%,\ i_2 = \frac{1.10 - 1.05}{1.05} = 4.76\%,\ i_3 = \frac{1.15 - 1.10}{1.10} = 4.54\%.$$

While the interest stays the same, regarded as an effective rate relative to the current account value, the interest rate decreases over time. To put it another way, the interest stays fixed while the denominator grows.

The compound case

If i is the compound rate of interest then $\mathsf{A}(n) = (1+i)^n$, so

$$i_n \;=\; \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} \;=\; \frac{(1+i)^{n-1}\big[(1+i)-1\big]}{(1+i)^{n-1}} \;=\; (1+i)-1 \;=\; i.$$

KEY IDEA

This does not depend on n. Under compound interest (and matching compounding periods), the effective rate of interest is the same every period and equals the compound rate. So C accumulates to $C(1+i)^n$.

For compound interest only, the two word agree. Conceptually, however, they are still different.

Henceforth, i means *effective rate of interest per unit time*, compound.

Your turn

EXERCISE 1.14

An account grows by 4% *over the first year and by* 5% *over the second. Find* A(2)*, then* i_1*,* i_2 *and* d_2*.*

Your turn

EXERCISE 1.14

An account grows by 4% *over the first year and by* 5% *over the second. Find* $\mathsf{A}(2)$*, then* i_1*,* i_2 *and* d_2*.*

Consistency gives $\mathsf{A}(2) = \mathsf{A}(0,1)\mathsf{A}(1,2) = 1.04 \times 1.05 = 1.092$. Hence

$$i_1 \;=\; 4\%, \qquad i_2 \;=\; \frac{1.092 - 1.04}{1.04} \;=\; 5\%, \qquad d_2 \;=\; \frac{1.092 - 1.04}{1.092} \;=\; 4.7619\%.$$

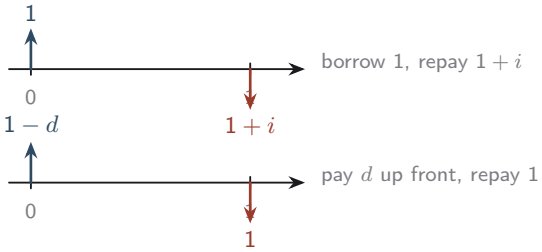
- i_2 is the 5% we were given, as the effective rate reads the growth of *that* period, irrelevant of what happened before.
- $d_2 = 0.05/1.05$, the same interest measured against the closing balance instead of the opening one.

Equivalent rates

DEFINITION 1.15 · Equivalent

Two rates of interest and/or discount are **equivalent over a fixed period** if the same principal, invested for the same length of time, produces the same accumulated value under each.

Compare two ways of borrowing for one period:



Conversion

Recall that v, the sum of money required to be invested now to accumulate to 1 after one time interval (i.e. the present value of 1), is $1 - d$. Hence

$$d = 1 - \mathsf{v} = 1 - \frac{1}{1+i} = \frac{i}{1+i} = i\mathsf{v} \qquad \text{equivalently} \qquad \boxed{i = \frac{d}{1-d}}.$$

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EXAMPLE 1.16

$i = 8\%$ gives $d = 0.08/1.08 = 7.4074\%$, and $d = 5\%$ gives $i = 0.05/0.95 = 5.2632\%$.

Sanity checks on $d = i/(1+i)$

Always do sanity checks (like checking units in physics). For positive interest:

- $d < i$: Money received *early* is worth more, so a smaller number does the same job.
- $d \approx i - i^2$ for small i: $i = 2\%$ gives $d = 1.9608\%$ against $i - i^2 = 1.96\%$; but $i = 50\%$ gives $d = 33.\bar{3}\%$ against $i - i^2 = 25\%$.
- $d < 1$ however large i is — you cannot deduct more than the whole value in advance.

> WHERE THE APPROXIMATION COMES FROM
>
> $\dfrac{i}{1+i} = i(1+i)^{-1} = i\left(1 - i + i^2 - \cdots\right) = i - i^2 + i^3 - \cdots$, a series that is true for $-1 < i < 1$. It's truncation is close to the original value while $i << 1$ is small.

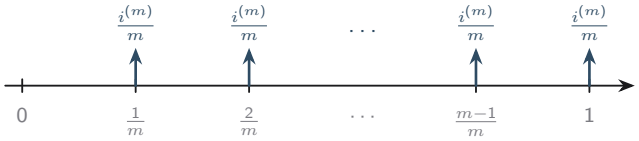
SECTION 5

Nominal rates

Nominal rates of interest

DEFINITION 1.17 · Nominal rate of interest

The **nominal rate of interest payable m times per period**, written $i^{(m)}$, means a rate of $i^{(m)}/m$ applied in each m-th of a period. Meaning, $i^{(m)}$ is equivalent to an effective rate of $i^{(m)}/m$ per period of length $1/m$.



Also called the rate **convertible** or **compounded m-thly**. A nominal 6% p.a. convertible quarterly means 1.5% per quarter, nothing more.

The brackets around the m is to differentiate from the m-th power of i.

Nominal $\to$ effective

Each m-th of a period multiplies the balance by $1 + \frac{i^{(m)}}{m}$, and there are m of them. For a consistent market, we have

$$\boxed{1 + i \;=\; \left(1 + \frac{i^{(m)}}{m}\right)^m} \qquad \text{i.e.} \qquad i^{(m)} \;=\; m\left[(1+i)^{\frac{1}{m}} - 1\right].$$

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CAUTION

$i^{(m)}$ is a quotation convention, not a rate anyone earns. It is m instances of the rate actually applied. As interest compounds during the year, $i^{(m)} < i$ for $m > 1$, and $i^{(1)} = i$.

Example (The "Caution")

EXAMPLE 1.18

Let $i^{(12)} = 12\%$ and $C = €100.00$. The nominal rate is convertible monthly, so the rate applied each month is $\frac{i^{(12)}}{12} = 1\%$. Hence, after twelve months,

$$100(1.01)^{12} = €112.68 \qquad \Longrightarrow \qquad i = \frac{112.68 - 100}{100} = 12.68\%.$$

The nominal rate 12% is not the amount earned over the year. It is twelve applications of the monthly rate 1%.
The amount earned over year is the effective rate, and amounts to 12.68%.

Example

EXAMPLE 1.19

If the nominal rate of interest convertible monthly is 3% p. a., calculate the nominal rate of interest per annum convertible quarterly.

Route 1: Through the effective rate. We are given $i^{(12)} = 0.03$, so

$$1 + i = \left(1 + \frac{0.03}{12}\right)^{12} = 1.0025^{12} = 1.030416, \qquad i = 0.030416,$$

$$i^{(4)} = 4\left[(1+i)^{1/4} - 1\right] = 4\left[1.030416^{1/4} - 1\right] = 0.030075.$$

The same example, without ever computing i

A quarter is three months, so three monthly steps make one quarterly step: $\left(\left(1+\frac{0.03}{12}\right)^{12}\right)^{1/4} = \left(1+\frac{0.03}{12}\right)^{3}$, and therefore

$$i^{(4)} \;=\; 4\left[\left(1+\tfrac{0.03}{12}\right)^{3}-1\right] \;=\; 4\left[1.0025^{3}-1\right] \;=\; 0.030075.$$

KEY IDEA

You can convert by matching *periods* instead of recalling formulas. Ask how many of the small periods fit into the one you want, and compound that many times. It is shorter, and can even be more precise due to rounding.

Your turn

EXERCISE 1.20

A nominal 8% *per annum convertible quarterly is offered. Find (a) the effective annual rate, and (b) the equivalent nominal rate convertible monthly.*

Your turn

EXERCISE 1.20

A nominal 8% *per annum convertible quarterly is offered. Find (a) the effective annual rate, and (b) the equivalent nominal rate convertible monthly.*

(a) The quarterly rate is 2%, so $1 + i = 1.02^4 = 1.082432$ and $i = 8.2432\%$.
(b) Three months make one quarter, so one month is a third of a quarter:

$$1 + \frac{i^{(12)}}{12} \;=\; 1.02^{1/3} \;=\; 1.006623, \qquad i^{(12)} \;=\; 12 \times 0.006623 \;=\; 7.9472\%.$$

Sanity check: $7.9472\% = i^{(12)} < i^{(4)} = 8\% < i = 8.2432\%$.

The credit card looks cheaper that it actually is

EXAMPLE 1.21

A card charges a nominal 18% per annum convertible monthly. What do you actually pay?

$$\frac{i^{(12)}}{12} = 1.5\% \text{ per month} \implies 1+i = 1.015^{12} = 1.195618 \implies i = 19.56\%.$$

RATE NAMES IN THE WILD

Nominal ("per annum convertible monthly") is the number *before* compounding, which lenders like to advertise. **AER**, or effective annual rate, is the number *after*, which savings accounts like to advertise.

Nominal rates of discount

For discounts, everything repeats with the payment moved to the front of each subperiod.

DEFINITION 1.22 · Nominal rate of discount

The **nominal rate of discount payable m times per period**, $d^{(m)}$, means a rate of discount of $d^{(m)}/m$ in each m-th of a period. Thus, $d^{(m)}$ is equivalent to an effective rate of discount of $d^{(m)}/m$ per period of length $1/m$.

Each subperiod multiplies the present value by $1 - \frac{d^{(m)}}{m}$, so

$$\boxed{1 - d \;=\; \left(1 - \frac{d^{(m)}}{m}\right)^m} \qquad \text{i.e.,} \qquad d^{(m)} \;=\; m\left[1 - (1-d)^{1/m}\right].$$

Two short examples

EXAMPLE 1.23

The nominal discount rate convertible every other month is 1.2% per annum. Find the effective annual discount rate.

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$$d = 1 - \left(1 - \frac{0.012}{6}\right)^6 = 1 - 0.998^6 = 0.01194.$$

Two short examples

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$$d = 1 - \left(1 - \frac{0.012}{6}\right)^6 = 1 - 0.998^6 = 0.01194.$$

EXAMPLE 1.24

The nominal rate of interest convertible monthly is 3% per annum. Find the effective rate of discount per annum.

Two short examples

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EXAMPLE 1.24

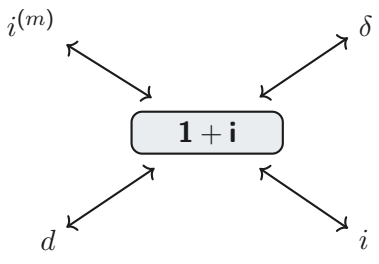
The nominal rate of interest convertible monthly is 3% per annum. Find the effective rate of discount per annum. We found $i = 0.030416$ in Examples 1.19, so

$$d = \frac{i}{1+i} = \frac{0.030416}{1.030416} = 0.029518.$$

One language for all rates: the accumulation factor

ONE-PERIOD ACCUMULATION FACTOR — "THE BRIDGE" $1+i$

Instead of learning conversion formulas for every pair, convert everything to $1+i$.



Convert *to* $1+i$, then convert *from* $1+i$.

$$
\begin{aligned}
1+i &= \left(1+\frac{i^{(m)}}{m}\right)^{m} \\
1+i &= (1-d)^{-1} \\
1+i &= \left(1-\frac{d^{(m)}}{m}\right)^{-m} \\
1+i &= \mathsf{v}^{-1} \\
1+i &= \mathsf{e}^{\delta} \qquad \text{soon to be defined} \\
(1+i)^{n} &= \mathsf{A}(n) \quad \text{for acc. of compound rate.}
\end{aligned}
$$

A pleasant identity

Combining the bridges connecting $i^{(m)}$ and $d^{(m)}$ with $i+1$, we get

$$\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i = \left(1-\frac{d^{(m)}}{m}\right)^{-m} \implies \left(1+\frac{i^{(m)}}{m}\right)\left(1-\frac{d^{(m)}}{m}\right) = 1.$$

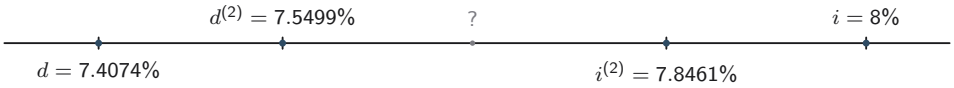
Rearranged, this gives us $\boxed{\frac{d^{(m)}}{m} = \frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}} = \frac{i^{(m)}}{m+i^{(m)}}}$, a direct formula if we want.

BEHIND THE JARGON

Compare this with $d = \dfrac{i}{1+i}$. By plugging $m=1$ we see that there is only ever one relationship here.

Where the four rates sit

At $i = 8\%$, with the four quotations of the same account:



KEY IDEA

For $i > 0$, $n < m$, we have $d < d^{(n)} < d^{(m)} < i^{(m)} < i^{(n)} < i$, with $d^{(m)}$ increasing and $i^{(m)}$ decreasing in m. All these numbers describe the same balance change, and they are clearly converging onto something in the middle. This is next lecture.

Lecture 1 on one slide

EVERY QUOTATION PASSES THROUGH $1+i$

$$\left(1+\frac{i^{(m)}}{m}\right)^{m} = \frac{1}{1-d} = \left(1-\frac{d^{(m)}}{m}\right)^{-m} = \frac{1}{\mathsf{v}} = 1+i.$$

- $\mathsf{A}(t_1,t_2)$: what 1 at t_1 becomes at t_2; $\mathsf{v}(t)=1/\mathsf{A}(t)$.
- Consistency: $\mathsf{A}(t_0,t_2)=\mathsf{A}(t_0,t_1)\mathsf{A}(t_1,t_2)$. Compound rate yes, simple no.
- $i_n=\dfrac{\mathsf{A}(n)-\mathsf{A}(n-1)}{\mathsf{A}(n-1)}$; d_n is the same but over $\mathsf{A}(n)$.
- **Effective**: what one whole period does. **Nominal**: $i^{(m)}$ means $i^{(m)}/m$, applied m times.
- One period, two ends: $d=i\mathsf{v}=\dfrac{i}{1+i}<i$.
- $d<d^{(m)}<i^{(m)}<i$. Nominal rates do not compare directly.

Exercises

EXERCISE 1.25

An account pays a nominal 4.8% *per annum convertible monthly. Find the following:*

(a) *Effective annual rate of interest;*

(b) *Accumulated value of* €2,500.00 *after* 30 *months;*

(c) *Equivalent nominal rate convertible quarterly.*

EXERCISE 1.26

(a) *A bill for* €20,000.00 *due in* 91 *days is bought at a simple discount rate of* 3.5% *per annum (*365*-day year). Find the price paid and the equivalent effective annual rate of interest.*

(b) *A shop offers "*2% *off if you pay today instead of in three months". What effective annual rate of interest is it charging the customers who wait?*

Exercises, continued

EXERCISE 1.27

A bank offers 6% *simple interest. That is* $\mathsf{A}(t) = 1 + 0.06t$ *for a deposit made at time* 0*, and the same terms to anyone depositing at any later date.*

(a) *Find the effective rates of interest* i_1*,* i_2*,* i_3 *for the first three years.*

(b) *Show that the market is not consistent, and say how much risk-free profit* 1 *of borrowed capital can be made to yield over two years.*

(c) *Find the effective compound rate that would accumulate* 1 *to the same amount over two years.*

Solutions are in the backup slides with some steps to guide you. Try a few before checking solutions, but feel free to also just browse them

END OF LECTURE 1

The more often you compound (as m increases in $i^{(m)}$) the higher the effective interest. But it does not grow to infinity (diverge); it grows assymptotically (converges to a limit). Next lecture deals with that number: The force of interest. Moreover, we will allow it to move with time.

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Solution 1

The rate actually applied is $\dfrac{i^{(12)}}{12} = \dfrac{0.048}{12} = 0.004$ per month; 4.8% itself is never applied to anything.

(a) Twelve monthly steps make one year, so

$$1 + i \;=\; 1.004^{12} \;=\; 1.049070, \qquad i \;=\; 4.9070\%.$$

(b) Thirty months is exactly thirty monthly periods — no conversion to an annual rate is needed:

$$2500 \times 1.004^{30} \;=\; 2500 \times 1.127227 \;=\; €\,2{,}818.07.$$

(c) Three months make a quarter, so the quarterly factor is $1.004^3 = 1.012048$ and

$$i^{(4)} \;=\; 4\left(1.004^3 - 1\right) \;=\; 4 \times 0.012048 \;=\; 4.8193\%.$$

Sanity check: We have the ordering $i^{(12)} = 4.8\% < i^{(4)} = 4.8193\% < i = 4.9070\%$.

Solution 2

(a) A discount rate is deducted from the *face* value, so

$$\text{price} \;=\; 20000\left(1 - 0.035 \times \tfrac{91}{365}\right) \;=\; 20000(1 - 0.008726) \;=\; €19{,}825.48.$$

The *interest* earned is measured against the price paid, so the growth over the 91 days is $20000/19825.48 = 1.008803$. Ninety-one days is $\frac{91}{365}$ of a year, and $\frac{365}{91}$ such periods make one year:

$$1 + i \;=\; 1.008803^{365/91} \;=\; 1.035779, \qquad i \;=\; 3.5779\%.$$

A 3.5% discount rate is not a 3.5% yield: the denominators differ.

(b) Paying 0.98 now in place of 1 in three months makes 0.98 the discount factor for a quarter-year, i.e. $\mathrm{v}(1/4) = 0.98$. Hence $(1+i)^{-1/4} = 0.98$ and

$$1 + i \;=\; \left(\tfrac{1}{0.98}\right)^4 \;=\; 1.084166, \qquad i \;=\; 8.4166\%.$$

The customer who waits is borrowing at over 8%.

Solution 3

(a) Here $\mathsf{A}(1) = 1.06$, $\mathsf{A}(2) = 1.12$, $\mathsf{A}(3) = 1.18$, so

$$i_1 \;=\; \frac{1.06 - 1}{1} \;=\; 6\%, \qquad i_2 \;=\; \frac{0.06}{1.06} \;=\; 5.6604\%, \qquad i_3 \;=\; \frac{0.06}{1.12} \;=\; 5.3571\%.$$

The interest is the same 0.06 every year; only the denominator grows.

(b) Leaving 1 alone for two years gives $\mathsf{A}(2) = 1.12$. Withdrawing at time 1 and reinvesting on the same terms gives $\mathsf{A}(0,1)\mathsf{A}(1,2) = 1.06 \times 1.06 = 1.1236 \neq 1.12$, so consistency fails. Borrowing 1 on the two-year deal and lending it out one year at a time returns 1.1236 against a debt of 1.12: a certain 0.0036 per unit borrowed, from no capital and no risk.

(c) We need $(1+i)^2 = 1.12$, so

$$i \;=\; 1.12^{1/2} - 1 \;=\; 1.058301 - 1 \;=\; 5.8301\%,$$

which sits between $i_1 = 6\%$ and $i_2 = 5.6604\%$, as an average of the two years' growth should.