LECTURE 22

Revision II: geometry and decompositions

Lectures 15–20, and how the whole module fits together

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Where we got to

LAST LECTURE

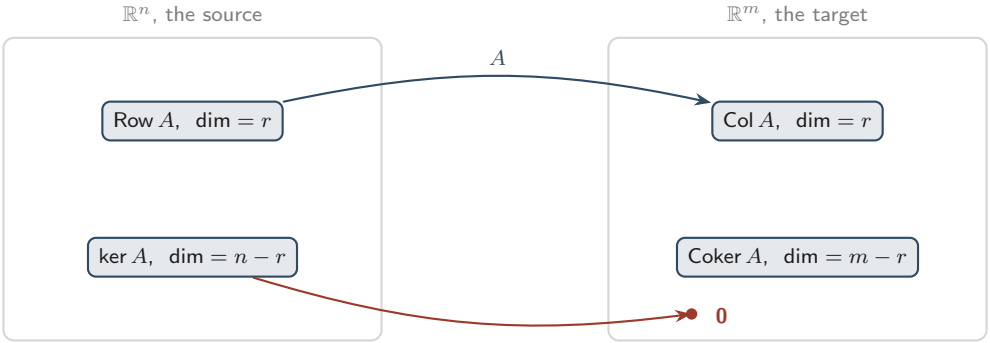
- Matrices, determinants, inverses; the subspace test; span, Col A and ker A.
- Row reduction preserves ker A and Row A, destroys Col A; rref(A) answers nearly everything.
- Linear maps are matrices; bases give coordinates; $\dim V$ is well defined and the Basis Theorem halves the work.
- rank A + nullity $A = n$, and the six-way equivalence for invertibility.

Today: the four subspaces, orthogonality, least squares, the SVD — and a guide to picking the right tool under exam conditions.

SECTION 1

The four fundamental subspaces

All four at once



In $\mathbb{R}^n$, Row A and ker A are orthogonal complements: $(\text{Row } A)^\perp = \ker A$. In $\mathbb{R}^m$, so are Col A and Coker $A = \ker A^\mathsf{T}$.

SECTION 2

Orthogonality

The toolkit

INNER PRODUCT	$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathsf{T}}\mathbf{v} = \sum u_i v_i$; $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$; $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$.
ORTHOGONAL	$\mathbf{u} \perp \mathbf{v}$ means $\mathbf{u} \cdot \mathbf{v} = 0$. Then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.
INEQUALITIES	$\|\mathbf{u} \cdot \mathbf{v}\| \leq \|\mathbf{u}\|\|\mathbf{v}\|$ and $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.
COMPLEMENT	$H^\perp$ is a subspace; test only a spanning set of H. $\dim H + \dim H^\perp = n$.
INDEPENDENCE	An orthogonal set of non-zero vectors is automatically independent — so n of them in $\mathbb{R}^n$ form a basis.

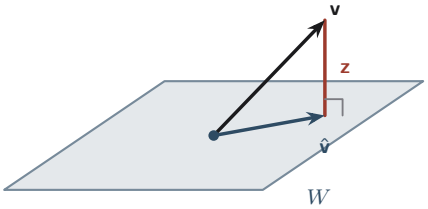
L16–17

Projection and decomposition

With an *orthogonal* basis $\mathbf{u}_1, \ldots, \mathbf{u}_k$ of W,

$$\text{proj}_W(\mathbf{v}) = \sum_i \frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \mathbf{u}_i.$$

Every $\mathbf{v}$ splits uniquely as $\hat{\mathbf{v}} + \mathbf{z}$ with $\hat{\mathbf{v}} \in W$ and $\mathbf{z} \in W^\perp$, and $\hat{\mathbf{v}} = \text{proj}_W(\mathbf{v})$ is the *closest* point of W to $\mathbf{v}$.



The proof is Pythagoras — the engine of least squares.

L17–18

GRAM–SCHMIDT L17

$$\mathbf{u}_j = \mathbf{x}_j - \text{proj}_{\text{span}\{\mathbf{u}_1,\ldots,\mathbf{u}_{j-1}\}}(\mathbf{x}_j)$$

Spans are preserved at every stage, so the result is an orthogonal basis of the same space. Normalise at the end if orthonormal is wanted.

Gram–Schmidt, and orthogonal matrices

GRAM–SCHMIDT L17

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ORTHOGONAL MATRICES

For square A, these agree: columns orthonormal; $A^{-1} = A^{\mathsf{T}}$; $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$; $\|A\mathbf{x}\| = \|\mathbf{x}\|$. Then $\det A = \pm 1$: a rotation or a reflection.

SECTION 3

Least squares

When there is no solution

KEY IDEA

If $A\mathbf{x} = \mathbf{b}$ is inconsistent, seek $\hat{\mathbf{x}}$ minimising $\|\mathbf{b} - A\mathbf{x}\|$. Since $A\mathbf{x}$ runs over $\operatorname{Col} A$, this is asking for $\operatorname{proj}_{\operatorname{Col} A}(\mathbf{b})$ — so the residual must be orthogonal to every column:

$$A^{\mathsf{T}} A\,\hat{\mathbf{x}} \;=\; A^{\mathsf{T}}\mathbf{b}.$$

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LINE OF BEST FIT — rows $(1, x_i)$, right-hand side y_i; then $A^{\mathsf{T}}A = \begin{bmatrix} m & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$ and $A^{\mathsf{T}}\mathbf{b} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$.

UNIQUENESS — $A\hat{\mathbf{x}}$ is always unique; $\hat{\mathbf{x}}$ is unique exactly when $\ker A = \{\mathbf{0}\}$.

L18

SECTION 4

Complex, and the SVD

The complex dictionary

TRANSPOSE	$A^{\mathsf{T}} \longrightarrow A^*$, conjugate transpose
INNER PRODUCT	$\mathbf{u}^{\mathsf{T}}\mathbf{v} \longrightarrow \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^*\mathbf{v}$, so $\|\mathbf{u}\|^2 = \sum \lvert u_i \rvert^2$ is real
SYMMETRIC	$A^{\mathsf{T}} = A \longrightarrow$ **Hermitian**, $A^* = A$
ORTHOGONAL	$A^{\mathsf{T}}A = I \longrightarrow$ **unitary**, $U^*U = I$

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KEY IDEA

Two facts carry the whole of the SVD: A^*A is Hermitian, so it has an orthonormal eigenbasis with real eigenvalues; and those eigenvalues are ≥ 0, since $\lambda = \|A\mathbf{v}\|^2 / \|\mathbf{v}\|^2$.

The SVD

THEOREM 22.1

Every $A \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ factors as $A = U\Sigma V^*$, with U and V unitary and Σ carrying the singular values $s_1 \geq \cdots \geq s_n \geq 0$ on its diagonal.

1. Form A^*A; find its eigenvalues; $s_i = \sqrt{\lambda_i}$, ordered decreasing.
2. Orthonormal eigenvectors, in that order, are the columns of V.
3. Σ has the shape of A, with the s_i on the diagonal.
4. $\mathbf{u}_i = A\mathbf{v}_i/s_i$ for $s_i > 0$; extend to an orthonormal basis of $\mathbb{C}^m$ by Gram–Schmidt.

The shape of Σ matches A, so it is rectangular whenever A is — and step 4 is where the extra columns of U come from. L20

What one SVD gives you

With r non-zero singular values:

RANK	$\operatorname{rank} A = r$
Col A	$\mathbf{u}_1, \ldots, \mathbf{u}_r$
ker A	$\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$
Coker A	$\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m$
GEOMETRY	A is a rotation, a stretch by s_i along axes, then another rotation
LEAST SQUARES	$\hat{\mathbf{x}} = A^+\mathbf{b}$ with $A^+ = V\Sigma^+U^*$, inverting only the non-zero s_i

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Orthonormal bases of all four fundamental subspaces, the rank, the geometry and the least-squares solution — from one factorisation.

SECTION 5

Choosing the tool

Which method answers which question

IS THIS A SUBSPACE?	(U1), (U2), (U3) — simplify first
IS $\mathbf{b}$ IN Col A?	reduce $[\,A \mid \mathbf{b}\,]$; look for $0 = c$
DESCRIBE ker A	rref(A); one vector per free column
BASIS OF Col A	pivot columns *of* A
INDEPENDENT? SPANNING?	count against $\dim V$ first; then determinant or row reduction
IS A INVERTIBLE?	$\det A$, or $\operatorname{rank} A = n$, or rref$(A) = I$
CLOSEST POINT IN W	proj_W, after Gram–Schmidt if the basis is not orthogonal
NO SOLUTION EXISTS	$A^{\mathsf{T}}A\hat{\mathbf{x}} = A^{\mathsf{T}}\mathbf{b}$, or $A^{+}\mathbf{b}$

SECTION 6

One matrix, everything

A full example (1/3)

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, and answer everything.

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Rank and the subspaces. $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, so both columns are pivot columns:

$$\operatorname{rank} A = 2, \qquad \operatorname{nullity} A = 2 - 2 = 0, \qquad \ker A = \{\mathbf{0}\}.$$

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$$\operatorname{Col} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \dim \operatorname{Coker} A = m - r = 3 - 2 = 1.$$

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So A is injective but not surjective — a 3×2 matrix never can be.

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The bottom row reads $0 = -2$: inconsistent, so **No.**

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Then find the least-squares solution. $A^{\mathsf{T}}A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $A^{\mathsf{T}}\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so

$$2\hat{x}_1 + \hat{x}_2 = 1, \quad \hat{x}_1 + 2\hat{x}_2 = 1 \qquad \Longrightarrow \qquad \hat{\mathbf{x}} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}.$$

A full example (3/3)

Check the geometry. $A\hat{\mathbf{x}} = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, and the residual is

$$\mathbf{b} - A\hat{\mathbf{x}} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix},$$

whose inner product with each column of A is $\frac{2}{3} - \frac{2}{3} = 0$. **Yes.**

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And the SVD. $A^{\mathsf{T}}A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ has eigenvalues $3, 1$, so $s_1 = \sqrt{3}$, $s_2 = 1$ — both non-zero, confirming $\operatorname{rank} A = 2$ and $\ker A = \{\mathbf{0}\}$.

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Four questions from three lectures, one matrix, and every answer consistent with every other. That consistency is the module.

Errors that cost marks

CAUTION

- Using the projection formula with a basis that is not orthogonal. Run Gram–Schmidt first.
- Forgetting to normalise when a question asks for *orthonormal*.
- In the SVD: listing the $\mathbf{v}_i$ in a different order from the s_i, or forgetting to extend U to m columns.
- Dividing by a zero singular value. The pseudoinverse leaves zeros as zeros.
- Treating $\langle \mathbf{u}, \mathbf{v} \rangle$ as symmetric over $\mathbb{C}$ — it conjugates.
- Concluding $\mathbf{b} \notin \operatorname{Col} A$ from a row of zeros: only $[\,0 \;\cdots\; 0 | c\,]$, $c \neq 0$, proves that.

The module in one page

KEY IDEA

A vector space is a set where you may add and scale. Every finite one is $\mathbb{R}^n$ in disguise, and every linear map between them is a matrix.

Row reduction answers the computational questions; dimension and rank count what is there; orthogonality adds length and angle, which the axioms alone do not provide.

And the SVD says what a matrix *is*: a rotation, a stretch along axes, and another rotation — with the four fundamental subspaces readable off the factors.

Everything else in these twenty-two lectures is detail, technique, or proof.

Final revision questions

AT HOME

1. Let $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Find an orthogonal basis of W, then $\text{proj}_W \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and the distance from that vector to W.
2. Find the line of best fit through $(1, 2)$, $(2, 3)$, $(3, 5)$.
3. Find an SVD of $A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$, and read off $\text{rank}\, A$, $\text{Col}\, A$ and $\ker A$.

END OF LECTURE 22

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Solution: question 1

SOLUTION

Gram–Schmidt: $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and with $\mathbf{x}_2 \cdot \mathbf{u}_1 = 1$, $\mathbf{u}_1 \cdot \mathbf{u}_1 = 2$,

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \tfrac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Check: $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1 - 1 + 0 = 0$. **Yes.**

Solution: question 1

SOLUTION

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Check: $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1 - 1 + 0 = 0$. **Yes.**

For $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$: $\mathbf{v} \cdot \mathbf{u}_1 = \mathbf{v} \cdot \mathbf{u}_2 = 2$, with $\mathbf{u}_2 \cdot \mathbf{u}_2 = 6$, so $\operatorname{proj}_W(\mathbf{v}) = \frac{1}{3} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{z} = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$, of length $\frac{1}{3}\sqrt{12} = \frac{2}{\sqrt{3}}$.

Solution: question 2

SOLUTION

$m = 3$, $\sum x_i = 6$, $\sum x_i^2 = 14$, $\sum y_i = 10$, $\sum x_i y_i = 23$, so

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 10 \\ 23 \end{bmatrix}.$$

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Doubling the first equation and subtracting gives $2d = 3$, so $d = 1.5$; then $3c = 10 - 9$, giving $c = \frac{1}{3}$.

Solution: question 2

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The line is $y = \frac{1}{3} + \frac{3}{2}x$, with residuals $\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}$ — which sum to 0, as the first normal equation requires.

Solution: question 3

SOLUTION

$A^*A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$, with eigenvalues 8 and 0. So $s_1 = 2\sqrt{2}$, $s_2 = 0$, and $\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\mathbf{u}_1 = \frac{1}{2\sqrt{2}} A\mathbf{v}_1 = \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ completes $U = I_2$.

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One non-zero singular value, so $r = 1$: $\operatorname{rank} A = 1$, $\operatorname{Col} A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$, $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$.

Check: $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. **Yes.**