LECTURE 21

Revision I: matrices to dimension

Lectures 1–14, in one place

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? February 2027 · Winter 2026

How to use these two lectures

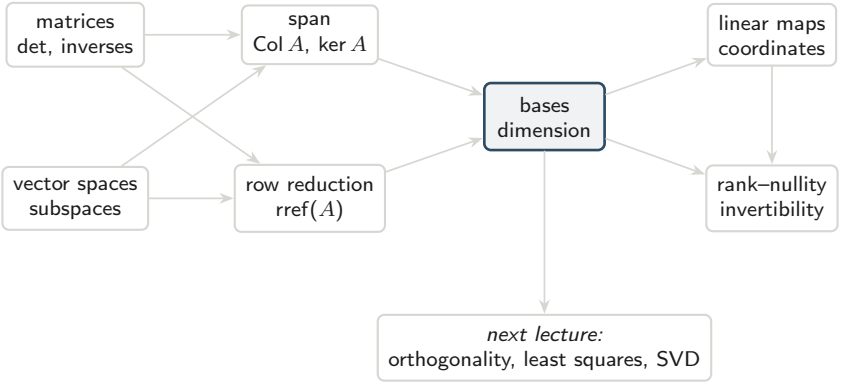
KEY IDEA

Nothing here is new. These two lectures are a map: what we proved, how the pieces connect, and which tool answers which question.

- Today: lectures 1–14 — matrices, vector spaces, span, row reduction, linear maps, bases, dimension, rank.
- Next: lectures 15–20 — the four subspaces, orthogonality, least squares, and the SVD.
- Each part ends with the errors that cost most marks, and a worked example that uses everything at once.

If a slide here is unfamiliar, that is the signal to go back to the lecture it came from — each is cited.

The shape of the whole module



Everything funnels through **basis** and **dimension**: that is where the counting starts, and every later theorem is a count.

SECTION 1

Matrices

Matrices: the three things to know

PRODUCT $(AB)_{i,j} = a_{i,-} \cdot b_{-,j}$; shapes $(m \times n)(n \times p) = (m \times p)$.
$AB \neq BA$, but $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$ and $\det(AB) = \det A \det B$.

COLUMNS $A\mathbf{x}$ is a linear combination of the *columns* of A, with the entries of $\mathbf{x}$ as coefficients.

L5

INVERSE Exists iff $\det A \neq 0$; found by reducing $[\,A \mid I\,] \to [\,I \mid A^{-1}\,]$.
$(AB)^{-1} = B^{-1}A^{-1}$, $\det(A^{-1}) = 1/\det A$.

L2

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$|\det A|$ is the factor by which A scales area or volume; $\det A = 0$ means a collapse, which is why it cannot be undone.

SECTION 2

Vector spaces and subspaces

Vector spaces: what to remember

Nine axioms: (A1)–(A5) make $(V, +)$ an abelian group, (S0)–(S4) govern scaling. In practice you will never check all nine — you will check the *subspace* conditions instead.

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$U \subseteq V$ is a subspace if

(U1) $\mathbf{0} \in U$, (U2) closed under $+$, (U3) closed under scaling.

Then U is a vector space in its own right: the six axioms that are *equations* come free.

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Examples to have ready: $\mathbb{R}^n$, $\mathrm{Mat}_{m\times n}(\mathbb{R})$, P_n, $\mathbb{R}[x]$, $\mathcal{F}(\mathbb{R},\mathbb{R})$, and $\mathbb{C}$ over $\mathbb{R}$.

Subspace or not: the four failure modes

LINEAR, $= 0$

Yes. and systems of such equations too, being intersections.

AFFINE

No. $x + y = 1$ misses $\mathbf{0}$: (U1) fails.

INEQUALITY

No. $x + y \geq 0$ breaks at $\lambda < 0$: (U3).

CURVED

No. $x^2 + y^2 = 1$ fails both tests.

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SIMPLIFY FIRST

$x^2 + y^2 + 1 = 1$ *is* a subspace — it says $x = y = 0$; and $x^2 + y^2 \geq 0$ says nothing, so gives the whole space. Read the condition before testing it.

L4

SECTION 3

Span, kernel, column space

Two ways to name a subspace

FROM INSIDE

span(S): all combinations of S, the smallest subspace containing it. Col $A \subseteq \mathbb{R}^m$ is the span of the columns, also written im A.

FROM OUTSIDE

$\ker A = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n$, cut out by equations.

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Every subspace is *both* an image and a kernel L6, L9

Choose whichever makes the question easy: spanning sets are cheap for Col, membership for ker.

SECTION 4

Row reduction

What row reduction does and does not preserve

PRESERVED	$\ker A$, and $\operatorname{Row} A$. Also all linear relations *among the columns*.
DESTROYED	$\operatorname{Col} A$.

$\begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}1\\0\end{bmatrix}$: different lines.

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CONSEQUENCES L8, L11

- $\ker A = \ker \operatorname{rref}(A)$: one spanning vector per *free* column.
- The pivot columns of A are a basis of $\operatorname{Col} A$; the non-zero rows of $\operatorname{rref}(A)$ are a basis of $\operatorname{Row} A$.
- $\mathbf{b} \in \operatorname{Col} A$ iff $[\,A \mid \mathbf{b}\,]$ reduces with no row $[\,0 \cdots 0 | c\,]$, $c \neq 0$.

SECTION 5

Linear transformations

Maps and matrices are the same thing

T is **linear** if $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$. Then $T(\mathbf{0}) = \mathbf{0}$ — a one-glance test that rules out every affine map.

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STANDARD MATRIX	$[T] = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$
$\ker T$, $\operatorname{im} T$	$\ker A$, $\operatorname{Col} A$
INJECTIVE	$\ker T = \{\mathbf{0}\}$, i.e. nullity $A = 0$
ISOMORPHISM	A invertible, $\det A \neq 0$
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And a linear map is determined by its values on a spanning set — which is why bases matter so much. L9, L12

Coordinates and change of basis

UNIQUE REPRESENTATION L11

If $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ is a basis, every $\mathbf{x}$ is $c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ in exactly one way. Write $[\mathbf{x}]_{\mathcal{B}}$ for that list.

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With $P_{\mathcal{B}} = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_n\end{bmatrix}$:

$$\mathbf{x} = P_{\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}\,, \qquad [\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}, \qquad P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}.$$

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$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an isomorphism $V \to \mathbb{R}^n$: any n-dimensional space *is* $\mathbb{R}^n$ in different clothes, so problems may be moved there and back. L12

SECTION 6

Independence, bases, dimension

The central definitions

INDEPENDENT	$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ only when all $c_i = 0$. Equivalently: none of them is a combination of the others.
BASIS	A spanning set that is independent. Never unique — but every basis of V has the same size.
DIMENSION	$\dim V$ is that size. $\dim \mathbb{R}^n = n$, $\dim P_n = n+1$, $\dim \mathrm{Mat}_{m\times n}(\mathbb{R}) = mn$, $\dim\{\mathbf{0}\} = 0$.

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BASIS THEOREM L13

Given *exactly* $\dim V$ vectors, basis $\iff$ independent $\iff$ spanning. So with the right count, half the definition suffices.

Counting arguments you should reach for

- More than $\dim V$ vectors: always dependent. (Four vectors in $\mathbb{R}^3$; three polynomials in P_1.)
- Fewer than $\dim V$: cannot span.
- $H \subseteq V$ gives $\dim H \leq \dim V$, with equality only when $H = V$.
- Any independent set extends to a basis; any finite spanning set contains one.

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> KEY IDEA
>
> Most "show that…" questions about independence or spanning are settled by counting first and computing only if the count is inconclusive.

SECTION 7

Rank

Rank–nullity, and the equivalence

THEOREM 21.1 · Rank–Nullity L14

For A of shape $m \times n$: $\operatorname{rank} A + \operatorname{nullity} A = n.$

Every column of $\operatorname{rref}(A)$ is a pivot column or a free one.

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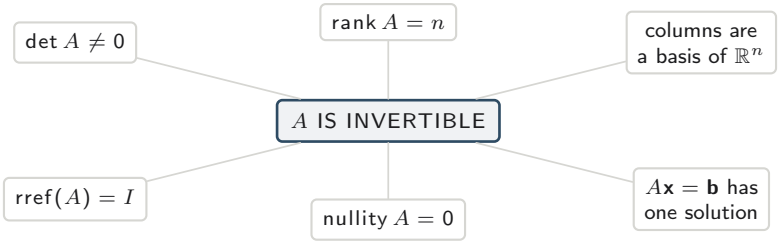

Six statements from six different lectures — and every one of them is the same statement.

SECTION 8

One matrix, everything

A worked example (1/2)

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix} \xrightarrow{\text{reduce}} \operatorname{rref}(A) = \begin{bmatrix} \mathbf{1} & 2 & 0 & 1 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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RANK two pivots, so $\operatorname{rank} A = 2$

NULLITY $4 - 2 = 2$, matching the two free columns

Col A pivot columns *of* A: $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Row A non-zero rows of $\operatorname{rref}(A)$: $\operatorname{span}\{(1,2,0,1),(0,0,1,1)\}$

A worked example (2/2)

For $\ker A$, set the free variables $x_2 = s$, $x_4 = t$. The pivot rows read $x_1 + 2s + t = 0$ and $x_3 + t = 0$, so

$$\ker A \;=\; \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}, \qquad \dim = 2. \quad \textbf{Yes.}$$

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Is $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ in $\operatorname{Col} A$? Reduce $[\,A \mid \mathbf{b}\,]$: the bottom row becomes $[\,0\ 0\ 0\ 0 | 0\,]$, so the system is consistent and **Yes.**

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Is $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ in $\operatorname{Col} A$? The same operations give $[\,0\ 0\ 0\ 0 | 1\,]$, reading $0 = 1$: **No.**

One reduction answered five separate questions.

Errors that cost marks

CAUTION

- Taking the pivot columns of rref(A) instead of *of* A for a basis of Col A. Row operations change Col A.
- Claiming a set is a subspace without checking $\mathbf{0} \in U$.
- Writing B/A for a matrix quotient. There is no division: write $A^{-1}B$ or BA^{-1}, and say which.
- Assuming $AB = BA$, or that $(AB)^{\mathsf{T}} = A^{\mathsf{T}}B^{\mathsf{T}}$.
- Confusing dim with however many vectors you wrote down — they agree only for a basis.

Revision questions

AT HOME

1. For $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}$: find $\operatorname{rref}(A)$, $\operatorname{rank} A$, bases of $\operatorname{Col} A$ and $\ker A$, and say whether A is invertible — in two different ways.
2. Is $\{1 + x,\ x + x^2,\ 1 + x^2,\ x\}$ independent in P_2? Answer without computing anything.
3. Let $T\colon P_2 \to P_2$ be $T(p) = p' + p(0)$. Show T is linear, find $[T]$ in the basis $(1, x, x^2)$, and compute $\ker T$ and $\operatorname{rank} T$.

END OF LECTURE 21

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Solution: question 1

SOLUTION

$$A \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{\text{tidy}} \begin{bmatrix} \mathbf{1} & 0 & 1 \\ 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So rank $A = 2$, and $\text{Col}\, A = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ — columns $1, 2$ *of* A.

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Free column 3: $x_3 = t$ gives $x_1 = -t$, $x_2 = -2t$, so ker $A = \text{span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$.

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Not invertible: rank $A = 2 < 3$; equivalently $\text{rref}(A) \neq I$, or nullity $A = 1 \neq 0$, or $\det A = 0$.

Solution: question 2

SOLUTION

No. — by counting alone. $\dim P_2 = 3$, and any $3 + 1 = 4$ vectors in a space of dimension 3 are dependent.

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No determinant, no system, no arithmetic. If you found yourself reducing a matrix here, that is the habit this slide is meant to break.

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For the record, a dependence is $(1 + x) - (x + x^2) - (1 + x^2) \ldots$ — but the theorem told us one exists without our having to find it.

Solution: question 3

SOLUTION

Linear, since $(p+q)' + (p+q)(0) = (p' + p(0)) + (q' + q(0))$, and likewise for scalars. On the basis $T(1) = 1$, $T(x) = 1$, $T(x^2) = 2x$, so $[T] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

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Two pivots, so $\operatorname{rank} T = 2$ and $\operatorname{nullity} T = 1$; column 2 is free, giving $\ker T = \operatorname{span}\{x - 1\}$.

Check: $(x-1)' + (x-1)(0) = 1 + (-1) = 0$. **Yes.**