LECTURE 19

Eigenvalues and diagonalisation

The directions a matrix does not turn, and what they are good for

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
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Where we got to

LAST LECTURE

- $\mathrm{Proj}_W(\mathbf{v})$ is the closest point of W to $\mathbf{v}$ (Best Approximation).
- When $A\mathbf{x} = \mathbf{b}$ is inconsistent, the least-squares solution satisfies the normal equations $A^{\mathsf{T}}A\hat{\mathbf{x}} = A^{\mathsf{T}}\mathbf{b}$.
- This fits any curve whose unknowns appear linearly.

Today we ask a different question: what does a square matrix actually *do*? You have met eigenvalues before, but we start again from the definition.

Warm-up

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Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Compute $A\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Which of the two answers is a multiple of the vector we started with?

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SOLUTION

$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, a multiple. On this vector, A is nothing more than multiplication by 3.

$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, which points in a new direction.

SECTION 1

Eigenvectors

The definition, and what it means

DEFINITION 19.1

For $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, a non-zero vector $\mathbf{v}$ with

$$A\mathbf{v} \;=\; \lambda\mathbf{v}$$

is an **eigenvector** of A, and the number λ is its **eigenvalue**.

KEY IDEA

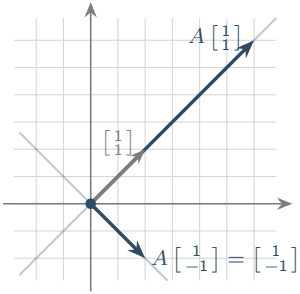
An eigenvector is a direction the matrix does not turn. A may stretch it, shrink it, or flip it, but the line $\mathrm{Span}\{\mathbf{v}\}$ is carried to itself.

The vector $\mathbf{0}$ is excluded, since $A\mathbf{0} = \lambda\mathbf{0}$ holds for every λ and would tell us nothing. The *number* 0 is perfectly allowed as an eigenvalue: it says that $A\mathbf{v} = \mathbf{0}$.

Seen in the plane

Take $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ from the warm-up.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is stretched by 3, and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is left exactly where it is. These are eigenvectors, with eigenvalues 3 and 1.



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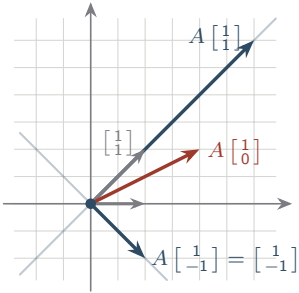
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not an eigenvector: it gets tilted to $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

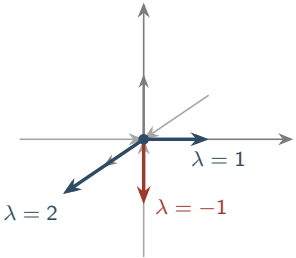
Along an eigendirection the matrix is just multiplication by a number. Finding enough such directions is what makes a matrix understandable.



Seen in $\mathbb{R}^3$

Take $A = \mathrm{Diag}(2, 1, -1)$: a stretch along x, nothing along y, a flip along z.

The three axes are eigendirections, with eigenvalues 2, 1 and -1. Every other direction gets tilted.



NOT EXAMINABLE

SECTION 2

Finding eigenvalues and eigenvectors

A reminder on determinants

$$\mathrm{Det}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc,$$

$$\mathrm{Det}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\,\mathrm{Det}\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\,\mathrm{Det}\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\,\mathrm{Det}\begin{bmatrix} d & e \\ g & h \end{bmatrix}.$$

The signs alternate $+, -, +$. One may expand along any row or column, with a chessboard of signs starting from $+$ at the top left, so pick the one with most zeros.

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WHAT IT IS FOR

For a square matrix B, the following say the same thing:

$$\text{Det}\, B = 0 \iff B \text{ is not invertible} \iff \text{Ker}\, B \neq \{\mathbf{0}\}.$$

From eigenvectors to a polynomial

Move everything in $A\mathbf{v} = \lambda\mathbf{v}$ to one side:

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \quad \iff \quad (A - \lambda I)\,\mathbf{v} = \mathbf{0}.$$

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DEFINITION 19.2

The **characteristic polynomial** of $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is $\chi_A(\lambda) \coloneqq \mathrm{Det}(A - \lambda I)$, a polynomial of degree n in λ. The eigenvalues of A are precisely its roots.

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CAUTION

$A - \lambda I$ has λ subtracted from the diagonal entries only.

A worked reminder

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$\lambda = 3$: $A - 3I = \left[\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$, kernel spanned by $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.
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Any non-zero multiple of an eigenvector is again one, so there is no single correct answer: $\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right]$ is just as good as $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.

Eigenspaces

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> CAUTION
>
> If $A - \lambda I$ row reduces to something with no free variable, then λ is not an eigenvalue, and there is a mistake further up.

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The eigenvalues are $\lambda = 6$ and $\lambda = 2$, the latter a double root.

No common factor? Multiply out and try small integers dividing the constant term. Here $\lambda^3 - 10\lambda^2 + 28\lambda - 24$ vanishes at $\lambda = 2$, and dividing by $\lambda - 2$ leaves a quadratic.

A 3×3 example (2/2)

$\lambda = 6$. $A - 6I = \begin{bmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$.

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Two free variables $y = s$, $z = t$ and $x = -s - t$, so the eigenspace has basis $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

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Check one: $A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. **Yes.** The double root gave a two-dimensional eigenspace. As we will see, that is not automatic.

Two cheap checks

PROPOSITION 19.5

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, listed with repetition. Then

$$\lambda_1 + \cdots + \lambda_n = \mathrm{Tr}\, A \qquad \textit{and} \qquad \lambda_1 \cdots \lambda_n = \mathrm{Det}\, A,$$

where Tr A *is the sum of the diagonal entries.*

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Also, a triangular matrix has its eigenvalues on the diagonal. A warning: a real matrix need not have any real eigenvalues. A rotation by 90° turns every direction, and its characteristic polynomial $\lambda^2 + 1$ has no real root. In this module, every eigenvalue we compute will be real.

SECTION 3

Diagonalisation

A basis of eigenvectors

Suppose $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ has n independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, with $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$. Put them in the columns of a matrix E, and the eigenvalues on the diagonal of D:

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$$AE = \begin{bmatrix} | & & | \\ A\mathbf{v}_1 & \cdots & A\mathbf{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\mathbf{v}_1 & \cdots & \lambda_n\mathbf{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = ED.$$

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DEFINITION 19.7

A is **diagonalisable** if $A = EDE^{-1}$ for an invertible matrix E and a diagonal matrix D.

This is a change of basis: in the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the map A simply scales each coordinate.

When is a matrix diagonalisable?

THEOREM 19.8

$A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is diagonalisable if and only if it has n independent eigenvectors, that is, if and only if the dimensions of its eigenspaces add up to n.

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THE METHOD

Collect a basis of every eigenspace. Eigenvectors from different eigenspaces are automatically independent, so all that matters is how many you have.

- n of them: put them in E, and the eigenvalues in D.
- Fewer than n: A is not diagonalisable.

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- Fewer than n: A is not diagonalisable.

In particular, n distinct eigenvalues give n independent eigenvectors, so such a matrix is always diagonalisable. Repeated eigenvalues need a closer look.

Two examples

EXAMPLE 19.9

For $A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$, with eigenvalues $3, 1$:

$$E = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},$$

and $E^{-1} = \frac{1}{2}\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

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EXAMPLE 19.10

For the 3×3 matrix, the eigenspaces have dimensions 1 and 2, and $1 + 2 = 3$, so

$$E = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

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ORDER MATTERS

Column i of E must be an eigenvector for the i-th diagonal entry of D. To check an answer, compare AE with ED: this avoids computing E^{-1}.

When it fails

EXAMPLE 19.11

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ has $\mathrm{Det}(A - \lambda I) = (2 - \lambda)^2$, so $\lambda = 2$ is the only eigenvalue.

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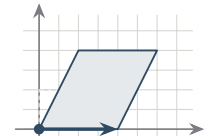
$A - 2I = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ has a one-dimensional kernel, spanned by $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. There is no second independent eigenvector, so A is not diagonalisable.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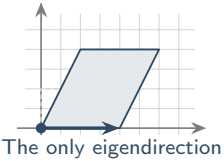

CAUTION

A repeated eigenvalue is not by itself a failure, as our 3×3 example shows. What counts is the dimension of the eigenspace.

SECTION 4

Powers and the long run

Powers of a matrix

Why go to all this trouble? Because the E^{-1} and the E cancel in the middle:

$$A^2 = ED\,(E^{-1}E)\,DE^{-1} = ED^2E^{-1}, \qquad \text{and in general} \qquad \boxed{A^k = ED^kE^{-1}}.$$

Powers of a matrix

Why go to all this trouble? Because the E^{-1} and the E cancel in the middle:

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Powers of a diagonal matrix are easy: $D^k = \mathrm{Diag}(\lambda_1^k, \ldots, \lambda_n^k)$.

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Powers of a diagonal matrix are easy: $D^k = \mathrm{Diag}(\lambda_1^k, \ldots, \lambda_n^k)$.

For example, for $A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$,

$$A^k = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{bmatrix}.$$

Check with $k = 1$: $\frac{1}{2}\left[\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right] = A$. And A^{10} is now no harder than A^2.

Systems that move in steps

Many things change in discrete steps by a fixed rule $\mathbf{x}_{k+1} = A\mathbf{x}_k$. Then

$$\mathbf{x}_1 = A\mathbf{x}_0, \quad \mathbf{x}_2 = A^2\mathbf{x}_0, \quad \ldots, \quad \boxed{\mathbf{x}_k = A^k\mathbf{x}_0 = ED^kE^{-1}\mathbf{x}_0}.$$

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EXAMPLE 19.12

A machine is either working or broken. A working machine breaks down during the day with probability 0.1, and a broken one is repaired with probability 0.5. If $\mathbf{x}_k$ lists the probabilities of working and of being broken on day k, then

$$\mathbf{x}_{k+1} = M\mathbf{x}_k, \qquad M = \begin{bmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The first column says where "working" goes, the second where "broken" goes, and each adds up to 1. The machine starts the first day in working order.

Solving it (1/2)

$$\text{Det}(M - \lambda I) = (0.9 - \lambda)(0.5 - \lambda) - 0.05 = \lambda^2 - 1.4\lambda + 0.4 = (\lambda - 1)(\lambda - 0.4).$$

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$\lambda = 1$: $M - I = \left[\begin{smallmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{smallmatrix}\right]$, so $x = 5y$, and the kernel is spanned by $\left[\begin{smallmatrix} 5 \\ 1 \end{smallmatrix}\right]$.
$\lambda = 0.4$: $M - 0.4I = \left[\begin{smallmatrix} 0.5 & 0.5 \\ 0.1 & 0.1 \end{smallmatrix}\right]$, so $x = -y$, and the kernel is spanned by $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$.

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$$E = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}, \qquad E^{-1} = \frac{1}{6}\begin{bmatrix} 1 & 1 \\ 1 & -5 \end{bmatrix}.$$

Two distinct eigenvalues, so M is diagonalisable without any further checking. For the inverse, recall that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$.

Solving it (2/2)

Work from the right, so that only vectors ever appear:

$$E^{-1}\mathbf{x}_0 = \frac{1}{6}\begin{bmatrix}1\\1\end{bmatrix}, \qquad D^k E^{-1}\mathbf{x}_0 = \frac{1}{6}\begin{bmatrix}1\\0.4^k\end{bmatrix}, \qquad \mathbf{x}_k = ED^k E^{-1}\mathbf{x}_0 = \frac{1}{6}\begin{bmatrix}5+0.4^k\\1-0.4^k\end{bmatrix}.$$

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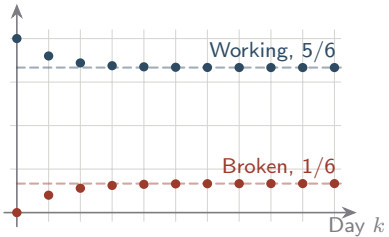
As $k \to \infty$, we have $0.4^k \to 0$, so

$$\mathbf{x}_k \longrightarrow \frac{1}{6}\begin{bmatrix}5\\1\end{bmatrix}.$$

In the long run the machine works 5 days out of 6. The limit is a multiple of the eigenvector for $\lambda = 1$.



Check with $k = 1$: $\frac{1}{6}\left[\begin{smallmatrix}5.4\\0.6\end{smallmatrix}\right] = \left[\begin{smallmatrix}0.9\\0.1\end{smallmatrix}\right] = M\mathbf{x}_0$.

What decides the long run

If $A\mathbf{v} = \lambda\mathbf{v}$, then $A^k\mathbf{v} = \lambda^k\mathbf{v}$. So when $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ in a basis of eigenvectors,

$$\mathbf{x}_k \;=\; c_1\lambda_1^k\,\mathbf{v}_1 + \cdots + c_n\lambda_n^k\,\mathbf{v}_n,$$

which is $ED^kE^{-1}\mathbf{x}_0$ with $E^{-1}\mathbf{x}_0$ written out as the c_i.

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$\lvert\lambda\rvert < 1$	That term dies out.
$\lambda = 1$	That term stays exactly as it is.
$\lvert\lambda\rvert > 1$	That term grows without bound, and the largest such $\lvert\lambda\rvert$ wins.
$\lambda < 0$	The sign of that term alternates from step to step.

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$\lambda < 0$	The sign of that term alternates from step to step.

KEY IDEA

The eigenvalues tell us the long-term behaviour, without any need to iterate.

A population that grows

A species has juveniles and adults. Each adult produces 2 juveniles per year, all juveniles become adults, and all adults survive one more year:

$$\mathbf{x}_{k+1} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{x}_k, \qquad \text{Det}(A - \lambda I) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

NOT EXAMINABLE

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The eigenvectors are $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ for $\lambda = 2$ and $\left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]$ for $\lambda = -1$. Starting from $\mathbf{x}_0 = \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right]$,

$$\mathbf{x}_k = 2^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1)^k \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

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$$\mathbf{x}_k = 2^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1)^k \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

The population roughly doubles every year, and the 2^k term soon swamps the other. Whatever the start, the ratio of juveniles to adults settles at $1 : 1$, the direction of the eigenvector with the largest $|\lambda|$.

NOT EXAMINABLE

Summary of Lecture 19

EIGENVECTOR	$A\mathbf{v} = \lambda\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$: a direction that A only scales.
EIGENVALUES	The roots of $\chi_A(\lambda) = \mathrm{Det}(A - \lambda I)$. Check with $\sum \lambda_i = \mathrm{Tr}\, A$.
EIGENSPACE	$\mathrm{Ker}(A - \lambda I)$: row reduce and read off a basis.
DIAGONALISABLE	$A = EDE^{-1}$ exactly when there are n independent eigenvectors. These are the columns of E, in the same order as the eigenvalues in D.
FAILURE	Occurs when the eigenspace dimensions add up to less than n.
POWERS	$A^k = ED^kE^{-1}$, so $\mathbf{x}_{k+1} = A\mathbf{x}_k$ has $\mathbf{x}_k = ED^kE^{-1}\mathbf{x}_0$.
LONG RUN	Terms with $\lvert\lambda\rvert < 1$ die out, those with $\lambda = 1$ remain, those with $\lvert\lambda\rvert > 1$ grow.

Before next time

AT HOME

1. Find the eigenvalues and eigenvectors of $A = \left[\begin{smallmatrix} 4 & 1 \\ 2 & 3 \end{smallmatrix}\right]$, and write $A = EDE^{-1}$.
2. Let $B = \left[\begin{smallmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{smallmatrix}\right]$. Find its characteristic polynomial, a basis of each eigenspace, and diagonalise B if possible.
3. Is $\left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ diagonalisable? Explain why, or why not.
4. Let $\mathbf{x}_{k+1} = A\mathbf{x}_k$ with $A = \left[\begin{smallmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{smallmatrix}\right]$ and $\mathbf{x}_0 = \left[\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right]$. Find a formula for $\mathbf{x}_k$, and describe what happens as $k \to \infty$.

END OF LECTURE 19

ilia.pirashvili@universityofgalway.ac.ie

Solution: question 1

SOLUTION

$\text{Det}(A - \lambda I) = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2)$.

$\lambda = 5$: $A - 5I = \left[\begin{smallmatrix} -1 & 1 \\ 2 & -2 \end{smallmatrix}\right]$, spanned by $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.

$\lambda = 2$: $A - 2I = \left[\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}\right]$, spanned by $\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]$.

$$E = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \qquad D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \qquad E^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Solution: question 1

SOLUTION

$\text{Det}(A - \lambda I) = (4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2)$.

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$$E = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \qquad D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}, \qquad E^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Check: $A\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 5 \\ 5 \end{smallmatrix}\right]$ and $A\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2 \\ -4 \end{smallmatrix}\right]$. **Yes.**

Solution: question 2

SOLUTION

Expanding along the second row, $\mathrm{Det}(B-\lambda I) = (1-\lambda)\big[(2-\lambda)^2-1\big] = -(\lambda-1)^2(\lambda-3)$.

$\lambda = 3$: $B - 3I = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$, so $y = 0$ and $x = z$. The eigenspace is spanned by $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$\lambda = 1$: $B - I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, so $x = -y - z$. A basis is $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

So $E = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \mathrm{Diag}(3,1,1)$. Check: $3+1+1 = 5 = \mathrm{Tr}\, B$. **Yes.**

Solution: question 3

SOLUTION

$\mathrm{Det}(A - \lambda I) = (3 - \lambda)^2$, so $\lambda = 3$ is the only eigenvalue. But $A - 3I = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ has a one-dimensional kernel. There is only one independent eigenvector and we need two, so it is **No.** diagonalisable.

Solution: question 4

SOLUTION

Det$(A-\lambda I)=\lambda^2-1.5\lambda+0.5=(\lambda-1)(\lambda-0.5)$, with eigenvectors $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$. So

$$E=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\qquad D=\begin{bmatrix}1 & 0\\ 0 & 0.5\end{bmatrix},\qquad E^{-1}=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\qquad E^{-1}\mathbf{x}_0=\begin{bmatrix}2\\ 2\end{bmatrix}.$$

Hence $\mathbf{x}_k=ED^kE^{-1}\mathbf{x}_0=\left[\begin{smallmatrix}2+2(0.5)^k\\ 2-2(0.5)^k\end{smallmatrix}\right]$.

Solution: question 4

SOLUTION

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Hence $\mathbf{x}_k=ED^kE^{-1}\mathbf{x}_0=\left[\begin{smallmatrix}2+2(0.5)^k\\ 2-2(0.5)^k\end{smallmatrix}\right]$.

As $k\to\infty$, $0.5^k\to 0$ and $\mathbf{x}_k\to\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$: the total of 4 gets shared out equally.

Check with $k=1$: $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]=A\mathbf{x}_0$. **Yes.**