LECTURE 16

Inner products and orthogonality

Putting length and angle back into a vector space

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
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Where we got to

LAST LECTURE

- Row $A \subseteq \mathbb{R}^n$ and Coker $A \subseteq \mathbb{R}^m$ join Col A and Ker A as the four fundamental subspaces.
- dim Row $A =$ dim Col $A = r$, while dim Ker $A = n - r$ and dim Coker $A = m - r$.
- Row operations preserve Row A and Ker A, but not Col A.

In Lecture 12 we noted that length is *not* a vector space notion, as it cannot be built from $+$ and scaling alone. But what if we added it as an extra strucutre?

Warm-up

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Compute $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ ▶ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ ▶. What is special about the second?

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SOLUTION

$3 + 0 - 1 = 2$, and $-2 + 2 = 0$.

A zero dot product meant *perpendicular*, back in Lecture 1. Today, we use it as a general definition rather than an observation.

SECTION 1

Inner products

Definition

DEFINITION 16.1 ▸ · Inner product on $\mathbb{R}^n$

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{u} \cdot \mathbf{v} \coloneqq u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i \in \mathbb{R}.$$

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A vector of $\mathbb{R}^n$ is an $n \times 1$ matrix, so this is a matrix product once one side is transposed ▸:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathsf{T}} \mathbf{v} \qquad \underbrace{(1 \times n)}_{\mathbf{u}^{\mathsf{T}}} \underbrace{(n \times 1)}_{\mathbf{v}} = 1 \times 1.$$

Also called the **dot product** or the **scalar product**. Scalar, since the answer is a number, not a vector.

The rules

PROPOSITION 16.2

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$*:*

SYMMETRY	$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	
DISTRIBUTIVITY	$((\mathbf{u} + \mathbf{v})) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$	*(Also called* ***additivity****)*
SCALARS	$(c\,\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\,\mathbf{v}) = c\,(\mathbf{u} \cdot \mathbf{v})$	
POSITIVITY	$\mathbf{u} \cdot \mathbf{u} \geq 0$, *with equality only for* $\mathbf{u} = \mathbf{0}$	

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The first three formulas are immediately checked coordinate by coordinate. For positivity, observe that $\mathbf{u} \cdot \mathbf{u} = \sum u_i^2$, a sum of squares, which vanishes only when every u_i does.

The first three are linearity in one variable, with the other fixed. The fourth will give meaning to lengths.

Length

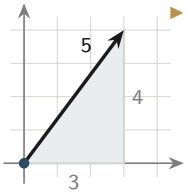
DEFINITION 16.3

The **length** (or **norm**) of $\mathbf{v} \in \mathbb{R}^n$ is

$$\|\mathbf{v}\| \;:=\; \sqrt{\mathbf{v}\cdot\mathbf{v}} \;=\; \sqrt{v_1^2 + \cdots + v_n^2}.$$

Positivity is what lets us take the square root, and it gives $\|\mathbf{v}\| = 0$ only for $\mathbf{v} = \mathbf{0}$.

In $\mathbb{R}^2$ this is Pythagoras: the vector $\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]$ is the hypotenuse of a triangle with sides 3 and 4, so its length is $\sqrt{9+16} = 5$.



Also $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$, since $(c\,\mathbf{v})\cdot(c\,\mathbf{v}) = c^2\,\mathbf{v}\cdot\mathbf{v} = c^2\,\mathbf{v}^2$.

Unit vectors

DEFINITION 16.4

A vector with $\|\mathbf{u}\| = 1$ is a called **unit vector**. For any $\mathbf{v} \neq \mathbf{0}$,

$$\hat{\mathbf{v}} := \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

is a unit vector in the same direction. This process producing it is called **normalising**.

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$\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ has $\|\mathbf{v}\| = 5$, so $\hat{\mathbf{v}} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$, and indeed $\frac{9}{25} + \frac{16}{25} = 1$.

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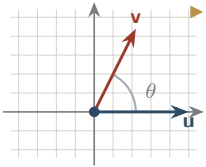
Normalising separates a vector into a direction and a size: $\mathbf{v} = \|\mathbf{v}\|\,\hat{\mathbf{v}}$. Rather a lot like polar coordinates.

Angles

For non-zero $\mathbf{u}, \mathbf{v}$, the angle θ between them satisfies

$$\mathbf{u} \cdot \mathbf{v} \;=\; \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta.$$



So the inner product measures how much the two vectors agree in direction, scaled by their sizes.

- $\theta < 90°$: positive.
- $\theta = 90°$: zero.
- $\theta > 90°$: negative.

Rearranged, $\cos\theta = \hat{\mathbf{u}} \cdot \hat{\mathbf{v}}$. So, normalise first, and the inner product *is* the cosine.

SECTION 2

Orthogonality

Definition, and why it is the right one

DEFINITION 16.6

$\mathbf{u}$ and $\mathbf{v}$ are **orthogonal**, written $\mathbf{u} \perp \mathbf{v}$, if $\mathbf{u} \cdot \mathbf{v} = 0$.

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Since $\cos 90^\circ = 0$, this agrees with the notion of "perpendicularity" in $\mathbb{R}^2$ and $\mathbb{R}^3$, but makes sense in $\mathbb{R}^{17}$ too.

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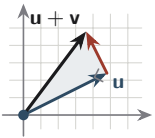
And $\mathbf{0}$ is orthogonal to *everything*, itself included. It *the only* vector orthogonal to itself.

Pythagoras in $\mathbb{R}^n$



Pythagoras, in any dimension. Three points span a plane, so this happens in a (possibly askew) plane.

THEOREM 16.8

$\mathbf{u} \perp \mathbf{v}$ *if and only if*

$$\|\mathbf{u}+\mathbf{v}\|^2 \;=\; \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

PROOF.

$\|\mathbf{u}+\mathbf{v}\|^2$

□

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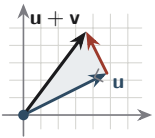
PROOF.

$\|\mathbf{u}+\mathbf{v}\|^2$ ← Definition: $\|\mathbf{w}\| = \sqrt{\mathbf{w}\cdot\mathbf{w}}$, here with $\mathbf{w} = \mathbf{u}+\mathbf{v}$.



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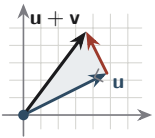

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PROOF.

$$\|\mathbf{u}+\mathbf{v}\|^2 = \left(\sqrt{((\mathbf{u}+\mathbf{v}))\cdot(\mathbf{u}+\mathbf{v})}\right)^2$$

□

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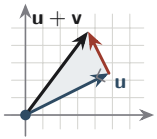

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PROOF.

$\|\mathbf{u}+\mathbf{v}\|^2 = ((\mathbf{u}+\mathbf{v})) \cdot (\mathbf{u}+\mathbf{v})$

The square root and the square cancel, since $\mathbf{w} \cdot \mathbf{w} \geqslant 0$ by positivity.

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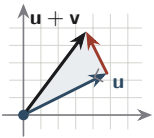

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PROOF.

$$\|\mathbf{u}+\mathbf{v}\|^2 = ((\mathbf{u}+\mathbf{v}))\cdot\underbrace{(\mathbf{u}+\mathbf{v})}_{\mathbf{w}}$$

Distributivity: $((\mathbf{a}+\mathbf{b}))\cdot\mathbf{w} = \mathbf{a}\cdot\mathbf{w}+\mathbf{b}\cdot\mathbf{w}$, here with $\mathbf{w} = \mathbf{u}+\mathbf{v}$.

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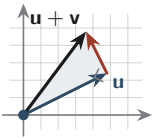

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$$\|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v})$$

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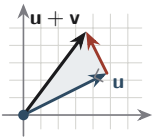

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PROOF.

$$\|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v})$$

Symmetry, used twice: $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.

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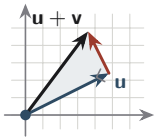

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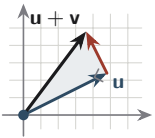

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PROOF.

$$\|\mathbf{u}+\mathbf{v}\|^2 = ((\mathbf{u}+\mathbf{v}))\cdot\mathbf{u} + ((\mathbf{u}+\mathbf{v}))\cdot\mathbf{v}$$

Distributivity again, in the first slot of each term.

Pythagoras in $\mathbb{R}^n$

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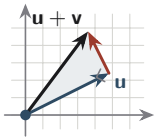

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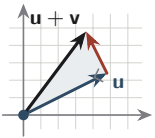

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PROOF.

$$\|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}\cdot\mathbf{u} + \boxed{\mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v}} + \mathbf{v}\cdot\mathbf{v}$$

Symmetry: $\mathbf{v}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{v}$, so the two middle terms are equal and can be grouped.

Pythagoras in $\mathbb{R}^n$

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$$\|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}\cdot\mathbf{u} + 2\,\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}$$

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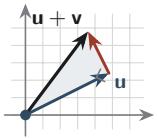

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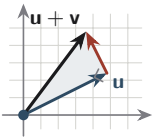

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PROOF.

$$\|\mathbf{u}+\mathbf{v}\|^2 = \boxed{\mathbf{u}\cdot\mathbf{u}} + 2\,\mathbf{u}\cdot\mathbf{v} + \boxed{\mathbf{v}\cdot\mathbf{v}}$$

Definition, reversed: $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2$, with $\mathbf{w} = \mathbf{u}$ and $\mathbf{w} = \mathbf{v}$.

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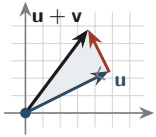

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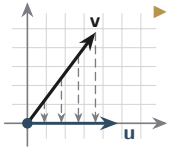
PROOF.

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\,\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$$

So the two sides agree exactly when $2\,\mathbf{u}\cdot\mathbf{v} = 0$, that is, when

The shadow of one vector on another

Imagine a horizontal light (tube) shining straight down. The rays hitting it at a right angle. The **shadow** of $\mathbf{v}$ is where it lands, which is clearly a multiple $t\mathbf{u}$ of $\mathbf{u}$. The ray reaching it from $\mathbf{v}$ is $\mathbf{v} - t\mathbf{u}$.



$\mathbf{p}$ is the shadow of $\mathbf{v}$ on the line through $\mathbf{u}$.

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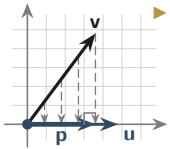


$\mathbf{p}$ is the shadow of $\mathbf{v}$ on the line through $\mathbf{u}$.

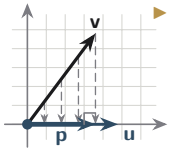
By linearity,

$$(\mathbf{v} - t\mathbf{u}) \cdot \mathbf{u} \;=\; \mathbf{v} \cdot \mathbf{u} - t\,\mathbf{u} \cdot \mathbf{u},$$

and since $\mathbf{u} \cdot \mathbf{u} \neq 0$ this is zero for exactly one t, namely $t = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$.

The shadow of one vector on another

Imagine a horizontal light (tube) shining straight down. The rays hitting it at a right angle. The **shadow** of $\mathbf{v}$ is where it lands, which is clearly a multiple $t\mathbf{u}$ of $\mathbf{u}$. The ray reaching it from $\mathbf{v}$ is $\mathbf{v} - t\mathbf{u}$.



$\mathbf{p}$ is the shadow of $\mathbf{v}$ on the line through $\mathbf{u}$.

By linearity,

$$(\mathbf{v} - t\mathbf{u}) \cdot \mathbf{u} \;=\; \mathbf{v} \cdot \mathbf{u} - t\,\mathbf{u} \cdot \mathbf{u},$$

and since $\mathbf{u} \cdot \mathbf{u} \neq 0$ this is zero for exactly one t, namely $t = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$.

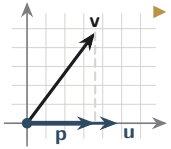
DEFINITION 16.9 ▸

The **shadow** of $\mathbf{v}$ on $\mathbf{u}$ is

$$\mathrm{Proj}_{\mathbf{u}}(\mathbf{v}) \;=\; \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\,\mathbf{u}.$$

Making a perpendicular from two vectors

Let $\mathbf{p} = t\mathbf{u}$ be the shadow of $\mathbf{v}$ on $\mathbf{u}$, and let $\mathbf{w} = \mathbf{v} - \mathbf{p}$ be what is left over, so that $\mathbf{v} = \mathbf{p} + \mathbf{w}$.



$\mathbf{v} = \mathbf{p} + \mathbf{w}$ with $\mathbf{w} \perp \mathbf{u}$ and $\mathbf{p} \perp \mathbf{w}$

Making a perpendicular from two vectors

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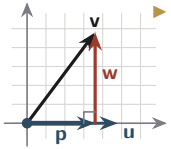
$$\mathbf{w} \cdot \mathbf{u} \;=\; \mathbf{v} \cdot \mathbf{u} - \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \cdot \mathbf{u} \;=\; 0$$

and hence, is orthogonal to $\mathbf{u}$.



$\mathbf{v} = \mathbf{p} + \mathbf{w}$ with $\mathbf{w} \perp \mathbf{u}$ and $\mathbf{p} \perp \mathbf{w}$

Making a perpendicular from two vectors

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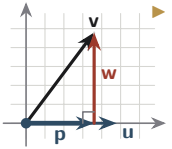


$\mathbf{v} = \mathbf{p} + \mathbf{w}$ with $\mathbf{w} \perp \mathbf{u}$ and $\mathbf{p} \perp \mathbf{w}$

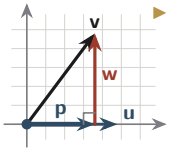
ORTHOGONAL DECOMPOSITION

So every $\mathbf{v}$ splits as $\mathbf{p} + \mathbf{w}$ with one piece along $\mathbf{u}$, one perpendicular to it. The pieces are also orthogonal to each other, since $\mathbf{p} \cdot \mathbf{w} = t\,\mathbf{u} \cdot \mathbf{w} = 0$.

The shadow is never longer than the vector

Since $\mathbf{p} \perp \mathbf{w}$, Pythagoras applies to $\mathbf{v} = \mathbf{p} + \mathbf{w}$:

$$\|\mathbf{v}\|^2 \;=\; \|\mathbf{p}\|^2 + \|\mathbf{w}\|^2 \;\geq\; \|\mathbf{p}\|^2.$$



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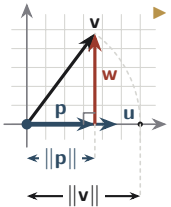
$$\|\mathbf{v}\|^2 \;=\; \|\mathbf{p}\|^2 + \|\mathbf{w}\|^2 \;\geq\; \|\mathbf{p}\|^2.$$

Geometric intuition: Swing $\mathbf{v}$ round until it lies along $\mathbf{u}$. Its tip lands at distance $\|\mathbf{v}\|$ from 0. The shadow will clearly stop short of that point, unless it is flush against $\mathbf{u}$.



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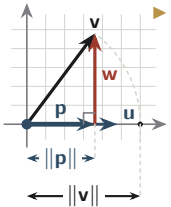

SHADOW $\leq$ VECTOR

$\|\mathbf{p}\| \leq \|\mathbf{v}\|$, with equality exactly when $\mathbf{w} = \mathbf{0}$.

SECTION 3

Two inequalities

Cauchy–Schwarz

THEOREM 16.10 ▸ · Cauchy–Schwarz

For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

Cauchy–Schwarz

THEOREM 16.10 · Cauchy–Schwarz

For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \;\le\; \|\mathbf{u}\|\,\|\mathbf{v}\|.$$

PROOF.

If $\mathbf{u} = \mathbf{0}$ both sides are 0. Otherwise let $\mathbf{p} = t\mathbf{u}$, $t = \mathbf{v}\cdot\mathbf{u}/\|\mathbf{u}\|^2$, be the shadow of $\mathbf{v}$ on $\mathbf{u}$. The shadow is never longer than the vector, so

$$\|\mathbf{v}\|^2 \;\ge\; \|\mathbf{p}\|^2 \;=\; t^2\|\mathbf{u}\|^2 \;=\; \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^4}\,\|\mathbf{u}\|^2 \;=\; \frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2}.$$

Multiplying by $\|\mathbf{u}\|^2 > 0$ and taking square roots of the two non-negative sides gives the result. □

Cauchy–Schwarz

THEOREM 16.10 · Cauchy–Schwarz

For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \;\le\; \|\mathbf{u}\|\,\|\mathbf{v}\|.$$

PROOF.

If $\mathbf{u} = \mathbf{0}$ both sides are 0. Otherwise let $\mathbf{p} = t\mathbf{u}$, $t = \mathbf{v}\cdot\mathbf{u}/\|\mathbf{u}\|^2$, be the shadow of $\mathbf{v}$ on $\mathbf{u}$. The shadow is never longer than the vector, so

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Multiplying by $\|\mathbf{u}\|^2 > 0$ and taking square roots of the two non-negative sides gives the result. □

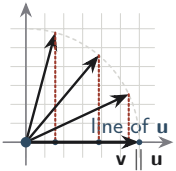
In words: the shadow of $\mathbf{v}$ on $\mathbf{u}$ has length $|\mathbf{u}\cdot\mathbf{v}|/\|\mathbf{u}\|$, and it cannot exceed $\|\mathbf{v}\|$, as stated 2 frames ago.

When is the shadow the whole vector?

For $\mathbf{u} \neq \mathbf{0}$, equality in Cauchy–Schwarz holds exactly when $\mathbf{w} = \mathbf{0}$ ▸:

$$\begin{array}{rrcccccc} & |\mathbf{u}\cdot\mathbf{v}| & = & \|\mathbf{u}\|\|\mathbf{v}\| & \iff & \|\mathbf{p}\| & = & \|\mathbf{v}\| \\ \iff & \|\mathbf{w}\|^2 & = & 0 & \iff & \mathbf{w} & = & \mathbf{0}. \end{array}$$

The first step holds because $\|\mathbf{p}\| = |\mathbf{u}\cdot\mathbf{v}|/\|\mathbf{u}\|$, the second because $\|\mathbf{v}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{w}\|^2$.



same length, bigger shadow as $\mathbf{v}$ turns to $\mathbf{u}$

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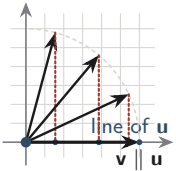


same length, bigger shadow as $\mathbf{v}$ turns to $\mathbf{u}$

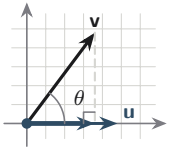
Note that $\mathbf{w} = \mathbf{0}$ means $\mathbf{v} = \mathbf{p} = t\mathbf{u}$, a multiple of $\mathbf{u}$. If $\mathbf{v} = s\mathbf{u}$ then $\mathbf{v}\cdot\mathbf{u} = s\mathbf{u}\cdot\mathbf{u}$, so $t = s$, $\mathbf{p} = \mathbf{v}$ and $\mathbf{w} = \mathbf{0}$. The shadow is the whole vector only when there is nothing sticking out sideways.

From shadows to angles

For $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, divide Cauchy–Schwarz by $\|\mathbf{u}\|\|\mathbf{v}\|$:

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1.$$

As such, it makes sense to define the angle θ between $\mathbf{u}$ and $\mathbf{v}$ as $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.



NOT EXAMINABLE

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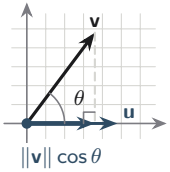


As such, it makes sense to define the angle θ between $\mathbf{u}$ and $\mathbf{v}$ as $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$.

The signed length of the shadow is $t = \frac{\mathbf{u}\cdot\mathbf{v}}{\|u\|}$. So, the ratio becomes

$$\cos\theta \;=\; \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \;=\; \frac{\text{signed length of the shadow}}{\|\mathbf{v}\|},$$

exactly the school definition of the cosine in a right triangle.

NOT EXAMINABLE

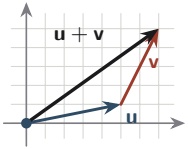
The triangle inequality

THEOREM 16.11

$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

PROOF.

Use the fact that $a \leq b \iff a^2 \leq b$ for $a, b \geq 0$, square and expand $\|\mathbf{u}+\mathbf{v}\|$, then apply Cauchy–Schwarz in the form $\mathbf{u}\cdot\mathbf{v} \leq |\mathbf{u}\cdot\mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|$, we get



Going straight is never longer than going via a corner.

$$\|\mathbf{u}+\mathbf{v}\|^2 \;=\; \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \;=\; \big(\|\mathbf{u}\| + \|\mathbf{v}\|\big)^2.$$

Both sides of the result are non-negative, so take square roots. □

Distance

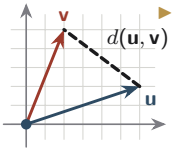
DEFINITION 16.12

The **distance** between $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is

$$d(\mathbf{u}, \mathbf{v}) \;:=\; \|\mathbf{u} - \mathbf{v}\|.$$

- $d(\mathbf{u}, \mathbf{v}) \geq 0$, zero only if $\mathbf{u} = \mathbf{v}$;
- $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$;
- $d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$.

The third is the triangle inequality applied to $\mathbf{u} - \mathbf{w} = (\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w})$.



Example: ▸

$$\begin{aligned} d\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}4\\6\end{bmatrix}\right) &= \left\|\begin{bmatrix}-3\\-4\end{bmatrix}\right\| \\ &= \sqrt{(-3)^2 + (-4)^2} = 5 \end{aligned}$$

SECTION 4

Orthogonal complements

Orthogonal to a whole subspace

DEFINITION 16.13

Let $H \subseteq \mathbb{R}^n$ be a subspace. A vector $\mathbf{v}$ is **orthogonal to H** if $\mathbf{v} \perp \mathbf{h}$ for every $\mathbf{h} \in H$. The set of all such vectors is the **orthogonal complement**

$$H^{\perp} = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{h} = 0 \text{ for all } \mathbf{h} \in H\}.$$

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$$H^{\perp} \;=\; \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{h} = 0 \;\text{ for all }\; \mathbf{h} \in H\}.$$

IF $H = \mathrm{Span}\{\mathbf{h}_1, \ldots, \mathbf{h}_k\}$ WITH $\mathbf{v} \perp \mathbf{h}_i$ FOR EACH i, THEN $\boxed{\mathbf{v} \perp H}$.

Indeed, for any linear combination, we have

$$\mathbf{v} \cdot \sum c_i \mathbf{h}_i \;=\; \sum c_i\, \mathbf{v} \cdot \mathbf{h}_i \;=\; 0.$$

In other words, it suffices to check on a spanning set, and in particular, on generators only.

$H^\perp$ is a subspace

THEOREM 16.14

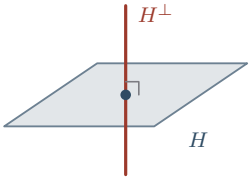
$H^\perp$ is a subspace of $\mathbb{R}^n$.

PROOF.

$\mathbf{0} \cdot \mathbf{h} = 0$ always, giving (U1). If $\mathbf{u}, \mathbf{v} \perp H$ then for every $\mathbf{h} \in H$ and ever $\lambda \in \mathbb{R}$,

$$((\mathbf{u}+\mathbf{v}))\cdot\mathbf{h} \;=\; 0+0 \;=\; 0, \quad (\lambda\mathbf{u})\cdot\mathbf{h} \;=\; \lambda\cdot 0 \;=\; 0,$$

which are (U2) and (U3). □



A plane and the line through $\mathbf{0}$ perpendicular to it.

Computing one

EXAMPLE 16.15 ▸

For $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$, a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ lies in $H^\perp$ exactly when $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$, meaning when

$$x + y = 0, \qquad y + z = 0.$$

Computing one

EXAMPLE 16.15 ▸

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$$x + y = 0, \qquad y + z = 0.$$

Setting $y = t$ gives $x = -t$ and $z = -t$, so $H^\perp = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}$, and

$$\dim H + \dim H^\perp = 2 + 1 = 3.$$

Computing one

EXAMPLE 16.15 ▸

For $H = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\} \subseteq \mathbb{R}^3$, a vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ lies in $H^\perp$ exactly when $\begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} = 0$ and $\begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}0\\1\\1\end{bmatrix} = 0$, meaning when

$$x + y = 0, \qquad y + z = 0.$$

Setting $y = t$ gives $x = -t$ and $z = -t$, so $H^\perp = \text{Span}\left\{\begin{bmatrix}-1\\1\\-1\end{bmatrix}\right\}$, and

$$\dim H + \dim H^\perp = 2 + 1 = 3.$$

KEY IDEA

The two conditions are the rows of $A = \begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}$, so $H^\perp = \text{Ker}\, A$ when the rows of A span H ▸. That is, $(\text{Row}\, A)^\perp = \text{Ker}\, A$.

Summary of Lecture 16

INNER PRODUCT	$\mathbf{u}\cdot\mathbf{v}=\sum u_iv_i=\mathbf{u}^{\mathsf{T}}\mathbf{v}$. Symmetric, linear in each entry, and $\mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|^2\geq 0$ with $\mathbf{u}\cdot\mathbf{u}=0$ iff $\mathbf{u}=\mathbf{0}$.
LENGTH	$\|\mathbf{v}\|=\sqrt{\mathbf{v}\cdot\mathbf{v}}$; normalising gives $\hat{\mathbf{v}}=\mathbf{v}/\|\mathbf{v}\|$, a unit vector.
ANGLE	$\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$, so $\mathbf{u}\perp\mathbf{v}$ means $\mathbf{u}\cdot\mathbf{v}=0$.
PYTHAGORAS	$\mathbf{u}\perp\mathbf{v}\iff\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2$.
INEQUALITIES	$\|\mathbf{u}\cdot\mathbf{v}\|\leq\|\mathbf{u}\|\|\mathbf{v}\|$ (Cauchy–Schwarz), and $\|\mathbf{u}+\mathbf{v}\|\leq\|\mathbf{u}\|+\|\mathbf{v}\|$.
COMPLEMENT	$H^\perp$ is a subspace; checking a spanning set of H suffices, and $(\text{Row}\,A)^\perp=\text{Ker}\,A$.

Before next time

AT HOME

1. For $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$: compute $\mathbf{u} \cdot \mathbf{v}$ ▸, $\|\mathbf{u}\|$ ▸, $\|\mathbf{v}\|$ ▸, $d(\mathbf{u}, \mathbf{v})$ ▸, and $\hat{\mathbf{u}}$ ▸. Are they orthogonal?
2. Write $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ as $\mathbf{p} + \mathbf{w}$ with $\mathbf{p}$ a multiple of ▸ $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} \perp \mathbf{u}$ ▸. Verify Pythagoras on your answer.
3. Find $H^\perp$ for $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$, and check $\dim H + \dim H^\perp = 4$.

END OF LECTURE 16

ilia.pirashvili@universityofgalway.ac.ie

Solution: question 1

SOLUTION

$\mathbf{u} \cdot \mathbf{v} = 2 - 2 + 0 = 0$ ▸, so **Yes.** they are orthogonal.

$\|\mathbf{u}\| = \sqrt{1+4+4} = 3$ ▸ and $\|\mathbf{v}\| = \sqrt{4+1+0} = \sqrt{5}$ ▸.

$\mathbf{u} - \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, so $d(\mathbf{u}, \mathbf{v}) = \sqrt{1+9+4} = \sqrt{14}$ ▸.

$\hat{\mathbf{u}} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Solution: question 1

SOLUTION

$\mathbf{u} \cdot \mathbf{v} = 2 - 2 + 0 = 0$ ▸, so **Yes.** they are orthogonal.

$\|\mathbf{u}\| = \sqrt{1+4+4} = 3$ ▸ and $\|\mathbf{v}\| = \sqrt{4+1+0} = \sqrt{5}$ ▸.

$\mathbf{u} - \mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, so $d(\mathbf{u}, \mathbf{v}) = \sqrt{1+9+4} = \sqrt{14}$ ▸.

$\hat{\mathbf{u}} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Check Pythagoras: $9 + 5 = 14$, as it must be, since the two are orthogonal.

Solution: question 2 ▸

SOLUTION

$\mathbf{v} \cdot \mathbf{u} = 3$ and $\mathbf{u} \cdot \mathbf{u} = 1$, so

$$\mathbf{p} = 3\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}, \qquad \mathbf{w} = \mathbf{v} - \mathbf{p} = \begin{bmatrix}0\\4\end{bmatrix}.$$

Indeed $\mathbf{w} \cdot \mathbf{u} = 0$.

Solution: question 2 ▸

SOLUTION

$\mathbf{v} \cdot \mathbf{u} = 3$ and $\mathbf{u} \cdot \mathbf{u} = 1$, so

$$\mathbf{p} = 3\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}, \qquad \mathbf{w} = \mathbf{v} - \mathbf{p} = \begin{bmatrix}0\\4\end{bmatrix}.$$

Indeed $\mathbf{w} \cdot \mathbf{u} = 0$.

And $\|\mathbf{p}\|^2 + \|\mathbf{w}\|^2 = 9 + 16 = 25 = \|\mathbf{v}\|^2$.

Here $\mathbf{u}$ was an axis, so the split is just the coordinates. The formula does the same job for any $\mathbf{u}$.

Solution: question 3 ▸

SOLUTION

$\mathbf{v} \in H^{\perp}$ exactly when $x_1 + x_3 = 0$ and $x_2 + x_4 = 0$. Setting $x_3 = s$, $x_4 = t$ gives $x_1 = -s$ and $x_2 = -t$, so

$$H^{\perp} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

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Both have dimension 2, and $2 + 2 = 4$.

That $\dim H + \dim H^{\perp} = n$ always holds is rank–nullity in disguise, since $(\operatorname{Row} A)^{\perp} = \operatorname{Ker} A$.