LECTURE 15

The four fundamental subspaces

Four invariants of a matrix

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Recall

LAST LECTURE

- Rank $A = \dim(\text{Im}\, A)$, Nullity $A = \dim(\text{Ker}\, A)$, and Rank A + Nullity $A = n$.
- Invertible $\iff$ Rank $A = n$ $\iff$ Nullity $A = 0$ $\iff$ columns are basis, if A is square.
- Rank $A =$ Rank A^{T}, hence Rank $A \leq \min(m, n)$.

Warm-up

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Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$. Find Rank A ▸, Nullity A ▸, and a basis of Im A ▸.

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SOLUTION

$R_2 \leftarrow R_2 - 2R_1$ gives $\mathrm{RREF}(A) = \left[\begin{smallmatrix} \mathbf{1} & 2 & 3 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, hence one pivot. So, Rank $A = 1$ and Nullity $A = 3 - 1 = 2$.

The pivot is in column 1, so Im $A = \mathrm{Span}\{[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}]\}$, a line in $\mathbb{R}^2$.

SECTION 1

Row space

The row space

A matrix has rows as well as columns, and each row of an $m \times n$ matrix is a $1 \times n$ matrix, called **row vector**.

DEFINITION 15.1 ▸ · Row space

The **row space** of A is the span of its rows,

$$\text{Row}\, A \;=\; \text{Span}\{a_{1,-}, \ldots, a_{m,-}\} \;\subseteq\; \mathbb{R}^n.$$

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Transposing turns rows into columns ▸, so

$$\operatorname{Row} A \;=\; \operatorname{Im} A^{\mathsf{T}}$$

once we agree to write the rows as columns. That is how every question about $\operatorname{Row} A$ will be answered: transpose, and use what we know.

Row operations preserve the row space

KEY IDEA

Each row operation replaces the rows by *combinations* of themselves, and is reversible. So the rows of the new and old lie in each others span, i.e., $\text{Row}\, A = \text{Row}\, \text{RREF}(A)$.

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KEY IDEA

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The exact opposite of the column space, which row operations destroy.

BASIS FOR $\text{Row}\, A$

The non-zero rows of $\text{RREF}(A)$ are a basis of $\text{Row}\, A$ as they span it, and each has a leading 1 where the others are zero, so they are independent (Proposition 8.4).

Row rank equals column rank

THEOREM 15.2

$\dim(\text{Row}\, A) = \dim(\text{Im}\, A) = \text{Rank}\, A.$

PROOF.

In $\text{RREF}(A)$ the non-zero rows number the pivots, since each has exactly one; that number is $\text{Rank}\, A$. And those rows are a basis of $\text{Row}\, \text{RREF}(A) = \text{Row}\, A$. This is effectively Proposition 8.4 again. □

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KEY IDEA

So $\text{Rank}\, A = \text{Rank}\, A^{\mathsf{T}}$, used last lecture. Row space and column space live in different spaces and are usually different sets, yet always have the same dimension.

The cokernel

Three subspaces so far: Im A, Ker A, Row A. The fourth is the transpose of Ker.

DEFINITION 15.3 · Cokernel

The **cokernel** of A is the set of row vectors $\mathbf{y}$ with $\mathbf{y}A = \mathbf{0}$, that is

$$\operatorname{Coker} A = \left\{ \mathbf{y} \;\middle|\; \mathbf{y}^{\mathsf{T}} \in \operatorname{Ker} A^{\mathsf{T}} \right\} \subseteq \mathbb{R}^m.$$

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Transposing $\mathbf{y}A = \mathbf{0}$ gives $A^{\mathsf{T}}\mathbf{y}^{\mathsf{T}} = \mathbf{0}$, so the two descriptions really are the same. In practice one computes Coker A by finding Ker A^{T}.

ASIDE

The cokernel can be defined more generally for vector spaces (or abelian groups) using the *quotient*. For $f : V \to W$, it is defined as

$$\operatorname{Coker} f := W / \operatorname{Im}(f).$$

This is, of course, also true for matrices. Note that image and kernel are thus enough; and the cokernel is nothing new.

NOT EXAMINABLE

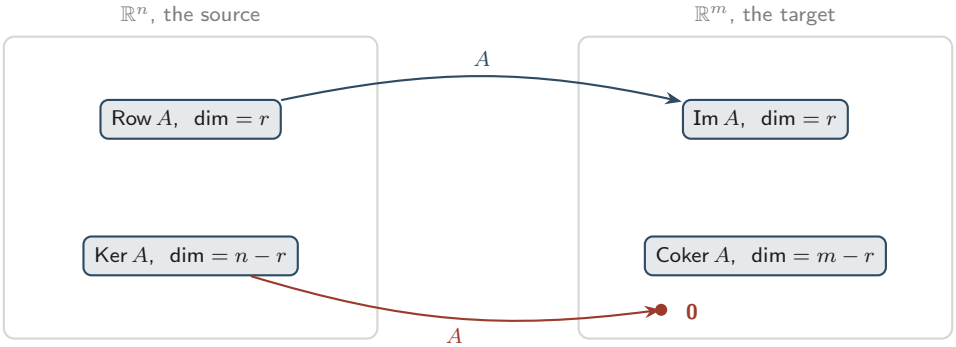
The prefix co is not a "one-off". The further one pursues algebra, the more often this will appear: limit/colimit, product/coproduct, dimension/codimension,... "co" signifies the deep duality living underneath all of algebra.

For example, one can also talk of the coimage, which would be $V / \operatorname{Ker}(f)$ ▸. For finitely dimensional spaces, it *can* be identified with the row space, but not in a *canonical* way.

SECTION 2

The four subspaces together

Four subspaces, two homes



A carries the row space onto the column space, and contracts the kernel to $\mathbf{0}$. The cokernel is what A never reaches. It is to A^{T} what the kernel is to A.

The dimension count

THEOREM 15.4

Let A be $m \times n$ of rank r. Then

dim(Im A)	r	$\subseteq \mathbb{R}^m$
dim(Row A)	r	$\subseteq \mathbb{R}^n$
dim(Ker A)	$n - r$	$\subseteq \mathbb{R}^n$
dim(Coker A)	$m - r$	$\subseteq \mathbb{R}^m$

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$\dim(\mathrm{Ker}\, A)$	$n - r$	$\subseteq \mathbb{R}^n$
$\dim(\mathrm{Coker}\, A)$	$m - r$	$\subseteq \mathbb{R}^m$

The first three are last lecture's results. The fourth is rank–nullity applied to A^{T}, which has shape $n \times m$ and rank r, hence

$$\dim(\mathrm{Ker}\, A)^{\mathsf{T}} \;=\; m - \mathrm{Rank}\, A^{\mathsf{T}} \;=\; m - r.$$

In each space the two dimensions add to that space's dimension: $r + (n - r) = n$ and $r + (m - r) = m$.

All four, computed

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix} \qquad \text{reduces to} \qquad \mathrm{RREF}(A) = \begin{bmatrix} \mathbf{1} & 2 & 0 & 1 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $m = 3$, $n = 4$, $r = 2$.

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So $m = 3$, $n = 4$, $r = 2$.

Im A: the pivot columns of A ▸,

$$\mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \dim = 2.$$

Row A: the non-zero rows of $\mathrm{RREF}(A)$,

$$\mathrm{Span}\{(1,2,0,1),(0,0,1,1)\}, \ \dim = 2.$$

Ker A: free columns 2 and 4 ▸,

$$\mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}, \quad \dim = 2.$$

Coker A: dimension $m - r = 1$, computed next.

The cokernel of the same matrix ▸

We need Ker A^{T}, so transpose and reduce:

$$A^{\mathsf{T}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{tidy}} \begin{bmatrix} \mathbf{1} & 0 & -1 \\ 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The cokernel of the same matrix ▸

We need $\operatorname{Ker} A^{\mathsf{T}}$, so transpose and reduce:

$$A^{\mathsf{T}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{tidy}} \begin{bmatrix} \mathbf{1} & 0 & -1 \\ 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Two pivots, one free column: setting $z = t$ gives $x = t$, $y = -t$, so

$$\operatorname{Coker} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}, \qquad \dim = 1 = m - r.$$

The cokernel of the same matrix ▸

We need $\operatorname{Ker} A^{\mathsf{T}}$, so transpose and reduce:

$$A^{\mathsf{T}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[R_4 \leftarrow R_4 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{tidy}} \begin{bmatrix} \mathbf{1} & 0 & -1 \\ 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Check: $(1, -1, 1)A = (1 - 2 + 1,\ 2 - 4 + 2,\ 0 - 1 + 1,\ 1 - 3 + 2) = \mathbf{0}$. The row combination $R_1 - R_2 + R_3$ really does vanish.

Summary of Lecture 15

ROW SPACE	$\operatorname{Row} A = \operatorname{Span}\{\text{rows}\} = \operatorname{Im} A^{\mathsf{T}} \subseteq \mathbb{R}^n$. Row operations *preserve* it, so the non-zero rows of $\operatorname{RREF}(A)$ are a basis.
ROW = COLUMN	$\dim \operatorname{Row} A = \dim(\operatorname{Im} A) = r$: the non-zero rows of $\operatorname{RREF}(A)$ number the pivots. Hence $\operatorname{Rank} A = \operatorname{Rank} A^{\mathsf{T}}$.
COKERNEL	$\operatorname{Coker} A = \{\mathbf{y} \mid \mathbf{y}A = \mathbf{0}\}$, the row vectors with $\mathbf{y}^{\mathsf{T}} \in \operatorname{Ker} A^{\mathsf{T}}$. Compute it as $\operatorname{Ker} A^{\mathsf{T}}$.
THE FOUR	In $\mathbb{R}^n$: $\operatorname{Row} A$ of dimension r, $\operatorname{Ker} A$ of $n - r$. In $\mathbb{R}^m$: $\operatorname{Im} A$ of r, $\operatorname{Coker} A$ of $m - r$.

Before next time

AT HOME

1. For $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 1 & 0 & 1 \end{bmatrix}$ find $\operatorname{Rank} A$ and a basis of each of the four fundamental subspaces. Check all four dimensions against the theorem.
2. An $m \times n$ matrix has $\dim \operatorname{Coker} A = 0$. What does that say about $\operatorname{Rank} A$, and about the solvability of $A\mathbf{x} = \mathbf{b}$?
3. Let $T\colon P_2 \to P_2$ be $T(p) = p' + p$. Write its matrix in the basis $(1, x, x^2)$, and decide whether T is an isomorphism.

END OF LECTURE 15

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Solution: question 1 ▸

SOLUTION

$$A \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{\text{tidy}} \begin{bmatrix} \mathbf{1} & 0 & 1 \\ 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Two pivots, so $r = 2$, with $m = n = 3$.

Solution: question 1 ▸

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Two pivots, so $r = 2$, with $m = n = 3$.

$\text{Im}\, A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ (columns $1, 2$ of A), $\text{Row}\, A = \text{Span}\{(1,0,1), (0,1,1)\}$,
$\text{Ker}\, A = \text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Dimensions $2, 2, 1$ and, since $m - r = 1$, a one-dimensional cokernel ▸, spanned by $(2, -1, 0)$.

Solution: question 2

SOLUTION

dim Coker $A = m - r = 0$ means $r = m$, so Rank $A = m$.

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Then Im A is a subspace of $\mathbb{R}^m$ of dimension m, hence all of $\mathbb{R}^m$ by lecture 13. So $A\mathbf{x} = \mathbf{b}$ is solvable for *every* $\mathbf{b} \in \mathbb{R}^m$.

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SOLUTION

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Then Im A is a subspace of $\mathbb{R}^m$ of dimension m, hence all of $\mathbb{R}^m$ by lecture 13. So $A\mathbf{x} = \mathbf{b}$ is solvable for *every* $\mathbf{b} \in \mathbb{R}^m$.

A vanishing cokernel means nothing is out of reach. That is what "co-" is for: the cokernel measures exactly what the image misses.

Solution: question 3

SOLUTION

$T(1) = 1$, $T(x) = 1 + x$, $T(x^2) = 2x + x^2$, so in the basis $(1, x, x^2)$

$$T \longleftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: question 3

SOLUTION

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$$T \longleftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix is triangular with 1s on the diagonal, so its determinant ▸ is $1 \neq 0$. Hence it is invertible, and T is an isomorphism.

Equivalently Nullity $= 0$: if $p' + p = 0$ with p polynomial, comparing top degrees forces $p = 0$.