LECTURE 14

Rank and nullity

One equation that ties the whole module together

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
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Recap

LAST LECTURE

- With a basis of n vectors, any $n+1$ vectors are dependent. So, every basis has the same size and $\dim V$ as number of basis vectors is well-defined.
- $\dim H \leq \dim V$ for a subspace, with equality only when $H = V$. Any independent set extend to bases, any spanning set contracts to a basis.
- **Basis Theorem**: For exactly $\dim V$ vectors, basis, independent and spanning are the same condition.

We now attach two numbers to a matrix, and find that they always add up to the same.

Warm-up

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SOLUTION

Three vectors and dim $P_2 = 3$, so by the Basis Theorem it is enough to check independence. Writing coordinates with respect to $(1, x, x^2)$ and taking the determinant ▸,

$\text{Det} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 1(1) - 0 + 1(1) = 2 \neq 0.$

So **Yes.**

SECTION 1

Rank and nullity

Two numbers

DEFINITION 14.1 ▸ · Rank and nullity

For an $m \times n$ matrix A,

$$\text{Rank}\, A \;:=\; \dim(\text{Im}\, A), \qquad \text{Nullity}\, A \;:=\; \dim(\text{Ker}\, A).$$

By our results on RREF(A) in Lecture 11:

- The *pivot* columns of A form a basis of Im A, so $\text{Rank}\, A = \text{number of the piots}$;
- The *free* columns of A form a basis of Ker A, so $\text{Nullity}\, A = \text{number of the free columns}$.

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By our results on $\operatorname{RREF}(A)$ in Lecture 11:

- The *pivot* columns of A form a basis of $\operatorname{Im} A$, so $\operatorname{Rank} A =$ number of the piots ;
- The *free* columns of A form a basis of $\operatorname{Ker} A$, so $\operatorname{Nullity} A =$ number of the free columns .

This immediately gives us the following theorem, as every column is a pivot or free.

THEOREM 14.2 · Rank–Nullity, matrix form

For every $m \times n$ matrix A, $\operatorname{Rank} A + \operatorname{Nullity} A = n$.

Note the "oddity". In a matrix, rank and nullity measure the number of columns of the same matrix. But the statement is deeper, and concearns subspaces of *different* spaces.

A small outlook towards homological algebra

A FAR ASIDE

The idea "what goes in is what comes out plus what is lost" recurs throughout algebra. That is why image and kernel together are so strong. The sequence

$$0 \longrightarrow \operatorname{Ker} A \longrightarrow \mathbb{R}^n \longrightarrow \operatorname{Im} A \longrightarrow 0$$

is an example of a so-called **short exact sequence**. In general, for A, B, C abelian groups

$$0 \longrightarrow A \xrightarrow{f} C \xrightarrow{g} B \longrightarrow 0$$

is short-exact if f is injective, g surjective and $\operatorname{Ker} g = \operatorname{Im} f$.

When this balance fails, but $\operatorname{Im} f \subseteq \operatorname{Ker} g$ (meaning $g \circ f = 0$), **homology** arises. This leads to **Betti numbers** and other beautiful things.

An example

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 7 \\ 1 & 2 & 1 & 4 \end{bmatrix} \quad \text{with} \quad \text{RREF}(A) = \begin{bmatrix} \mathbf{1} & 2 & 0 & 3 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Rank $A = 2$, with basis of $\operatorname{Im} A$ ▸ taken from A using the pivot indicies of $\mathrm{RREF}(A)$:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Nullity $A = 2$, with basis of $\operatorname{Ker} A$ ▸:

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

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Note the two subspaces live in different places: $\mathrm{Im}\, A \subseteq \mathbb{R}^3$, $\mathrm{Ker}\, A \subseteq \mathbb{R}^4$.

The same theorem without matrices

THEOREM 14.3 · Rank–Nullity, abstract form

Let $T\colon V \to W$ *be linear with* $\dim V = n < \infty$. *Then*

$$\dim(\operatorname{Ker} T) \;+\; \dim(\operatorname{Im} T) \;=\; \dim V.$$

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$$\dim(\operatorname{Ker} T) \;+\; \dim(\operatorname{Im} T) \;=\; \dim V.$$

PROOF.

Choose basis of V and W. Choosing coordinates turn T into a matrix A of shape $\dim W \times n$, and isomorphisms preserve dimension, so $\dim(\operatorname{Ker} T) = \text{Nullity}\, A$ and $\dim(\operatorname{Im} T) = \operatorname{Rank} A$. Now apply the matrix form. □

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EXAMPLE 14.4

$D\colon P_2 \to P_1$, $p \mapsto p'$. Here, $\operatorname{Ker} D$ is the constants and has dimension 1. The image $\operatorname{Im} D = P_1$ has dimension 2. Indeed $1 + 2 = 3 = \dim P_2$.

SECTION 2

Invertibility, one more time

Everything at once

THEOREM 14.5

For an $n \times n$ matrix A, the following are equivalent:

1. *A is invertible.*
2. Nullity $A = 0$.
3. Rank $A = n$.
4. *The columns of A form a basis of $\mathbb{R}^n$.*

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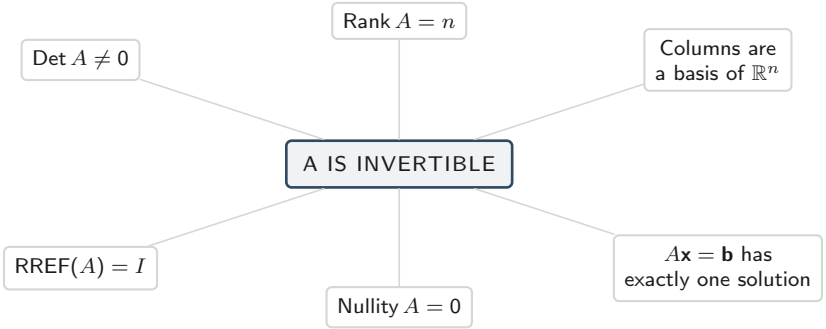
PROOF.

1. $\implies$ 2. Invertibility $\implies$ injectivity $\iff$ Nullity $A = 0$.
2. $\implies$ 3. Rand-nullity theorem (Theorem 14.2).
3. $\implies$ 4. Definition 14.1 of rank, including the discussion below, linking basis of Im A, pivot columns of A, and Rank A. Dimensionality (Lecture 13) now force Im $A = \mathbb{R}^n$.
4. $\implies$ 1. Columns form basis $\implies$ linearly independent. It is preserved under row reduction $\implies$ A diagonalises $\implies$ Det $A \neq 0$ $\implies$ A is inverstible.

Invertibility on one slide



All the following are equivalent.

Reading a system through the column space

For a general $m \times n$ matrix, the same two subspaces answer the two questions one can ask about $A\mathbf{x} = \mathbf{b}$:

SOLUTION EXISTS? Exactly when $\mathbf{b} \in \operatorname{Im} A$. **If** $\operatorname{Rank} A = m$ the column space is all of $\mathbb{R}^m$ and the answer is always **Yes.**

HOW MANY? If one solution $\mathbf{x}_0$ exists, the full solution set is $\mathbf{x}_0 + \operatorname{Ker} A$ ▸. So the solution is unique exactly when $\operatorname{Nullity} A = 0$.

The '$+ \operatorname{Ker} A$' is a special case of a **Quotient space**. You already one know another: $\mathbb{Z}/n\mathbb{Z}$, the numbers modulo n.

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Gauss–Jordan answers both at once, because RREF $\left[A \mid \mathbf{b}\right]$ shows the pivots (hence the rank), the free columns (hence the nullity), and whether the augmented column carries a pivot (hence consistency).

Linear algebra is realising that one method solves many questions. This lets us guess at the strength of *abstraction*.

SECTION 3

Transpose, and how large a rank can be

The transpose respects rank

Recall A^{T} from Lecture 1 and that $\operatorname{Det}(A^{\mathsf{T}}) = \operatorname{Det} A$ holds. Moreover, we have

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PROOF.

We have

$$\begin{aligned}
\operatorname{Rank} A &= \dim(\operatorname{Im} A) && \text{Definition 14.1}\\
&= \dim(\operatorname{Col} A) && \text{Definition of column space}\\
&= \dim(\operatorname{Row} A) && \text{Proposition 8.4}\\
&= \dim(\operatorname{Col} A^{\mathsf{T}}) && \text{Definition of transpose}\\
&= \dim(\operatorname{Im} A^{\mathsf{T}}) && \text{Definition of column space}\\
&= \dim(\operatorname{Rank} A^{\mathsf{T}}) && \text{Definition 14.1}
\end{aligned}$$

□

How large can a rank be?

$\mathrm{Im}\, A$ is a subspace of $\mathbb{R}^m$, so

$$\mathrm{Rank}\, A \;=\; \dim \mathrm{Im}\, A \;\leq\; \dim \mathbb{R}^m \;=\; m.$$

How large can a rank be?

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$$\operatorname{Rank} A \;=\; \dim \operatorname{Im} A \;\leq\; \dim \mathbb{R}^m \;=\; m.$$

On the other hand $\operatorname{Rank} A = \operatorname{Rank} A^{\mathsf{T}} = \dim(\operatorname{Im} A^{\mathsf{T}}) \leq n$, since $\operatorname{Im} A^{\mathsf{T}} \subseteq \mathbb{R}^n$. Putting the two together:

A WIDE MATRIX ALWAYS HAS A NON-TRIVIAL KERNEL

The rank is limited by the shorter side of the matrix as

$$\operatorname{Rank} A \;\leq\; \min(m, n).$$

In particular, if $n > m$ then $\operatorname{Nullity} A \geq n - m > 0$ ▸ by rank-nullity, i.e., $\dim(\operatorname{Ker}) > 0$.

Three quick questions

EXAMPLE 14.7

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If the kernel of a 4×7 matrix A is 3-dimensional, what is $\dim(\operatorname{Im} A)$? By rank–nullity, $\operatorname{Rank} A = 7 - 3 = 4$. It is consistent as $4 \leq \min(4, 7)$, and indeed, as large as possible.

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In particular, a 4×7 matrix cannot have nullity 2, as that would mean rank is $5 > 4$.

Summary of Lecture 14

DEFINITIONS	$\operatorname{Rank} A = \dim(\operatorname{Im} A)$ is the number of pivots; $\operatorname{Nullity} A = \dim(\operatorname{Ker} A)$ is the number of free columns.
THEOREM	$\operatorname{Rank} A + \operatorname{Nullity} A = n$, $A \in \operatorname{Mat}_{m\times n}(\mathbb{R})$, the columns are pivots or free. Abstractly, $\dim(\operatorname{Ker} T) + \dim(\operatorname{Im} T) = \dim V$, the source.
SQUARE CASE	A invertible $\iff \operatorname{Rank} A = n \iff \operatorname{Nullity} A = 0 \iff$ the columns are a basis of $\mathbb{R}^n \iff \det A \neq 0$.
SYSTEMS	$A\mathbf{x} = \mathbf{b}$ is solvable iff $\mathbf{b} \in \operatorname{Col} A$, and its solution set is $\mathbf{x}_0 + \operatorname{Ker} A$.
TRANSPOSE	$\operatorname{Rank} A^{\mathsf{T}} = \operatorname{Rank} A$: row rank equals column rank.
BOUND	$\operatorname{Rank} A \leq \min(m, n)$, so a wide matrix always has a non-trivial kernel.

Before next time

AT HOME

1. For $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 10 \\ 1 & 2 & 4 & 6 \end{bmatrix}$ find Rank A and Nullity A, a basis of each of Col A ▸ and Ker A ▸, and verify rank–nullity.
2. (a) Can a 3×5 matrix have nullity 1? (b) A 5×3 matrix have rank 4? (c) If a 6×4 matrix has Nullity $A = 0$, what is Rank A?
3. Let A be 5×5 with Rank $A = 5$. Explain, using Col A and Ker A separately, why $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b} \in \mathbb{R}^5$.

END OF LECTURE 14

ilia.pirashvili@universityofgalway.ac.ie

Solution: question 1 ▸

SOLUTION

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 10 \\ 1 & 2 & 4 & 6 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_1 \leftarrow R_1 - 3R_2]{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} \mathbf{1} & 2 & 0 & -2 \\ 0 & 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Pivots in columns $1, 3$: $\operatorname{Rank} A = 2$, with $\operatorname{Im} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} \right\}$. Free $x_2 = s$, $x_4 = t$ give $x_1 = -2s + 2t$, $x_3 = -2t$, so $\operatorname{Ker} A = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ and $\operatorname{Nullity} A = 2$.
$2 + 2 = 4 = n$. **Yes.**

Solution: question 2

(A) NO

Here $n = 5$ and Rank $A \leq \min(3, 5) = 3$, so Nullity $A \geq 2$.

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Here $n = 5$ and $\text{Rank}\, A \leq \min(3, 5) = 3$, so $\text{Nullity}\, A \geq 2$.

(B) NO

$\text{Rank}\, A \leq \min(5, 3) = 3 < 5$.

Solution: question 2

(A) NO

Here $n = 5$ and $\text{Rank}\, A \leq \min(3, 5) = 3$, so $\text{Nullity}\, A \geq 2$.

(B) NO

$\text{Rank}\, A \leq \min(5, 3) = 3 < 5$.

(C) $\text{Rank}\, A = 4$

$n = 4$, so $\text{Rank}\, A = 4 - 0 = 4 = \min(6, 4)$: the columns are independent but do not span $\mathbb{R}^6$.

Solution: question 3

EXISTENCE, FROM THE COLUMN SPACE

Rank $A = \dim(\text{Im}\, A) = 5$, and $\text{Im}\, A$ is a subspace of $\mathbb{R}^5$. By lecture 13, a subspace of the full dimension is the whole space, so $\text{Im}\, A = \mathbb{R}^5$. Every $\mathbf{b}$ therefore lies in $\text{Im}\, A$, which is exactly the statement that $A\mathbf{x} = \mathbf{b}$ has a solution.

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UNIQUENESS, FROM THE KERNEL

Rank–nullity gives $\operatorname{Nullity} A = 5 - 5 = 0$, so $\operatorname{Ker} A = \{\mathbf{0}\}$. If $A\mathbf{x} = A\mathbf{y}$ then $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$, forcing $\mathbf{x} = \mathbf{y}$.