LECTURE 13

Dimension

Why the size of a basis is a well-defined notion

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Recall

LAST LECTURE

- $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is linear and invertible, so $V \cong \mathbb{R}^n$ if V admits a basis of n vectors.
- $P_{\mathcal{B}}$ has the basis vectors as columns; $P_{\mathcal{B}}^{\mathcal{C}} = P_{\mathcal{C}}^{-1} \circ P_{\mathcal{B}}$.
- Isomorphic spaces are the same space as far as linear algebra can see.

In Lecture 10 we defined $\dim V$ as the number of vectors in a basis. But we also pointed out the problematic nature of this definition, as basis are not unique. Today, we finally make this well-defined by showing that all basis are of the same size.

Warm-up

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Let $\mathcal{B} = (\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix})$, a basis of $\mathbb{R}^2$. Find $P_{\mathcal{B}}$ ▸ and $[\mathbf{x}]_{\mathcal{B}}$ ▸ for $\mathbf{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

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SOLUTION

$P_{\mathcal{B}} = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$, the basis vectors as columns. Solving $a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$ gives $a = 3$ from the first row, then $6 + b = 5$, so $b = -1$: $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}3\\-1\end{bmatrix}$.

Equivalently $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x} = \begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix}$.

SECTION 1

What we want from dimension

What dimensions should satisfy

Before proving anything, let us say what a notion of dimension *ought* to do. It should assign to each finitely generated space a number in $\mathbb{N}$ such that:

- $\dim \mathbb{R}^n = n$, and more generally a space with a basis of n vectors has dimension n;

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More is actually true: Non-isomorphic spaces have non-equal dimension. This makes dim a *full invariant*.

The central theorem

THEOREM 13.1 ▸

Let $\mathcal{B}$ be a basis of V with n vectors. Any $n+1$ vectors in V are linearly dependent.

PROOF.

Let $\mathbf{w}_1, \ldots, \mathbf{w}_{n+1} \in V$ and consider the column matrix of shape $\times(n+1)$

$$M = \begin{bmatrix} [\mathbf{w}_1]_{\mathcal{B}} & \cdots & [\mathbf{w}_{n+1}]_{\mathcal{B}} \end{bmatrix}.$$

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There are $n+1$ columns but at most n pivots, one per row. This means at least one column is free, and thus $\operatorname{Ker} M \neq \{\mathbf{0}\}$. Hence, there exists $\mathbf{c} \neq \mathbf{0}$ with $M\mathbf{c} = \mathbf{0}$, that is

$$c_1 [\mathbf{w}_1]_{\mathcal{B}} + \cdots + c_{n+1} [\mathbf{w}_{n+1}]_{\mathcal{B}} \;=\; \mathbf{0}.$$

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As change of basis is an isomorphism, carrying this back to V gives $c_1\mathbf{w}_1 + \cdots + c_{n+1}\mathbf{w}_{n+1} = \mathbf{0}$ with the c_i not all zero: linear dependence. □

Dimension is well-defiend

COROLLARY 13.2 · Basis is an invariant

If V has **a** *basis with n vectors, then* **every** *basis of V has exactly n vectors.*

PROOF.

Let $\mathcal{C}$ be another basis with m vectors. By Theorem 13.1, if $m > n$, then $\mathcal{C}$ would be linearly dependent. This contradicts $\mathcal{C}$ being a basis. By symmetry (change the roles of $\mathcal{B}$ and $\mathcal{C}$), $n \leq m$. Thus, $n = m$. □

COROLLARY 13.3

$\dim V$ *as number of elements of a basis of V is well defined.*

SECTION 2

Dimensions we already know

Familiar spaces

$\mathbb{R}^n$	dim $= n$, with basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.
P_n	dim $= n+1$, with basis $(1, x, \ldots, x^n)$ ▸
	Note that $1 = x^0$ which makes the "+1" clear: We count from 0 here.
$\mathrm{Mat}_{m\times n}(\mathbb{R})$	dim $= mn$, with the matrix units E_{ij} ▸.
$\mathbb{C}$ OVER $\mathbb{R}$	dim $= 2$, with basis $(1, i)$ ▸.
$\{\mathbf{0}\}$	dim $= 0$. The empty set is a basis, and spans by convention.
	Technically, more than mere convention, but this would go too far...
$\mathbb{R}[x]$	Not finitely generated: no finite list can reach every degree.

Revisiting previous examples

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This is the pattern pointed out in Lecture 10: Two on a line, three in a plane, four in space.

SECTION 3

Subspaces

Subspaces inherit finiteness

THEOREM 13.6

Let $\dim V = n$ *and* $H \subseteq V$ *a subspace. Then* H *is finitely generated and* $\dim H \leq n$.

PROOF.

By Theorem 9.12 of Lecture 9 there exists a linear transformation $T\colon V \to W$ with $H \simeq \operatorname{Ker} T$. Let $\dim W = m$. We also know that the basis of a kernel is given by its free columns (non-pivot), which is clearly less than or equal to the number of columns = n. $\square$

This is by no means trivial and **not** true for abelian groups. It is scaling that does it, even if hidden in the argument.

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ALTERNATIVE/CONSTRUCTIVE — OPPOSITE OF THE *CASTING-OUT-LEMMA*.

If $H \neq \{\mathbf{0}\}$, choose $\mathbf{0} \neq \mathbf{v}_1 \in H$ and consider $H_1 := \operatorname{Span}\{\mathbf{v}_1\}$. If $H_1 \neq H$, test

$$H_2 := \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}, \quad \text{where } \mathbf{v}_2 \in H \setminus H_1 \text{ is any vector in } H \text{ but not in } H_1.$$

If $H_2 \neq H$ test $H_3 := \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with $\mathbf{v}_3 \in H \setminus H_2$ and so on. By Theorem 13.1 this must terminate latest for H_n, as otherwise we would have $n+1$ independent elements in H and thus in V. These elements are linearly independent by construction and span all of H, making them a basis of size $\leq n$. Hence, $\dim H \leq n$. □

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Extending an independent set

THEOREM 13.7

Let $\dim V = n$. *Any linearly independent set in* V *can be extended to a basis of* V.

PROOF.

This is our "constructive" prove from above rewritten.
Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be independent. If they span V, we are done. If not, pick $\mathbf{v}_{k+1}$ outside their span meaning $\mathbf{v}_{k+1} \in V \setminus \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. This larger set is still independent by construction and we repeat this until $k = n$ latest. □

When a subspace fills its space

THEOREM 13.8

Let $H \subseteq V$ with $\dim V = n$. Then $\dim H \leq n$, and $\dim H = n$ if and only if $H = V$.

PROOF.

The inequality is Theorem 13.6. If $H = V$ the dimensions agree trivially. Only one part needs proving: That equal dimensions give isomorphic spaces.

Let $\dim H = n$ and choose a basis $\mathcal{B} := (\mathbf{h}_1, \ldots, \mathbf{h}_n)$ of H. These are n independent vectors in V and span V since otherwise we could extend them to a basis of V by Theorem 13.7. But this would have more vectors than n, contradicting $\dim V = n$. So they span V, giving $V = \text{Span}\{\mathbf{h}_1, \ldots, \mathbf{h}_n\} = H$. □

SECTION 4

The Basis Theorem

Half the work is enough

THEOREM 13.9 · Basis Theorem

Let $1 \leq n = \dim V < \infty$, *and* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. *The following statements are equivalent:*

1. $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ *is a basis of* V;
2. $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *are linearly independent;*
3. $V = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

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3. $V = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

PROOF.

(1) implies both others by definition. If (2) holds, extend to a basis; the extension cannot add anything, since every basis has n vectors, so the set is already a basis. If (3) holds, cast out down to a basis; casting out cannot remove anything, for the same reason. □

Using it

KEY IDEA

Once the numbers agree, checking *either* condition gives the other. In practice, we count the vectors and test independence by looking at the determinant.

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EXAMPLE 13.10

Is $\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right)$ a basis of $\mathbb{R}^2$? Two vectors, and $\dim \mathbb{R}^2 = 2$. The determinant $-2 \neq 0$ gives independence ▸. So **Yes.** We did not need to check the spanning property.

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EXAMPLE 13.11

Is $(1+x,\ x+x^2,\ 1+x^2)$ a basis of P_2? Three vectors, $\dim P_2 = 3$. Their coordinate vectors with respect to $(1, x, x^2)$ have determinant ▸ $\operatorname{Det}\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix} = 2 \neq 0$. So **Yes.** .

Summary of Lecture 13

THE THEOREM	If V has a basis of n vectors, any $n+1$ vectors are dependent because n rows cannot carry $n+1$ pivots.
CONSEQUENCE	Every basis of V has the same size, so $\dim V$ is well defined at last.
DIMENSIONS	$\dim \mathbb{R}^n = n$, $\dim P_n = n+1$, $\dim \mathrm{Mat}_{m\times n}(\mathbb{R}) = mn$, $\dim \mathbb{C}_{\mathbb{R}} = 2$, $\dim\{\mathbf{0}\} = 0$.
SUBSPACES	A subspace of a finitely generated space is finitely generated, with $\dim H \leq \dim V$, and equality only when $H = V$.
EXTENSION	Any independent set extends to a basis; any finite spanning set reduces to one by casting out.
BASIS THEOREM	With exactly $\dim V$ vectors, basis $\iff$ independent $\iff$ spanning. Check one, get the rest.

Before next time

AT HOME

1. Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^3$? Use the Basis Theorem ▸, and if the answer is no, exhibit a dependence.
2. Extend $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ to a basis of $\mathbb{R}^3$, and justify that your answer is one.
3. Let $H = \{p \in P_2 \mid p(1) = 0\}$. Show that H is a subspace, find a basis, and state $\dim H$.

END OF LECTURE 13

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Solution: question 1 ▸

SOLUTION

Three vectors and $\dim \mathbb{R}^3 = 3$, so by the Basis Theorem it is enough to test independence. As columns of a matrix,

$$\text{Det} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = 1(-1-1) - 0 + 2(2-1) = -2 + 2 = 0,$$

so they are dependent, and **No.** .

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A dependence ▸: $2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

So the span is a plane, not $\mathbb{R}^3$: failing one condition of the Basis Theorem fails them all.

Solution: question 2 ▸

SOLUTION

Take $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$. These are three vectors in $\mathbb{R}^3$, and

$$\text{Det}\begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix} = 1 \neq 0,$$

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Many other answers work. The extension theorem promises that *some* completion exists; it does not claim it is unique, and here it certainly is not.

Solution: question 3 ▸

A SUBSPACE

H is the kernel of the evaluation map $P_2 \to \mathbb{R}$, $p \mapsto p(1)$, which we saw in lecture 9 is linear. Kernels are subspaces.

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A BASIS

Both $x - 1$ and $x^2 - 1$ vanish at 1, and neither is a multiple of the other, so they are independent. If $p = a + bx + cx^2$ has $p(1) = 0$ then $a = -b - c$, so $p = b(x - 1) + c(x^2 - 1)$: they span H, and $\dim H = 2$.

One linear condition on a 3-dimensional space, one dimension lost. That is next lecture in miniature.