LECTURE 12

Change of basis, and isomorphism

Translating between rulings/tiling, what it means for two spaces to be the same, and why these things are the same

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Where we got to

LAST LECTURE

- The pivot columns *of* A form a basis of $\operatorname{Col} A$, and $\dim \operatorname{Col} A + \dim \operatorname{Ker} A = n$.
- **Unique representation**: With a basis $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$, each $\mathbf{x}$ is $c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ in exactly one way.
- $[\mathbf{x}]_{\mathcal{B}}$ collects those coefficients, giving a map $V \to \mathbb{R}^n$ which straightens the tiling of $\mathcal{B}$ into squares.

We saw two bases, two tilings of the space, two coordinate vectors for the same $\mathbf{x}$. Today we recall basics of set maps, then pass between bases, and finally define *isomorphisms* and show that this is what being "the same space" means.

Warm-up

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Let $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$, a basis of $\mathbb{R}^2$. Find $[\mathbf{x}]_{\mathcal{B}}$ ▸ for $\mathbf{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Warm-up

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Let $\mathcal{B} = \left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right)$, a basis of $\mathbb{R}^2$. Find $[\mathbf{x}]_{\mathcal{B}}$ ▸ for $\mathbf{x} = \begin{bmatrix}4\\2\end{bmatrix}$.

SOLUTION

We need $a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$, that is $a + b = 4$ and $a - b = 2$. Adding and subtracting gives $a = 3$, $b = 1$, so $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}3\\1\end{bmatrix}$.

Notice that we did $\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$, where the basis vectors are the matrix columns.

SECTION 1

Recall: Injectivity, surjectivity and bijectivity

Injective, surjective, bijective

For this section only, let X, Y be sets and $T\colon X \to Y$ a set map, ignoring all vector space structures. The kernel no longer makes sense but we can still talk about the **image** as

$$\operatorname{im} T \;=\; \{T(x) \mid x \in X\} \subseteq Y.$$

DEFINITION 12.1 · Injective, surjective & bijective

We say T is

- **injective** if $T(x) = T(x')$ implies $x = x'$ (no two elements map to the same target);
- **surjective** if $\operatorname{im} T = Y$ (every element of Y is reached);
- **bijective** if it is both injective and surjective.

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- **bijective** if it is both injective and surjective.

Bijective maps match up the elements of X and Y perfectly, and can thus be run backwards.

Invertible maps

DEFINITION 12.2

A map $T\colon X \to Y$ is **invertible** if there exists a map $S\colon Y \to X$ with

$$S \circ T \;=\; \mathrm{Id}_X \qquad \text{and} \qquad T \circ S \;=\; \mathrm{Id}_Y,$$

where $\circ$ is the composition. The map S is called the **inverse** of T and denoted by T^{-1}.

A map is invertible exactly when it is bijective. The inverse $T^{-1}\colon Y \to X$ is defined by sending $y \in Y$ to an element $x \in X$ satisfying $T(x) = y$. Surjectivity gives us existence of at least one such x for every $y \in Y$ and injectivity gives us uniqueness (well-definidness).

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PROPOSITION 12.3

The inverse of a map $T\colon X \to Y$ is unique if it exists, making the notation T^{-1} sensible.

Let S and S' be inverses of T. Then
$S = S \circ \mathrm{Id} = S \circ (T \circ S') = (S \circ T) \circ S' = \mathrm{Id} \circ S' = S'$.

SECTION 2

Isomorphism

Definition

We return to vector spaces and linear transformations. As these are, in particular, sets and set maps, we can talk about injectivity, surjectivity and bijectivity. But we have more.

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A bijective linear map between vector spaces $T\colon V \to W$ is called an **isomorphism**, and V and W are called **isomorphic** and write $V \cong W$ in that case.

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An isomorphism is a bijection respecting the vector space structure: Addition and scaling. It has thus an inverse map $T^{-1}\colon W \to V$. For an isomorphism to be a sensible definition, the inverse needs to be a linear transformation as well, which the next proposition shows.

We have no special names for injective or surjective linear transformations.

The inverse of an isomorphism is linear

PROPOSITION 12.5

If $T\colon V \to W$ is an isomorphism, so is $T^{-1}\colon W \to V$.

PROOF.

The claim is that T^{-1} is linear, as the rest is just the definition of bijectivity. We have

$$T(\underbrace{T^{-1}(\mathbf{u}+\mathbf{v})}_{\mathbf{x}}) \;=\; \mathbf{u}+\mathbf{v} \;=\; T(T^{-1}(\mathbf{u}))+T(T^{-1}(\mathbf{v})) \;=\; T(\underbrace{T^{-1}(\mathbf{u})+T^{-1}(\mathbf{v})}_{\mathbf{y}}),$$

an expression of the form $T(\mathbf{x}) = T(\mathbf{y})$. Injectivity of T, by definition, now gives $\mathbf{x} = \mathbf{y}$. A like argument shows that T^{-1} respects scaling and we are done. □

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> FAR ASIDE: TRUE FOR ALL "*ALGEBRAIC STRUCTURES*", BUT NOT IN GENERAL.
>
> For example, the inverse of a bijective continuous map need not continuous. That is why continuity is explicitly required in the definition of a homeomorphism.

For linear maps, the kernel decides

While kernels were non-existent in sets, they come "back with a vengeance" in vector space (abelian groups in general). Recall $\operatorname{Ker} T = \{\mathbf{v} \in V : T\mathbf{v} = \mathbf{0}\}$.

PROPOSITION 12.6

A linear map T is injective if and only if $\operatorname{Ker} T = \{\mathbf{0}\}$.

PROOF.

$\Longrightarrow$ As $T(\mathbf{0}) = \mathbf{0}$, $T(\mathbf{v}) = \mathbf{0}$ forces $T(\mathbf{v}) = T(\mathbf{0})$ and thus $\mathbf{v} = \mathbf{0}$, meaning $\operatorname{Ker} T = \{\mathbf{0}\}$.

$\Longleftarrow$ From $T(\mathbf{u}) = T(\mathbf{v})$ follows $T(\mathbf{u}) - T(\mathbf{v}) = 0$ and thus $T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$. Hence $\mathbf{u} - \mathbf{v} = \mathbf{0}$ and so $\mathbf{u} = \mathbf{v}$, showing injectivity. □

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Consequently, $T\colon V \to W$ is an isomorphism if and only if $\operatorname{Ker} T = \{\mathbf{0}\}$ and $\operatorname{im} T = W$.

What isomorphism **truly** means

ISOMORPHISM IS PERFECT DICTIONARIES

Isomorphism of vector spaces match vectors with vectors and respect both operations.

Thus, everything that can *intrinsically* be expressed in the language of vector spaces, such as subspaces, spans, independence, bases, dimension, kernels, images, ... is preserved in full. They cross over intact, and can be restored in full.

Isomorphic spaces are *the same space* as far as linear algebra can see.

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EMBEDDINGS STILL SEE THE DIFFERENCE

If $W \subseteq W' \subseteq V$ are isomorphic subspaces, *how* the lie inside V will generally be different. Two line in the plane going through the origin are isomorphic, but not the same.

On their own, isomorphic objects can not be told apart. But in relation to other **fixed** things, they can.

SECTION 3

Coordinate isomorphism

The coordinate map is an isomorphism

THEOREM 12.7

The coordinate map $\phi : V \to \mathbb{R}^n$, *given by* $\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{b}_i \mapsto [\, c_1 \cdots c_n \,]^{\mathsf{T}} = [\mathbf{x}]_{\mathcal{B}}$, *is an isomorphism where* $\mathcal{B}$ *is an (ordered) basis of an* n*-dimensional space* V*. This isomorphism is called the* ***coordinate map*** *or* ***coordinate isomorphism****.*

PROOF. 1. ϕ IS **WELL-DEFINED**.

Every vector has (*span*) a unique (*linear independence*) expansion in a basis. □

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PROOF. 2. ϕ IS **LINEAR**.

For $\mathbf{x} = \sum_i c_i \mathbf{b}_i$, $\mathbf{y} = \sum_i d_i \mathbf{b}_i$, we have

$$\mathbf{x} + \mathbf{y} = \sum_i (c_i + d_i)\mathbf{b}_i, \quad \Longrightarrow \quad \phi(\mathbf{x} + \mathbf{y}) = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \phi(\mathbf{x}) + \phi(\mathbf{y}).$$

by uniqueness of coordinates, and similarly,

$$\lambda \mathbf{x} = \sum_i (\lambda c_i)\mathbf{b}_i, \quad \Longrightarrow \quad \phi(\lambda \mathbf{x}) = \begin{bmatrix} \lambda c_1 & \cdots & \lambda c_n \end{bmatrix}^\mathsf{T} = \lambda \phi(\mathbf{x}). \qquad \square$$

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PROOF. 3. ϕ IS **INJECTIVE**.

Let $\phi(\mathbf{x}) = \phi(\mathbf{y})$, with $\mathbf{x} = \sum_i c_i \mathbf{b}_i$, $\mathbf{y} = \sum_i d_i \mathbf{b}_i$. It implies

$$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \quad \text{and thus} \quad c_i = d_i \quad \text{for every } i.$$

Hence

$$\mathbf{x} = \sum_i c_i \mathbf{b}_i = \sum_i d_i \mathbf{b}_i = \mathbf{y}.$$

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PROOF. 4. ϕ IS **SURJECTIVE**.

For an arbitrary $\begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}^\mathsf{T} \in \mathbb{R}^n$, define $\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{b}_i$. Since B spans V, this vector belongs to V, and by definition $\phi(\mathbf{x}) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$. This proves surjectivity and Parts 1–4 prove isomorphism. □

To and from the standard basis ▸

Work in $\mathbb{R}^n$, and let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis. If $[\mathbf{x}]_{\mathcal{B}} = \mathbf{c}$, then by definition

$$\mathbf{x} \;=\; c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \;=\; \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_n\end{bmatrix}\mathbf{c}.$$

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$P_{\mathcal{B}}$ MAPS BASIS $\mathcal{B}$ TO THE STANDARD ONE, I.E., DEFINES $P_{\mathcal{B}} : \mathbb{R}^n_{\mathcal{B}} \to \mathbb{R}^n_{\mathcal{E}}$.

Write $P_{\mathcal{B}} = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_n\end{bmatrix}$, the matrix whose columns are the basis vectors. Then

$$\mathbf{x} \;=\; P_{\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}} \qquad \text{and} \qquad [\mathbf{x}]_{\mathcal{B}} \;=\; P_{\mathcal{B}}^{-1}\,\mathbf{x}.$$

$P_{\mathcal{B}}$ is invertible precisely because its columns are a basis.

CAUTION: EASY TO THINK THE OTHER WAY

$P_{\mathcal{B}}$ turns $\mathcal{B}$-coordinates into standard ones, and its inverse does the reverse.

SECTION 4

Composition

PROPOSITION 12.8

If $S\colon U \to V$ and $T\colon V \to W$ are linear, so is $T \circ S$.

PROOF.

$(T \circ S)(\mathbf{u}+\mathbf{v}) = T(S(\mathbf{u}+\mathbf{v})) = T(S\mathbf{u}+S\mathbf{v}) = T(S\mathbf{u}) + T(S\mathbf{v}) = (T \circ S)(\mathbf{u}) + (T \circ S)(\mathbf{v})$,

and $(T \circ S)(\lambda\mathbf{u}) = T(S(\lambda\mathbf{u})) = T(\lambda S(\mathbf{u})) = \lambda T(S\mathbf{u}) = \lambda(T \circ S)(\mathbf{u})$. □

Composites of linear maps ▸

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WHY MATRICES MULTIPLY AS THEY DO

$[T \circ S] = [T]\,[S]$. Matrix multiplication was *defined* so that this would hold! This is why AB means "do B, then A".

Composites of isomorphisms

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If S and T are isomorphisms, so is $T \circ S$, and $(T \circ S)^{-1} = S^{-1} \circ T^{-1}$.

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$T \circ S$ is linear by the previous slide. The computation that $S^{-1} \circ T^{-1}$ is its inverse is the *socks-and-shoes* argument of Lecture 2, with maps in place of matrices. Thus $T \circ S$ is invertible, hence bijective, hence an isomorphism.

That $S^{-1} \circ T^{-1}$ is itself linear was the previous section: Inverses of isomorphisms are isomorphisms.

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□

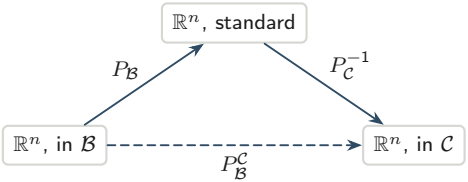
In matrices: $\left[T^{-1}\right] = [T]^{-1}$, and $P_{\mathcal{C}}^{\mathcal{B}} = (P_{\mathcal{B}}^{\mathcal{C}})^{-1}$. Being isomorphic is an *equivalence relation*: $V \cong V$ by Id_V, $V \cong W$ implies $W \cong V$, and $U \cong V$, $V \cong W$ imply $U \cong W$.

Change of basis, via the standard one

To convert $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates, go through the standard basis: First undo $\mathcal{B}$, then translate to $\mathcal{C}$.

$$P_{\mathcal{B}}^{\mathcal{C}} \;=\; P_{\mathcal{C}}^{-1} P_{\mathcal{B}}$$

This is called a **change of basis**.



The dashed arrow is the composite of the other two. In algebraic terms we would say *the diagram commutes*.

An example ▸

Let $\mathcal{B} = \left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right)$ and $\mathcal{C} = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$, both bases of $\mathbb{R}^2$.

$$P_{\mathcal{B}} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}, \qquad P_{\mathcal{C}} = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \qquad P_{\mathcal{C}}^{-1} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}.$$

$P_{\mathcal{B}} \colon \mathbb{R}^2_{\mathcal{B}} \to \mathbb{R}^2_{\mathcal{E}}$ $\qquad$ $P_{\mathcal{C}} \colon \mathbb{R}^2_{\mathcal{C}} \to \mathbb{R}^2_{\mathcal{E}}$ $\qquad$ $P_{\mathcal{C}}^{-1} \colon \mathbb{R}^2_{\mathcal{E}} \to \mathbb{R}^2_{\mathcal{C}}$

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$$P_{\mathcal{B}}^{\mathcal{C}} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}0 & 2\\1 & -1\end{bmatrix}.$$

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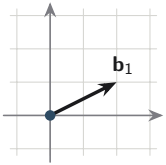
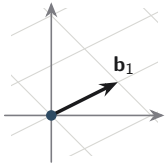
Recall also the other vector spaces we saw in Lecture 2. We have

$$P_2 \cong \mathbb{R}^3, \qquad \mathrm{Mat}_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^4, \qquad \mathbb{C} \cong \mathbb{R}^2 \text{ (over } \mathbb{R}).$$

A warning about length, area, volumne . . . ▸



In the "standard coordinate system", $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ has length $\sqrt{5}$.



In basis $\mathcal{B}$: $[\mathbf{b}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has length 1.

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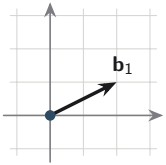

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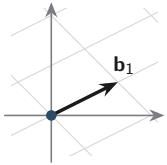

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The arrow never moved. Only the background tiling changed, and with it the numbers we read off.

REMARK 12.10

It would have been more precise to say $\mathbf{b}_1$ *has lenght* $\sqrt{5}$ *relative to the standard basis/coordinate system. Then it becomes clear why it changes as the basis changes.*

Length is not part of the structure

CAUTION

A change of bases does not (in general) preserve lengths, or its high-dimensional analogues (area, volumne,...). In the above example, $\|\mathbf{b_1}\| = \sqrt{5}$ while $\|[\mathbf{b_1}]_{\mathcal{B}}\| = 1$.

This does not contradict what we said about isomorphisms being perfect dictionaries, as length is *not* a vector space notion. It cannot be written purely with $+$ and scaling.

For example, regarding $\mathbb{C}$ as a 2-dimensional vector space, the distance is the same thing as the (complex) norm, being $|z| := \sqrt{z \cdot \overline{z}} = \sqrt{(a+bi)(a-bi)}$. The squre root is not the problem here, but the complex conjugate. It might seem natural, but if you think about it, there is an implicit dependence on the basis, as we write a vector z as $a + bi$, and thus 1 and i are our choice of basis.

Consider the second basis $\{1, j\}$ where $j = 1 + i$, and let us look at j. Under the new basis $\{1, j\}$, conjugate of j is $-j$, exatly as the conjugate of i is $-i$ in the basis $\{1, i\}$.
But the conjugate of $j = 1 + i$ in the basis $\{1, i\}$ is $1 - i$. But $1 - i = 2 - j \neq -j$! As such, the norm *depends* on the choice of basis and is *not* an intrinsic part of a vector space.

Why this is useful

KEY IDEA

We can use isomorphisms to move *intrinsic* problem to a setting where they are easier to solve.

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EXAMPLE 12.11 ▸ · Are $1+x$, $1-x$, x^2 independent in P_2?

The coordinates of $1+x$, $1-x$ and x^2 with respect to $(1, x, x^2)$ are $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\-1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$. The determinant of the coordinate map matrix (the column matrix) is $-2 \neq 0$, so they are independent as vectors in $\mathbb{R}^3$ and thus in P^2.

Obsreve that no polynomial manipulation was needed.

Summary of Lecture 12

BIJECTIVE	Injective ($T(a) = T(b) \Rightarrow a = b$) and surjective ($\operatorname{im} T$ is the full target). Equivalently invertible: $T^{-1} \circ T = \mathrm{Id}$ and $T \circ T^{-1} = \mathrm{Id}$.
COORDINATES	$\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is linear and bijective.
STANDARD BASIS	$P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$ sends $\mathcal{B}$-coords to standard ones; $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$.
CHANGE	$P_{\mathcal{B}}^{\mathcal{C}} = P_{\mathcal{C}}^{-1} \circ P_{\mathcal{B}}$: Translate to the standard basis and then to $\mathcal{C}$.
ISOMORPHISM	A bijective linear map. Inverse is automatically linear. For linear T, injective means $\operatorname{Ker} T = \{\mathbf{0}\}$, so T is an isomorphism iff $\operatorname{Ker} T = \{\mathbf{0}\}$ and $\operatorname{im} T = W$. Choosing a basis means $V \cong \mathbb{R}^n$.
ISOMORPHIC	Same space, different names. Everything sayable purely with $+$ and scaling crosses over. Extra structure (e.g., length) does not.
COMPOSITION	Composites of linear maps are linear, with $[T \circ S] = [T]\,[S]$. Composites of isomorphisms are isomorphisms, as are bijections.

Before next time

AT HOME

1. With $\mathcal{B} = \left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right)$ and $\mathcal{C} = \left(\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right)$, find $P_{\mathcal{B}}^{\mathcal{C}}$, and use it to convert $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}2\\1\end{bmatrix}$ into $\mathcal{C}$-coordinates. Check your answer directly. ▸
2. Show that $T\colon P_2 \to \mathbb{R}^3$, $T(p) = \begin{bmatrix}p(0) & p(1) & p(2)\end{bmatrix}^{\mathsf{T}}$, is an isomorphism.
3. Let $S = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ and $T = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$. Compute $[T \circ S]$ ▸ and $[S \circ T]$ ▸, confirm they differ, and find $[T \circ S]^{-1}$ ▸.

END OF LECTURE 12

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Solution: question 1

SOLUTION

$P_{\mathcal{B}} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ and $P_{\mathcal{C}} = \left[\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}\right]$, so $P_{\mathcal{C}}^{-1} = \frac{1}{6}\left[\begin{smallmatrix} 3 & -1 \\ 0 & 2 \end{smallmatrix}\right]$ and

$$P_{\mathcal{B}}^{\mathcal{C}} = \frac{1}{6}\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 4 \\ 2 & -2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Solution: question 1

SOLUTION

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Then $P_{\mathcal{B}}^{\mathcal{C}}\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right] = \frac{1}{3}\left[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\right]$. Directly: $[\mathbf{x}]_{\mathcal{B}} = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$ means $\mathbf{x} = 2\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$, and $a\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right] + b\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$ gives $b = \frac{1}{3}$, $a = \frac{4}{3}$. **Yes.**

Solution: question 2 ▸

SOLUTION

T is linear because evaluation is: $(p+q)(t) = p(t) + q(t)$ and $(\lambda p)(t) = \lambda\, p(t)$, at each of $t = 0, 1, 2$.

For invertibility, take coordinates on P_2 with respect to $(1, x, x^2)$ and compute the matrix of T column by column:

$$T(1) = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad T(x) = \begin{bmatrix}0\\1\\2\end{bmatrix}, \quad T(x^2) = \begin{bmatrix}0\\1\\4\end{bmatrix}, \qquad \text{Det}\begin{bmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 2 & 4\end{bmatrix} = 2 \neq 0.$$

Solution: question 2 ▸

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So the matrix is invertible, and T is an isomorphism.

A quadratic is determined by its values at three points. This is interpolation, seen from the side.

Solution: question 3 ▸

SOLUTION

$$[T \circ S] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad [S \circ T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

They differ: rotating then shearing is not shearing then rotating.

Solution: question 3 ▸

SOLUTION

$$[T \circ S] = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right], \quad [S \circ T] = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right].$$

They differ: rotating then shearing is not shearing then rotating.

Det $[T \circ S] = 0 + 1 = 1$, so

$$[T \circ S]^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = [S]^{-1}[T]^{-1}.$$