LECTURE 11

Pivot columns, and coordinates

A basis for the image; and what a basis is good for

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Revision

LAST LECTURE

- $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are **independent** if $\sum c_i \mathbf{v}_i = \mathbf{0}$ only for $c_1 = \cdots = c_p = 0$; dependent exactly when one of them is a combination of the others.
- A **basis** is a spanning set that is independent, and $\dim V$ is its size.
- Row reduction gives a basis of $\operatorname{Ker} A$: One vector per free column, independent because of the free slots.
- **Casting out** turns any finite spanning set into a basis.

So we can produce a basis of $\operatorname{Ker} A$ on demand. What about $\operatorname{Im} A$?

Warm-up

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Are $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$ independent? ▸ What is the dimension of their span?

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SOLUTION

As columns of a matrix, $\text{Det}\begin{bmatrix}1&0&1\\2&1&1\\1&1&0\end{bmatrix} = 1(0-1) - 0 + 1(2-1) = 0$, so they are dependent: Indeed $\begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$. Casting the third out leaves two vectors, neither is a multiple of the other, so the span is a 2 dimensional plane.

SECTION 1

A basis for the column space

What we have, and what we want

Ker A: DONE

Reduce A, give each free variable a parameter, collect. The vectors that appear are a basis.

Im A: IN THEORY

The columns span Im A. If they are dependent, cast one out and repeat. What is left is a basis.

What we have, and what we want

Ker A: DONE

Reduce A, give each free variable a parameter, collect. The vectors that appear are a basis.

Im A: IN THEORY

The columns span Im A. If they are dependent, cast one out and repeat. What is left is a basis.

The casting-out argument is what tells us a basis *exists*, and that is no small thing. But as a method it is poor. At each step we must check for (in)dependence, which means solving a system, and it never says which vector to discard.

But row reductions will help here as well.

Row operations preserve dependencies among columns ▸

RECALL 11.1

A linear relation among the columns of A,

$$c_1 a_{-,1} + \cdots + c_n a_{-,n} \;=\; \mathbf{0},$$

is exactly the statement $A\mathbf{c} = \mathbf{0}$, i.e. $\mathbf{c} \in \operatorname{Ker} A$.

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And $\operatorname{Ker} A = \operatorname{Ker} \operatorname{RREF}(A)$, by lecture 8. We can summerise this observation as follows:

> RREF CAPTURS LINEAR INDEPENDENCE.
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> The columns of A satisfy exactly the same linear relations as the columns of $\operatorname{RREF}(A)$.

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> The columns of A satisfy exactly the same linear relations as the columns of $\operatorname{RREF}(A)$.

Which columns are independent, and how the others are built from them, can therefore be read off the reduced form, even though $\operatorname{Im} A$ itself changes under row operations.

What the reduced form shows ▸

$$\mathrm{RREF}(A) = \begin{bmatrix} \mathbf{1} & 2 & 0 & 3 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The pivot columns are $\mathbf{e}_1$ and $\mathbf{e}_3$, which are visibly independent.
- Each non-pivot column is a combination of the pivot columns to its left, and the coefficients are its own entries:

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

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Both statements are about relations among columns and so, both hold for A itself.

The theorem

THEOREM 11.2

The pivot columns of A form a basis of $\operatorname{Im} A$.

PROOF.

Relations among columns survive row operations. The pivot columns of RREF(A) are independent. This implies that the corresponding columns of A are likewise independent. Moreover, every non-pivot column of RREF(A) is a combination of pivot columns, and the same must thus hold for A. Hence the pivot columns of A span $\operatorname{Im} A$. □

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CAUTION

$\operatorname{Im} A$ is spanned by the columns of A corresponding to the pivot columns of $\operatorname{RREF}(A)$, **not** by the privot columns of $\operatorname{RREF}(A)$ dorectly. They tell us *which* columns to take, not what to take.

I.e., they tell us the indicies, no the values.

An example ▸

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 7 \\ 1 & 2 & 1 & 4 \end{bmatrix} \quad \text{reduces to} \quad \text{RREF}(A) = \begin{bmatrix} \mathbf{1} & 2 & 0 & 3 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Pivots are in columns 1 and 3 of RREF(A). Thus columns 1 and 3 of A span the image of Im A. Meaning

Im $A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, of dimension 2.

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$\mathrm{Im}\, A = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, of dimension 2. The discarded columns are combinations of these, with coefficients read off the reduced form:

$$\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

A small observation hiding a big theorem ▸

Note the following in our example: For our matrix A of shape 3×4, we have

PIVOT COLUMNS	2, so $\dim \operatorname{Col} A = 2$
FREE COLUMNS	2, so $\dim \operatorname{Ker} A = 2$
ALL COLUMNS	4

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ALL COLUMNS	4

This is no accidedn. Indeed very column is either a pivot column or a free one:

$$\dim \operatorname{Col} A \; + \; \dim \operatorname{Ker} A \; = \; n.$$

The number of pivots is called the **rank** of A. We will return to this identity and start to see just how deep it runs.

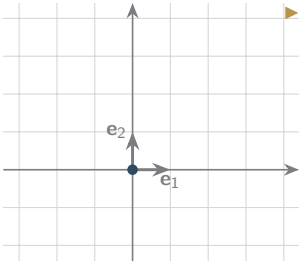
SECTION 2

Coordinates

A basis is a coordinate system

A basis subdivides the line (1D)/plane (2D)/space (3D or higher) into segments/squares/cubes (or higher dimensional units). The coordinates of a point say how far we have to walk along each axis to reach that point.

This is true for **any** basis. Indeed, there are two ways of thinking about it.



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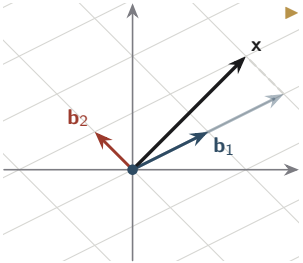

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Way 1. Stay in the standard basis and lay the new one over it as a lattice. With $\mathbf{b}_1 = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$, $\mathbf{b}_2 = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ the grid is skew, but every point still sits at a unique (not-ncessarily integer) crossing. Here $\mathbf{x} = 2\mathbf{b}_1 + \mathbf{b}_2$: two steps along $\mathbf{b}_1$, one along $\mathbf{b}_2$.

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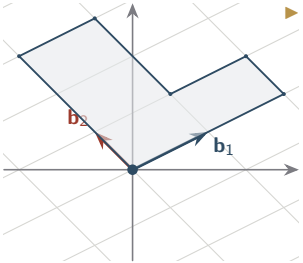

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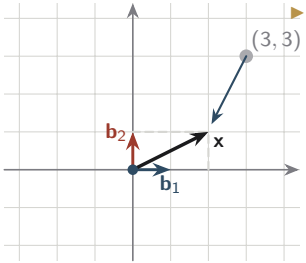

Way 2. Always stay on the standard grid and let the object move instead: Reshape the picture so that $\mathbf{b}_1, \mathbf{b}_2$ become the axes.

Now $\mathbf{x}$ sits at $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Grey shows where it was.

A basis is a coordinate system

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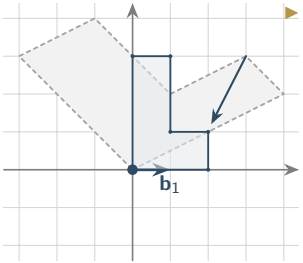

Way 2. Always stay on the standard grid and let the object move instead: Reshape the picture so that $\mathbf{b}_1, \mathbf{b}_2$ become the axes.

Every corner moves the same way, so the polygon is now drawn with its $\mathcal{B}$-coordinates. Grey shows where it was.

Unique representation

THEOREM 11.3 · Unique Representation Theorem

Let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis of V. For every $\mathbf{x} \in V$, there exists exactly one sequence $c_1, \ldots, c_n \in \mathbb{R}$ with

$$\mathbf{x} \;=\; c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

In mathematics, $\exists!$ is the common notation for "there exists exactly one", indicating existence $\exists$ and uniqueness.

PROOF.

Existence follows since a basis is, in particular, a spanning set. For *uniqueness* suppose

$$c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \;=\; d_1\mathbf{b}_1 + \cdots + d_n\mathbf{b}_n.$$

Subtracting gives $(c_1 - d_1)\mathbf{b}_1 + \cdots + (c_n - d_n)\mathbf{b}_n = \mathbf{0}$, and $\mathcal{B}$ is independent, so every $c_i - d_i = 0$. □

Spanning gives existence, independence gives uniqueness. Each half of the definition of a basis does exactly one job.

Coordinate vectors

DEFINITION 11.4 · Coordinate vector

Let $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis of V and $\mathbf{x} \in V$. The **coordinate vector** of $\mathbf{x}$ with respect to $\mathcal{B}$ is

$$[\mathbf{x}]_{\mathcal{B}} \coloneqq \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n,$$

where $c_1, \ldots, c_n$ is the unique sequence with $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

We need Theorem 11.3 for well-definedness, as otherwise, there would be no single vector to write down.

The order of the basis now matters, which is why we write $\mathcal{B}$ as a list "()/[]" rather than a set "{}".

The coordinate map

Every basis of an n-dimensional V therefore gives a function

$$V \longrightarrow \mathbb{R}^n, \qquad \mathbf{x} \longmapsto [\mathbf{x}]_{\mathcal{B}}\,.$$

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EXAMPLE 11.5 · The standard basis changes nothing

In $\mathbb{R}^n$ with $\mathcal{E} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ we have $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$, since $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$.

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EXAMPLE 11.6 ▸ · A different basis of $\mathbb{R}^2$

With $\mathcal{B} = \left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]\right)$ and $\mathbf{x} = \left[\begin{smallmatrix}5\\1\end{smallmatrix}\right]$, we have $\mathbf{x} = 3\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] + 2\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$ (Lecture 5) and thus $[\mathbf{x}]_{\mathcal{B}} = \left[\begin{smallmatrix}3\\2\end{smallmatrix}\right]$.

Coordinates outside $\mathbb{R}^n$

EXAMPLE 11.7 ▸

In P_2 with $\mathcal{B} = (1, x, x^2)$,

$$\left[a + bx + cx^2\right]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Coordinates outside $\mathbb{R}^n$

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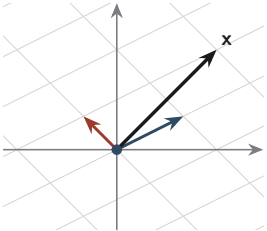
EXAMPLE 11.8 ▸

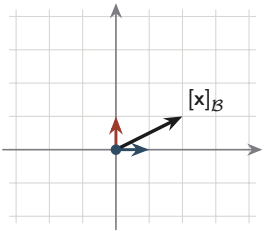
In P_2 with $\mathcal{C} = (1,\ 1+x,\ 1+x+x^2)$, take $p = 2 + 3x + 4x^2$. Comparing coefficients from the top down gives $c_3 = 4$, then $c_2 = -1$, then $c_1 = -1$, so $[p]_{\mathcal{C}} = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}$.

The same polynomial, expressed in two coordinate vectors. The morale is this: Coordinates belong to the pair (vecotr, basis). They are not intrinsic to the vector.

What the coordinate map does to the picture ▸



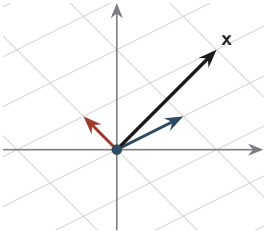
In V, the vector $\mathbf{x}$ has coordinate $(2, 1)$ in the "skew ruling".



The same vector if we make the new coordinate system "standard". Note, its again $(2, 1)$.

You can see this live in action at
https://academic-webpage-2aa4f4.gitlab.io/interactive_teaching/vector-spaces/vector-spaces.html
in the "Change of Basis" tab, by pressing "Run" in "Demonstration" mode.

What the coordinate map does to the picture ▸



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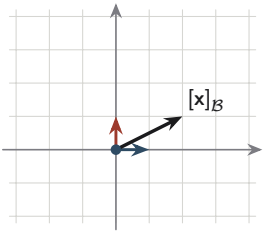

The same vector if we make the new coordinate system "standard". Note, its again $(2, 1)$.

The map straightens the grid: $\mathbf{b}_1$ and $\mathbf{b}_2$ go to $\mathbf{e}_1$ and $\mathbf{e}_2$, and the crossings correspond one to one.

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Summary of Lecture 11

KEY FACT	A and RREF(A) have the same column relations as any relation among the columns of A is a vector of Ker A, and Ker $A =$ Ker RREF(A)
THEOREM	The columns of A, corresponding to the pivot columns RREF(A) form a basis of Col A.
COUNT	dim Col $A +$ dim Ker $A = n$, as each column is a pivot or free.
UNIQUENESS	If $\mathcal{B}$ is a basis of V, each $\mathbf{x} \in V$ is $c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ in exactly one way.
COORDINATES	$[\mathbf{x}]_{\mathcal{B}}$ is that unique list of coordinates. $\mathbf{x} \to [\mathbf{x}]_{\mathcal{B}}$ gives a map $V \to \mathbb{R}^n$.
PICTURE	A basis "tiles" the space, and the coordinate map "straightens" that tiling into the standard squares.

Before next time

AT HOME

1. For $A = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 6 & 1 & 5 \\ 1 & 3 & 1 & 3 \end{bmatrix}$ find RREF(A), a basis of Col A ▸ and a basis of Ker A ▸. Check the count of the third summary line.
2. $\mathcal{B} = (\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix})$ is a basis of $\mathbb{R}^2$. Find $[\mathbf{e}_1]_{\mathcal{B}}$ ▸ and $[\mathbf{e}_2]_{\mathcal{B}}$ ▸.
3. In P_2 with $\mathcal{C} = (1,\ 1+x,\ 1+x+x^2)$, find $\left[3-2x+5x^2\right]_{\mathcal{C}}$ ▸.

END OF LECTURE 11

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Solution: question 1 ▶

SOLUTION

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 6 & 1 & 5 \\ 1 & 3 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} \mathbf{1} & 3 & 0 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: question 1 ▸

SOLUTION

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Pivots in columns $1, 3$, so taking those columns *of* A, $\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Free variables $x_2 = s$, $x_4 = t$ give $x_1 = -3s - 2t$, $x_3 = -t$, so $\operatorname{Ker} A = \operatorname{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

$2 + 2 = 4$, the number of columns.

Solution: question 2

SOLUTION

For $\mathbf{e}_1$ ▸: $a\begin{bmatrix}1\\2\end{bmatrix}+b\begin{bmatrix}3\\5\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ gives $a+3b=1$ and $2a+5b=0$. The second gives $a=-\frac{5}{2}b$, so $\frac{1}{2}b=1$, $b=2$ and $a=-5$: $[\mathbf{e}_1]_{\mathcal{B}}=\begin{bmatrix}-5\\2\end{bmatrix}$.

Solution: question 2

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SOLUTION

For $\mathbf{e}_2$ ▸: $a + 3b = 0$ and $2a + 5b = 1$ give $a = -3b$, then $-b = 1$, so $b = -1$ and $a = 3$, giving $[\mathbf{e}_2]_{\mathcal{B}} = \begin{bmatrix}3\\-1\end{bmatrix}$.

Check: $3\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$.

Solution: question 3 ▸

SOLUTION

We need $c_1 + c_2(1+x) + c_3(1+x+x^2) = 3 - 2x + 5x^2$. Comparing coefficients from the highest power down:

$$c_3 = 5, \qquad c_2 + c_3 = -2, \qquad c_1 + c_2 + c_3 = 3,$$

so $c_2 = -7$ and $c_1 = 3 + 7 - 5 = 5$, giving $\left[3 - 2x + 5x^2\right]_{\mathcal{C}} = \begin{bmatrix} 5 \\ -7 \\ 5 \end{bmatrix}$.

Solution: question 3 ▸

SOLUTION

We need $c_1 + c_2(1+x) + c_3(1+x+x^2) = 3 - 2x + 5x^2$. Comparing coefficients from the highest power down:

$$c_3 = 5, \qquad c_2 + c_3 = -2, \qquad c_1 + c_2 + c_3 = 3,$$

so $c_2 = -7$ and $c_1 = 3 + 7 - 5 = 5$, giving $\left[3 - 2x + 5x^2\right]_{\mathcal{C}} = \begin{bmatrix} 5 \\ -7 \\ 5 \end{bmatrix}$.

Check: $5 - 7(1+x) + 5(1+x+x^2) = 3 - 2x + 5x^2$.

Working from the top power downwards makes each coefficient fall out in turn, as the system is already triangular.