LECTURE 10

Linear independence, bases and dimension

Minimal spanning sets

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Recall

LAST LECTURE

- T is **linear** if it respects addition and scaling. In that case $T(\mathbf{0}) = \mathbf{0}$ and

$$T(\sum c_i \mathbf{v}_i) \;=\; \sum c_i T(\mathbf{v}_i).$$

- A linear map is determined by a spanning set.
- Every linear $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is multiplication by $[T] = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$.
- $\operatorname{Ker} T$ and $\operatorname{Im} T$ are subspaces, and every subspace is both an image and a kernel.

Today we consider how small we can make spanning sets.

Warm-up

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Let $T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x-y \\ 2x \end{smallmatrix}\right]$. Is T linear? If so, find $[T]$ and $\operatorname{Ker} T$.

WARM-UP

Let $T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x-y \\ 2x \end{smallmatrix}\right]$. Is T linear? If so, find $[T]$ and $\operatorname{Ker} T$.

SOLUTION

It is linear since each coordinate of the output is a combination of x and y, with no constants and no products. Feeding in the standard vectors ▸, $T(\mathbf{e}_1) = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and $T(\mathbf{e}_2) = \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]$, we get

$$[T] \;=\; \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \;=\; \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}.$$

Its determinant is $0 + 2 = 2 \neq 0$, so the matrix is invertible and $\operatorname{Ker} T = \{\mathbf{0}\}$ ▸.

SECTION 1

Linear independence

Definition

DEFINITION 10.1 · Linear independence

We say that vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$ are **linearly independent** if

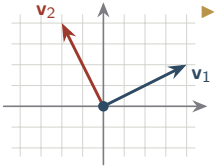
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p \;=\; \mathbf{0}$$

implies $c_1 = c_2 = \cdots = c_p = 0$. Otherwise they are called **linearly dependent**.

The combination with all coefficients zero always gives the zero vector $\mathbf{0}$. The question is whether there is *another*. Dependence immediately implies that one (or more) of them can be expressed by the remaining ones, see Proposition 10.6.

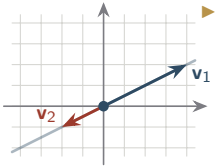
Note the important fact: Independence is a global property, meeaning of the whole set, not of any single vector of it.

The two pictures



Independent. There is no way you can add (a non-zero scaler of) $\mathbf{v}_2$ to (a non-zero scaler of) $\mathbf{v}_1$ to get to the origin.

Once you go the direction of $\mathbf{v}_1$, the only direction that brings you back is the opposite of $\mathbf{v}_1$.



Dependent. Here $\mathbf{v}_1 + 2\mathbf{v}_2 = \mathbf{0}$, which is a genuine cancellation, with coefficients not all zero.

Examples

EXAMPLE 10.2 ▸ · Independent

$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$: if $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{0}$ then $c_1 = c_2 = 0$ at once.

Examples

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EXAMPLE 10.3 ▸ · Dependent

$\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\4\end{bmatrix}$: here $2\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}2\\4\end{bmatrix} = \mathbf{0}$, and the coefficients $2, -1$ are not both zero.

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EXAMPLE 10.4 ▸ · Dependent, less obviously

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$: no two of them are parallel, yet $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix} = \mathbf{0}$.

Three facts to have at hand

PROPOSITION 10.5

1. *A single vector* $\mathbf{v}$ *is linearly independent if and only if* $\mathbf{v} \neq \mathbf{0}$.
2. *No collection containing* $\mathbf{0}$ *is linearly independent.*
3. *Two vectors* $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ *are linearly independent if and only if* $\mathbf{u} \neq c\,\mathbf{v}$ *for any* $c \in \mathbb{R}$.

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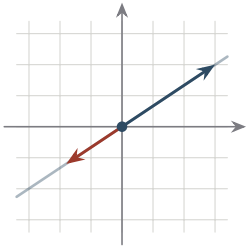
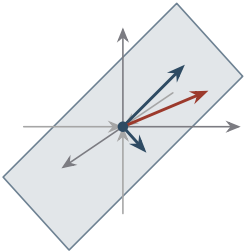
PROOF.

(1) If $\mathbf{v} \neq \mathbf{0}$ then $c\mathbf{v} = \mathbf{0}$ forces $c = 0$; and $1 \cdot \mathbf{0} = \mathbf{0}$ shows the converse.
(2) Take 1 as the coefficient of $\mathbf{0}$ and 0 as the coefficient of every other vector.
(3) Assume there exists a coefficicent c with $\mathbf{u} = c\mathbf{v}$. Then $1, -c$ is a system of non-trivial coefficients with $\mathbf{u} - c\mathbf{v} = \mathbf{0}$. Conversely, if $a\mathbf{u} + b\mathbf{v} = \mathbf{0}$, w.l.o.g $a \neq 0$ and $\mathbf{u} + \frac{b}{a}\mathbf{v} = \mathbf{0}$, thus $\mathbf{u} = -\frac{b}{a}\mathbf{v}$. □

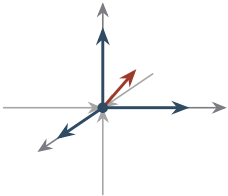
Dependence, geometrically



Two vectors ▸
on one line



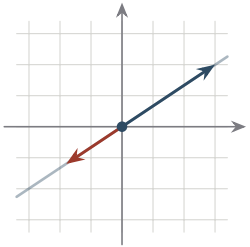
Three vectors ▸
in one plane



Four vectors ▸
in $\mathbb{R}^3$

Each picture shows a dependent collection.

Dependence, geometrically



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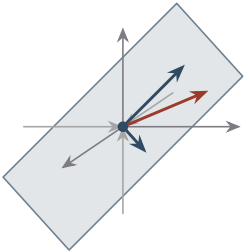
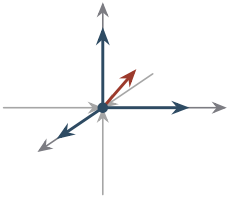


Three vectors ▸
in one plane



Four vectors ▸
in $\mathbb{R}^3$

Each picture shows a dependent collection. Two on a line, three in a plane, four in space.

Seeing a pattern?

Dependence means redundancy

PROPOSITION 10.6

A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ *with* $n \geq 2$ *is linearly dependant if and only if there exist a member amongst them that can be expressed as a linear combination of the others.*

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PROOF.

$\implies$ Asssume $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_n = \mathbf{0}$ with some $c_j \neq 0$. Divide by c_j and move the rest across:

$$\mathbf{v}_j = -\frac{c_1}{c_j}\mathbf{v}_1 - \cdots - \frac{c_{j-1}}{c_j}\mathbf{v}_{j-1} - \frac{c_{j+1}}{c_j}\mathbf{v}_{j+1} - \cdots - \frac{c_p}{c_j}\mathbf{v}_n.$$

On the RHS, we omit the term with index j.

□

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PROOF.

$\Longleftarrow$ Conversely, assume $\mathbf{v}_j \neq \mathbf{0}$ with

$$\mathbf{v}_j \;=\; c_1\mathbf{v}_1 + \cdots + c_{j-1}\mathbf{v}_{j-1} + c_{j+1}\mathbf{v}_{j+1} + \cdots c_n\mathbf{v}_n.$$

Note that on the RHS, we omit the term with index j.

Then

$$c_1\mathbf{v}_1 + \cdots + c_{j-1}\mathbf{v}_{j-1} + \mathbf{v}_j + c_{j+1}\mathbf{v}_{j+1} + \cdots c_n\mathbf{v}_n \;=\; 0.$$

□

Using the proposition

EXAMPLE 10.7 ▸ · In P_2

$p_1 = 1 + x$, $p_2 = x + x^2$, $p_3 = 1 + 2x + x^2$ are dependent, because $p_3 = p_1 + p_2$.

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EXAMPLE 10.8 ▸ · In P_1

$1 + x$ and $1 - x$ are independent as neither is a multiple of the other.

Or directly: $a(1 + x) + b(1 - x) = (a + b) + (a - b)x = 0$ forces $a + b = 0$ and $a - b = 0$, so $a = b = 0$.

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EXAMPLE 10.9 ▸ · In $\mathrm{Mat}_{2\times 2}(\mathbb{R})$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $I = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are dependent, since $I = A + B$.

SECTION 2

Bases

Definition

DEFINITION 10.10 · Basis

A subset $\mathcal{B} \subseteq V$ is a **basis** of V if

1. $\mathcal{B}$ spans V, and
2. $\mathcal{B}$ is linearly independent.

Note, another (maybe more) common term for *spans* V is *generates* V.

Meaning, exactly the right amount of vectors to give all of V, and not a single more. This is the "*a* best spanning set" promised in Lecture 7. Note, its "a" not "the" as the word "best" generally deserves, because, as we will see, even basis are not unique.

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EXAMPLE 10.11

$\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$; $\{1, x, \ldots, x^n\}$ of P_n; the mn matrix units of $\mathrm{Mat}_{m\times n}(\mathbb{R})$. We checked the spanns in Lecture 7. Independence is easily verified as well.

Span and liner independece

We note two simple observations on spans and linear independence: If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans V, adding a vector to S will still span V. But if we take away a vector, the rest might no longer span V.

If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, *throwing away* one of the $\mathbf{v}_j$ will still leave the rest linearly independent. But adding vectors might make the bigger set linearly dependent.

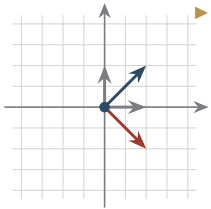
SPAN	Stable under adding elements, but not removing.
LINEAR INDEPENDENCE	Stable under removing elements, but not adding.

Bases are not unique

Already $\mathbb{R}$ has infinitely many. Indeed $\{1\}$ is a basis, $\{2\}$ is a basis, ... and so is $\{c\}$ for every $c \neq 0$.

In $\mathbb{R}^2$, both $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ are bases.

In P_2, $\{1, 1+x, 1+x+x^2\}$ is a basis. See Lecture 7 for the span argument.



Two bases of the plane. Onc set is the grey, the other consists of the blue and red

A (far) aside on the infinite case

ASIDE

Everything today has been stated for finite set, but the notions generalise. $\mathbb{R}[x]$ has the basis $\{1, x, x^2, \ldots\}$. This is an infinite set, yet none can be discarded. Note also that there is a myrriad of meanings to the word *infinite*.

For example, in $\mathbb{R}[x]$, each element only uses a finite set of generators, but things can be different. For example, analysis often uses infinite sums that converge. Analysing the notion of a basis in this case leads towards Fourier series and Hilbert spaces. That junction of algebra and analysis is where operator algebras and infinite-dimensional representation theory live. Beautiful, and far past the edge of this module.

There are also other ways why *infinite* is not such a clear term; that leads towards cardinality theory, a branch of set theory.

Three questions

- Does every vector space have a basis?

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- If so, how do we find one?

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- Does every vector space have a basis?
- If so, how do we find one?
- And what are bases actually good for?

SECTION 3

Bases of kernels

How to create a basis

From Lecture 8 we already have an approach: Row reduce A, give each free variable a parameter, and collect. That produces a *spanning* set of $\operatorname{Ker} A$, which is, in fact a basis.

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WHY IT IS INDEPENDENT

Each free variable gets its own spanning vector, and that vector carries a 1 in its own free slot and a 0 in every other free slot. As such, its a subset of the *standard basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and thus, linearly independent.

Remember, linear independence is preserved for subsets.

An example (1/2) ▸

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 7 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

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$$\xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A)$$

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Pivots in columns 1 and 3, so x_1 and x_3 are basic, x_2 and x_4 are free.

An example (2/2) ▸

Set $x_2 = s$ and $x_4 = t$. The pivot rows read

$$x_1 + 2s + 3t = 0, \qquad x_3 + t = 0,$$

so $x_1 = -2s - 3t$ and $x_3 = -t$.

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \;=\; s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \operatorname{Ker} A \;=\; \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

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Look at the highlighted free slots. The pattern $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ against $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ is exactly the independence argument of the last slide. So this spanning set is a **basis** of $\operatorname{Ker} A$, and $\operatorname{Ker} A$ needs two vectors. No more, no less.

SECTION 4

Dimension

The casting-out lemma

LEMMA · Casting out

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be linearly dependent. There exsts $\mathbf{v}_j$ that may be removed without changing the span. In formulas:

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \;=\; \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n\}.$$

PROOF.

By Proposition 10.6, there exists a $\mathbf{v}_j$ that is a combination of the others. Such a $\mathbf{v}_j$ may then be thrown out since the other elements already reach it and any combination using $\mathbf{v}_j$ may thus be reached by the other as well (transitivity is the algebraic term for this behaviour). □

Note, to avoid always writint $\mathbf{v}_{j-1}, \mathbf{v}_{j+1}$, to indicate that $\mathbf{v}_j$ is missing (which is not always ideal anyway since its easy to miss the jump), it is an accepted notation in maths to write $\hat{\mathbf{v}_j}$ to mean the same. Meaning, we could have written the RHS as $\mathrm{Span}\{\mathbf{v}_1, \ldots, \hat{\mathbf{v}_j}, \ldots \mathbf{v}_n\}$.

A small warning

CAUTION

Note, if two elements $\mathbf{v}_i$ and $\mathbf{v}_j$ from a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ can be expressed by a linear combination of the others not involving the given element, you can thrown out either **one**, but not nessecarily both. You must to the test again for the other element.

For example ▸, if $S = \{[\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}], [\begin{smallmatrix}1\\1\end{smallmatrix}]\}$, then

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}$$

but you can not throw out both $[\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}]$ at the same time.

Note, if you can express $\mathbf{v}_i$ and $\mathbf{v}_j$ with the others not involving either of them, then you can throw out both.

Casting out until it stops

KEY IDEA

Start with any finite spanning set of V. If it is dependent, cast **one** vector out that can be reached by the others. This leaves the span unchanged, and one size smaller. Repeating this clearly gives us a linearly independent set.

What remains is a set that spans V and is independent, meaning a **basis**.

Why? A finite set cannot shrink for ever, so the process must halt. And it will do so exactly when the remaining set is independent.

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THEOREM 10.12 ▸ · *First of three questions*

Every finite spanning set of V contains a basis of V. In particular, every vector space with a finite spanning set has a basis.

Dimension

DEFINITION 10.13

V is **finitely generated**, or **finite-dimensional**, if it has a finite spanning set (or basis). Its **dimension** $\dim V$ is the number of vectors in its basis.

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HOLD ON. IS THIS WELL DEFINED?

We already said that bases were not unique. So, for this definition to be well defined, we need the number of elements inside any basis of a given vector space V to be the same.
That is true! And we shall prove it, shortly. As such, sayign "*the* dimension" is allowed.

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$$\dim \mathbb{R}^n = n, \qquad \dim P_n = n+1, \qquad \dim \mathrm{Mat}_{m\times n}(\mathbb{R}) = mn, \qquad \dim\{\mathbf{0}\} = 0.$$

Summary of Lecture 10

INDEPENDENT	$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \implies c_1 = \cdots = c_p = 0$.
DEPENDENT	If vectors are not independent, they are **dependent**.
SIMPLE CHECKS	$\{\mathbf{v}\}$ is independent iff $\mathbf{v} \neq \mathbf{0}$; anything containing $\mathbf{0}$ is dependent; two non-zero vectors, iff neither is a multiple of the other.
REDUNDANCY	Dependent $\iff$ one is a combination of the others.
BASIS	A linearly independent spanning set. Never unique (unless for $\{\mathbf{0}\}$).
KERNELS	Row-reduction gives a basis as the free slots make its vectors independent.
DIMENSION	Casting out turns any finite spanning set into a basis; $\dim V$ is its size. $\dim \mathbb{R}^n = n$, $\dim P_n = n + 1$.

Before next time

AT HOME

1. Independent or dependent? (a) ▸ $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\4\\6\end{bmatrix}$; (b) ▸ $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$; (c) $1+x,\ 1-x,\ x^2$ in P_2.
2. Find a basis of $\operatorname{Ker} A$ ▸ for $A = \begin{bmatrix}1 & 3 & 0 & 2\\2 & 6 & 1 & 5\end{bmatrix}$, and state $\dim \operatorname{Ker} A$.
3. Show that $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ is a basis of $\mathbb{R}^3$. Adding $\begin{bmatrix}2\\2\\0\end{bmatrix}$ still gives a spanning set. Which old vector may be cast out, and why?

END OF LECTURE 10

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Solution: Question 1

THREE COLLECTIONS

(a) Dependent. The second is twice the first ▸, so $2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}2\\4\\6\end{bmatrix} = \mathbf{0}$.

Solution: Question 1

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(b) Independent. Writing them as the columns of a matrix ▸, $\operatorname{Det}\begin{bmatrix}1&0&1\\0&1&1\\1&1&0\end{bmatrix} = 1(0-1) - 0 + 1(0-1) = -2 \neq 0$, so the only solution of $A\mathbf{c} = \mathbf{0}$ is $\mathbf{c} = \mathbf{0}$.

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(c) Independent. $a(1+x) + b(1-x) + cx^2 = 0$ ▸ gives $a + b = 0$, $a - b = 0$ and $c = 0$, hence $a = b = c = 0$.

Solution: Question 2 ▸

SOLUTION

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 2 & 6 & 1 & 5 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \text{RREF}(A).$$

Pivots in columns $1, 3$; $x_2 = s$, $x_4 = t$ are free, giving $x_1 = -3s - 2t$ and $x_3 = -t$.

Solution: Question 2 ▸

SOLUTION

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Pivots in columns $1, 3$; $x_2 = s$, $x_4 = t$ are free, giving $x_1 = -3s - 2t$ and $x_3 = -t$.

$$\operatorname{Ker} A = \operatorname{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}, \qquad \dim \operatorname{Ker} A = 2,$$

the two vectors being independent by the free-slot argument.

Solution: Question 3

A BASIS

As columns of a matrix ▸, Det $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 1(1-0) - 0 + 1(1-0) = 2 \neq 0$. So the three are independent; and three independent vectors in $\mathbb{R}^3$ span it, since $\dim \mathbb{R}^3 = 3$. Hence a basis.

Solution: Question 3

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CASTING OUT

$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, so the four are dependent ▸ and the lemma applies. You can cast out $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, since its is a multiple of the new one. What we may *not* do is remove one of the other initial two.