LECTURE 9

Linear transformations

Structure preserving maps between vector spaces; also known as matrices

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Recall

LAST LECTURE

- $A\mathbf{x} = \mathbf{b}$ is a system of equations. As row operations leave its solutions alone, $\operatorname{Ker} A = \operatorname{Ker}(\operatorname{RREF}(A))$.
- In $\operatorname{RREF}(A)$ the pivot columns give **basic** variables and the rest **free** ones. One free column, one spanning vector of $\operatorname{Ker} A$.
- $\mathbf{b} \in \operatorname{Im} A$ exactly when $[\,A \mid \mathbf{b}\,]$ reduces without a row $[\,0 \cdots 0|c\,]$, $c \neq 0$.

We have seen that a matrix is both a rectangle of numbers, and a rule $\mathbf{x} \mapsto A\mathbf{x}$.

Warm-up

WARM-UP

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix}$.

1. Write $\operatorname{Ker} A$ as a span ▸.
2. Is $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathsf{T}} \in \operatorname{Im} A$? ▸

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2. Is $\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathsf{T}} \in \operatorname{Im} A$? ▸

1 $R_2 - 2R_1$ kills the second row, leaving $x + 2y - z = 0$. With $y = s$, $z = t$ this gives $x = -2s + t$, so $\operatorname{Ker} A = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2 Yes — it is the first column. Every column lies in $\operatorname{Im} A$, by definition.

SECTION 1

Linear transformations

Definition

DEFINITION 9.1 · Linear transformation

Let V and W be vector spaces. A map $T\colon V \to W$ is a **linear transformation** if for all $\mathbf{u}, \mathbf{v} \in V$ and all $\lambda \in \mathbb{R}$,

$$T(\mathbf{u}+\mathbf{v}) \;=\; T(\mathbf{u}) + T(\mathbf{v}) \qquad \text{and} \qquad T(\lambda \mathbf{v}) \;=\; \lambda T(\mathbf{v}).$$

In other words, it does not matter whether we combine first and apply T, or apply T first and combine. We say that the map *respects* the two operations.

> ASIDE: GROUP HOMOMORPHISM
>
> A map $f : G \to H$ of groups is called a **group homomorphism** if for all $g, g' \in G$
>
> $$f(g+g') \;=\; f(g) + g(g').$$

Homomorphisms respect special elements

LINEAR TRANSFORMATIONS RESPECT THE ZERO VECTOR

Every linear transformation satisfies $T(\mathbf{0}) = \mathbf{0}$, since

$$T(\mathbf{0}) \;=\; T(0 \cdot \mathbf{0}) \;=\; 0 \cdot T(\mathbf{0}) \;=\; \mathbf{0}.$$

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In general, we want maps to preserve all structure, including "special" elements such as $\mathbf{0}$. We did not include this in the axioms of a linear transformation because it follows for free.

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ASIDE: GROUP HOMOMORPHISMS RESPECT THE IDENTITY

Recall that (A3) requires the existence of the neutral element $\mathbf{0}$. Although the proof above uses scalar multiplication, addition alone is enough. More generally, every group homomorphism $f\colon G \to H$ satisfies

$$f(1_G) = 1_H,$$

where 1_G and 1_H are the neutral elements of G and H.

Aside: A word on the term

Its more correct name is a **vector space homomorphism**, but we will use them interchangibly. Indeed, for this module, I will more often refer to it as a linear transformation.

> FAR ASIDE: MORHISMS AND HOMOMORPHISMS
>
> Any type of algebraic/mathematical object comes with its associated notion of map. The universal term used for these is a **morphism**, or **homomorphism** in more "structured" settings.
>
> Examples are groups and group homorphisms, rings and ring homomorphisms, modules and module homomorphisms, ... and countless others. Some exceptions are (continous, differentiable, ...) maps (instead of (set, differentiable manifold, ...) homomorphisms).
>
> It is not immediate why, and indeed sounds counterintuitive, but it is a true fact that morphisms are the more important of the two, not objects. We learn vector spaces and think of vector space homomorphisms as an afterthought. In truth, vector spaces are (nearly) irrelevant; it's the vector space homomorphisms that see and carry all!

A first example, checked

EXAMPLE 9.2

Is $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$, a linear transformation?

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Take $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then

$$T(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} (u_1+v_1)-(u_2+v_2) \\ (u_1+v_1)+(u_2+v_2) \end{bmatrix} = \begin{bmatrix} u_1-u_2 \\ u_1+u_2 \end{bmatrix} + \begin{bmatrix} v_1-v_2 \\ v_1+v_2 \end{bmatrix} = T(\mathbf{u})+T(\mathbf{v}),$$

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Take $\mathbf{u} = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right]$ and $\mathbf{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$. Then

$$T(\mathbf{u}+\mathbf{v}) \;=\; \begin{bmatrix} (u_1+v_1)-(u_2+v_2) \\ (u_1+v_1)+(u_2+v_2) \end{bmatrix} \;=\; \begin{bmatrix} u_1-u_2 \\ u_1+u_2 \end{bmatrix} + \begin{bmatrix} v_1-v_2 \\ v_1+v_2 \end{bmatrix} \;=\; T(\mathbf{u})+T(\mathbf{v}),$$

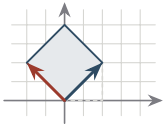
and $T(\lambda\mathbf{v}) = \begin{bmatrix} \lambda v_1 - \lambda v_2 \\ \lambda v_1 + \lambda v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 - v_2 \\ v_1 + v_2 \end{bmatrix} = \lambda T(\mathbf{v})$.

So **Yes.** Both conditions hold.

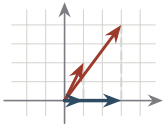
Two examples one can see

T of the last slide rotates the plane by 45° and stretches it by $\sqrt{2}$ ▸.

The **projection** $P\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ flattens the plane onto the horizontal axis ▸. It too is linear as projecting a sum is the sum of the projections.



T: rotate and stretch



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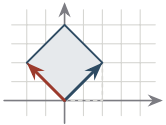
Note that P destroys information, as everything on a vertical line has the same image, while T does not. We shall measure that with the kernel.



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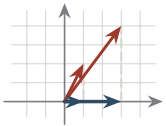

P: project

Two that are not linear

EXAMPLE 9.3 · Affine

$T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x+1 \\ y \end{smallmatrix}\right]$ merely shifts the plane. But

$$T(\mathbf{0}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \mathbf{0},$$

and

$$T(2\mathbf{v}) = \left[\begin{smallmatrix} 2x+1 \\ 2y \end{smallmatrix}\right] \neq 2T(\mathbf{v}) = \left[\begin{smallmatrix} 2x+2 \\ 2y \end{smallmatrix}\right].$$ No.

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EXAMPLE 9.4 · Quadratic

$T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \begin{bmatrix} x^2 \\ y \end{bmatrix}$ has $T(2\mathbf{v}) = \begin{bmatrix} 4x^2 \\ 2y \end{bmatrix}$, while $2T(\mathbf{v}) = \begin{bmatrix} 2x^2 \\ 2y \end{bmatrix}$. **No.**

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Squares are out, too. This is why it's called *linear* algebra. Shifts are subtler, the affine world is sort of the same as the linear world, but not direclty.

Beyond $\mathbb{R}^n$

Nothing in the definition mentioned coordinates.

EXAMPLE 9.5 · Differentiation

$D\colon P_2 \to P_1$, $D(p) = p'$, is linear, since $(p+q)' = p' + q'$ and $(\lambda p)' = \lambda p'$.

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EXAMPLE 9.6 · Trace

$\mathrm{Tr}\colon \mathrm{Mat}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$, $\mathrm{Tr}\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = a + d$, is linear as the diagonal of a sum is the sum of the diagonals.

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EXAMPLE 9.7 · Evaluation

$E_3\colon P_2 \to \mathbb{R}$, $E(p) = p(3)$, is linear, because $(p+q)(3) = p(3) + q(3)$.

Of course, it works for any number.

Linear maps respect linear combinations

PROPOSITION 9.8

If $T\colon V \to W$ is linear, then for all $\mathbf{v}_1, \dots, \mathbf{v}_p \in V$ and $c_1, \dots, c_p \in \mathbb{R}$,

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) \;=\; c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p).$$

PROOF.

Use the fact that T respects sums $p-1$ times to split the sum, then use the fact that it respects the scalers to the c_i's. I.e., Step 1:

$$T(c_1\mathbf{v}_1+\cdots+c_p\mathbf{v}_p) \;=\; T(c_1\mathbf{v}_1)+T(c_2\mathbf{v}_2+\cdots+c_p\mathbf{v}_p) \;=\; c_1T(\mathbf{v}_1)+T(c_2\mathbf{v}_2+\cdots+c_p\mathbf{v}_p)$$

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KEY IDEA

A linear map is determined by its spanning set. Know T for $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, and you know T for Span(S).

SECTION 2

Matrices, again

Every matrix is a linear transformation

MATRIX $\Longrightarrow$ LINEAR TRANSFORMATION

Given an $m \times n$ matrix A, define

$$T_A \colon \mathbb{R}^n \to \mathbb{R}^m \qquad \text{given by} \qquad T_A(\mathbf{x}) \;=\; A\mathbf{x}.$$

This is linear, because matrix multiplication distributes:

$$A(\mathbf{u}+\mathbf{v}) \;=\; A\mathbf{u} + A\mathbf{v}, \qquad A(\lambda\mathbf{v}) \;=\; \lambda\, A\mathbf{v}.$$

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The interesting direction is the other one. *Every* linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ arises this way, from exactly one matrix.

Every linear transformation is a matrix

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear. Write $\mathbf{x}$ in the standard spanning set and use Proposition 9.8

$$T(\mathbf{x}) \;=\; T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \;=\; x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n).$$

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The right-hand side is a linear combination of fixed vectors with coefficients $x_1, \ldots, x_n$.

LINEAR TRANSFORMATION $\Longrightarrow$ MATRIX

Thus T can be expressed as $T(\mathbf{x}) = A\mathbf{x}$, where

$$A \;=\; [T] \;=\; \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}.$$

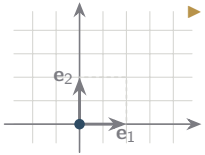
Where the columns come from

For $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$:

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so

$$[T] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$



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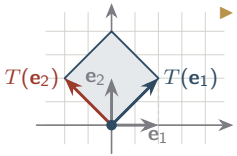

The unit square is carried to the parallelogram on those two columns, and $\text{Det}\,[T] = 2$. Areas are doubled, exactly as promised in Lecture 1.

SECTION 3

Kernel and image, in general

Two familiar sets

DEFINITION 9.9 ▸

For a linear transformation $T\colon V \to W$, define

$$\operatorname{Ker} T \;:=\; \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}\} \;\subseteq\; V, \qquad \operatorname{Im} T \;:=\; \{T(\mathbf{v}) \mid \mathbf{v} \in V\} \;\subseteq\; W.$$

For $T = T_A$ these are exactly $\operatorname{Ker} A$ and $\operatorname{Col} A$, so nothing new has happened, except that V and W may now be polynomials, matrices, or anything else.

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EXAMPLE 9.10

For $D\colon P_2 \to P_1$, $\operatorname{Ker} D$ is the constants and $\operatorname{Im} D = P_1$. For $E(p) = p(3)$, $\operatorname{Ker} E$ is the polynomials vanishing at 3, and $\operatorname{Im} E = \mathbb{R}$.

Both are subspaces

THEOREM 9.11

$\operatorname{Ker} T$ *is a subspace of* V*, and* $\operatorname{Im} T$ *is a subspace of* W*.*

PROOF.

For $\operatorname{Ker} T$: $T(\mathbf{0}) = \mathbf{0}$ gives (U1). If $T(\mathbf{u}) = T(\mathbf{w}) = \mathbf{0}$ then $T(\mathbf{u} + \mathbf{w}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$ and $T(\lambda \mathbf{u}) = \lambda \mathbf{0} = \mathbf{0}$, giving (U2) and (U3).

For $\operatorname{Im} T$: $\mathbf{0} = T(\mathbf{0})$ is attained; and $T(\mathbf{u}) + T(\mathbf{w}) = T(\mathbf{u} + \mathbf{w})$, $\lambda T(\mathbf{u}) = T(\lambda \mathbf{u})$ are attained too. □

In both sides, the hardest part is understanding *what* we have to prove. Proving it then is a single line.

Every subspace can be realised as both

THEOREM 9.12

Let V be a vector space and $H \subseteq V$ a subspace. There are vector spaces U and W and linear transformations $S\colon U \to V$ and $T\colon V \to W$ with

$$\operatorname{Im} S \;=\; H \;=\; \operatorname{Ker} T.$$

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WHAT IT IS SAYING

Lecture 7 gave two ways of presenting a subspace: "Inside out" from a list of vectors, and "the outside in" by restricting to the elements that a list of equations permits.

The theorem says *both are always available*, for every subspace. Neither description is more general, so we may always take whichever makes the question at hand easy.

The theorem, concretely

Let $H = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\} \subseteq \mathbb{R}^3$, the line through $\begin{bmatrix}1\\0\\1\end{bmatrix}$.

The theorem, concretely

Let $H = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\} \subseteq \mathbb{R}^3$, the line through $\begin{bmatrix}1\\0\\1\end{bmatrix}$.

As an image. ▸ Take $U = \mathbb{R}$ and

$$S = \begin{bmatrix}1\\0\\1\end{bmatrix},$$

a 3×1 matrix. Then $S\colon \mathbb{R} \to \mathbb{R}^3$ is given by $(t) = t\begin{bmatrix}1\\0\\1\end{bmatrix}$, so $\text{Im}\, S = H$.

As a kernel. ▸ Take $W = \mathbb{R}^2$ and

$$T = \begin{bmatrix}0 & 1 & 0\\1 & 0 & -1\end{bmatrix}.$$

Then $T\mathbf{x} = \mathbf{0}$ says $y = 0$ and $x = z$, so $\text{Ker}\, T = H$.

The theorem, concretely

Let $H = \mathrm{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\} \subseteq \mathbb{R}^3$, the line through $\begin{bmatrix}1\\0\\1\end{bmatrix}$.

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Then $T\mathbf{x} = \mathbf{0}$ says $y = 0$ and $x = z$, so $\mathrm{Ker}\, T = H$.

The columns of S span H, the rows of T are the equations cutting H out. Passing between the two is precisely the work of the last two lectures.

For a general V, the same construction works once we have a basis. We finally come to it in the next lecture.

Summary of Lecture 9

DEFINITION	T is **linear** if $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$. The more "structural" term is **vector space homomorphism.**
QUICK FACTS	$T(\mathbf{0}) = \mathbf{0}$ always. So affine maps are out. So is everything "non-linear" such as squares, even if they respect $\mathbf{0}$.
COMBINATIONS	As $T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$, T is determined by a spanning set.
SAME THING	$T_A(\mathbf{x}) = A\mathbf{x}$ is linear, and every linear $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is T_A for $A = [T] = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$.
KERNEL, IMAGE	$\operatorname{Ker} T$ and $\operatorname{Im} T$ are subspaces of V and W. For T_A, they are $\operatorname{Ker} A$ and $\operatorname{Col} A$.
THEOREM	Every subspace is an image *and* a kernel, for appropriate maps.

Before next time

AT HOME

1. Which of these are linear? (a) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x-y \\ 0 \end{bmatrix}$; (b) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x \end{bmatrix}$; (c) $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$; (d) $T\colon P_2 \to P_2$, $T(p) = p'' + p$.
2. Find $[T]$ for $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-z \\ 2y+z \end{bmatrix}$, then describe $\operatorname{Ker} T$ as a span ▸ and say what $\operatorname{Im} T$ is.
3. Let $H = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\} \subseteq \mathbb{R}^3$. Find a matrix S with $\operatorname{Im} S = H$, and a matrix T with $\operatorname{Ker} T = H$ ▸.

END OF LECTURE 9

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Solution: question 1

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(d) Yes. $(p+q)'' + (p+q) = (p'' + p) + (q'' + q)$, and likewise for scalars.

Solution: question 2 ▸

SOLUTION

Feeding in the standard vectors, $T(\mathbf{e}_1) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $T(\mathbf{e}_2) = \left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$, $T(\mathbf{e}_3) = \left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$, so

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And $\operatorname{Im} T = \operatorname{Col}[T] = \mathbb{R}^2$, since the first two columns are already not parallel.

Solution: question 3 ▸

AS AN IMAGE

Put the spanning vectors in as columns: $S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, a map $\mathbb{R}^2 \to \mathbb{R}^3$ with $\operatorname{Im} S = \operatorname{Col} S = H$.

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AS A KERNEL

We need one equation $\alpha x + \beta y + \gamma z = 0$ satisfied by both spanning vectors: $\alpha + \beta = 0$ and $\beta + \gamma = 0$. Taking $\alpha = 1$ gives $\beta = -1$, $\gamma = 1$, so $T = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$ has $\operatorname{Ker} T = H$.

One equation, because H is a plane. A line would have needed two.