LECTURE 8

Row echelon form

An algorthm that keeps on giving

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Where we got to

LAST LECTURE

- S **spans** V means $\mathrm{Span}(S) = V$. Equvialently: V is **generated by** S.
- $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ spans $\mathbb{R}^n$, $\{1, x, x^2\}$ spans P_2, the matrix units span $\mathrm{Mat}_{m\times n}(\mathbb{R})$. None of these are unique.
- $\mathrm{Col}\, A = \mathrm{Span}\{a_{-,1}, \ldots, a_{-,n}\} = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$, also written $\mathrm{Im}\, A$.
- $\mathrm{Ker}\, A$: Membership is easy but spanning set is hard. $\mathrm{Col}\, A$ is the other way round.

Today, we explore the difficult sides.

Warm-up

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Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. Is $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ in $\operatorname{Col} A$? ▸ Is $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T$? ▸

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SOLUTION

A vector of $\operatorname{Col} A$ is $a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$. The first two entries are free, the third must be their sum. So $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in $\operatorname{Col} A$, with $a = 1$, $b = 2$, and $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ is not, since $1 + 2 \neq 4$.

The answer fell out by inspection here. For a bigger matrix, things get exponentially harder, which is why an algorithm is so useful.

SECTION 1

Row operations, revisited

$A\mathbf{x} = \mathbf{b}$ is a system of equations ▸

Matrices and systems linear equations are effectively the same thing.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \Longleftrightarrow \quad \begin{aligned} x + 2y \quad\;\; &= 1 \\ 2x + 4y + z &= 3 \\ x + 2y + z &= 2 \end{aligned}$$

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- Taking $\mathbf{b} = \mathbf{0}$ and asking for *all* solutions describes $\operatorname{Ker} A$.
- Taking $\mathbf{b}$ general and asking whether *a* solution exists decides membership of $\operatorname{Col} A$.

So a good method for solving systems answers both questions at once.

The three elementary row operations, again ▸

1. **Swap** two rows: $R_i \leftrightarrow R_j$.
2. **Scale** a row by $\lambda \neq 0$: $R_i \leftarrow \lambda R_i$.
3. **Add** a multiple of one row to another: $R_i \leftarrow R_i + \lambda R_j$, $i \neq j$.

WHY WE MAY USE THEM

Each operation rewrites the equations of the system without changing which **x** satisfy them. This is because every operation can be undone: Swap back, scale by $1/\lambda$, subtract what was added.

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In particular, applying them to A leaves Ker A exactly as it was.

But the column space *does* change

CAUTION

Row operations preserve $\operatorname{Ker} A$. They do not preserve $\operatorname{Im} A$.

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} A' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \operatorname{Im} A = \operatorname{Span}\{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\}, \quad \operatorname{Im} A' = \operatorname{Span}\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\}.$$

Two different lines in $\mathbb{R}^2$. Both matrices have kernel $\{0\}$, as promised, but their images are not the same set.

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Two different lines in $\mathbb{R}^2$. Both matrices have kernel $\{0\}$, as promised, but their images are not the same set.

This is why we shall use row reduction on $\operatorname{Ker} A$ directly, but on $\operatorname{Col} A$ only through the question "is $A\mathbf{x} = \mathbf{b}$ solvable?" This is a question about the system, not about the image itself.
To put it another way, row operations preserve images only up to **ismorphisms**, not equalties.

SECTION 2

Echelon forms

Row echelon form

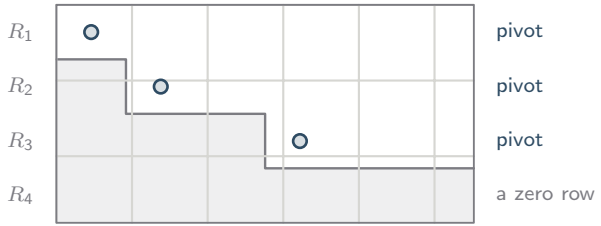
DEFINITION 8.1 ▸ · Row echelon form

A matrix is in **row echelon form** (REF) if:

1. Every row of zeros is at the bottom;
2. In each non-zero row, the first non-zero entry, call its **pivot**, lies strictly to the right of the pivot of the row above.

The second condition is the main difficulty, and main reward. The pivots descend in a staircase, and everything below the staircase is zero.

The staircase



Pivots strictly right and strictly down; zero rows at the foot. The columns carrying a pivot will matter a great deal in a moment.

Reduced row echelon form

DEFINITION 8.2 ▸ · Reduced row echelon form

A matrix is in **reduced** row echelon form (RREF) if it is in row echelon form and, in addition:

1. Every pivot equals 1;
2. Every pivot is the *only* non-zero entry in its column.

So, REF clears everything below the staircase, and RREF clears above it as well. This is precisely what Gauss–Jordan did to $[\,A \mid I\,]$ back in Lecture 2.

Three matrices, three verdicts

Decide if the following are REF, RREF or neither.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix}}_{\text{(a)}} \qquad \underbrace{\begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{(b)}} \qquad \underbrace{\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{(c)}}$$

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(a) **No.** This is neither, as a zero row sits above a non-zero one, and the pivot of the last row is to the left of the one above it.

(b) **Yes.** Echelon (REF), but not reduced (not RREF). ▸ The pivots are 2 and 5, not 1.

Even after scaling them to 1, the entry 1 above the second pivot would still have to go.

(c) **Yes.** This is even reduced (RREF). The pivots are 1, and each pivot column is otherwise zero. The non-pivot columns may be anything at all.

The theorem that makes this worth doing

THEOREM 8.3 · Existence and uniqueness of the RREF

Every matrix can be brought to a reduced row echelon form by a finite sequence of elementary row operations, and that reduced form is ***unique****. We denote it by* $\mathrm{RREF}(A)$.

Saying that it unique is another way of saying that it does not depend on which operations we use in which order. The result is always the same.

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> CAUTION
>
> The unreduced echelon form is not unique as scaling any row gives another one. It is the *reduced* form that is an *invariant* of A.

The procedure is Gaussian elimination for the echelon form, and Gauss–Jordan when we go on to clear above the pivots.

SECTION 3

A spanning set for the kernel

The plan ▸

KEY IDEA

Row operations leave $\operatorname{Ker} A$ unchanged, so

$$\operatorname{Ker} A \;=\; \operatorname{Ker}\left(\operatorname{RREF}(A)\right).$$

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A simple algorithm:

- Reduce A to $\operatorname{RREF}(A)$. Variables in pivot columns are called **basic** and the rest **free**.

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- Express the basic variables in term of the parameters (free terms).

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A simple algorithm:

- Reduce A to $\operatorname{RREF}(A)$. Variables in pivot columns are called **basic** and the rest **free**.
- Give each free variable its own parameter (not needed, but practical).
- Express the basic variables in term of the parameters (free terms).
- Consider the space this spans.

An example, slowly (1/2) ▸

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

The matrix is 4×3, so $\operatorname{Ker} A \subseteq \mathbb{R}^3$.

An example, slowly (1/2) ▸

$$A = \begin{bmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{2} & \boxed{4} & \boxed{6} \\ 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} A_1 = \begin{bmatrix} 1 & 2 & 3 \\ \boxed{0} & \boxed{0} & \boxed{0} \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

A zero row has appeared already, since the second equation was twice the first and said nothing new.

An example, slowly (1/2) ▸

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

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$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ \boxed{0} & \boxed{1} & \boxed{2} \\ 0 & 0 & 0 \\ \boxed{0} & \boxed{1} & \boxed{2} \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 - R_2} A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} \end{bmatrix}$$

Rows 2 and 4 agree, so their difference is a second zero row.

An example, slowly (1/2) ▸

$$A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ \boxed{0} & \boxed{0} & \boxed{0} \\ \boxed{0} & \boxed{0} & \boxed{0} \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The left matrix is a row echelon form, with pivots in columns 1 and 2 and two zero rows at the foot. Clearing above the second pivot makes it reduced.

An example, slowly (2/2) ▸

$$\mathrm{RREF}(A) = \begin{bmatrix} \mathbf{1} & 0 & -1 \\ 0 & \mathbf{1} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivots in columns $1, 2$, so x and y are **basic** and z is **free**. Set $z = t$; the two pivot rows then read

$$x - t = 0, \qquad y + 2t = 0.$$

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\mathrm{Ker}\, A = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

One free variable $\Longrightarrow$ One spanning vector $\Longrightarrow$ A line in $\mathbb{R}^3$.

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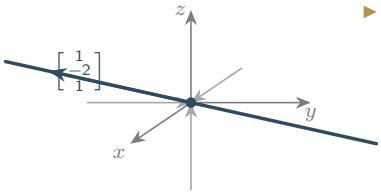
One free variable $\implies$ One spanning vector $\implies$ A line in $\mathbb{R}^3$.

To check your answer, multiply A by generating vector, it should give you zero. Indeed $A \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-4+3 \\ 2-8+6 \\ 1-6+5 \\ 0-2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

What the shape told us

The matrix was 4×3, meaning, regarded as a map, it was

$$A \colon \mathbb{R}^3 \to \mathbb{R}^4.$$



Thus, $\operatorname{Ker} A$ sits in $\mathbb{R}^3$.

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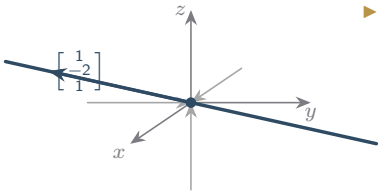

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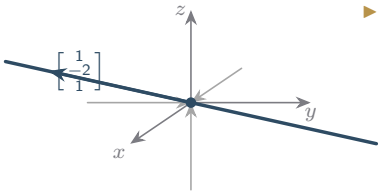

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> KEY IDEA
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> The number of spanning vectors of Ker is the number of columns without a pivot.

A second example, at speed ▸

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Pivots in columns 1 and 3; columns 2 and 4 are free. Set $y = s$ and $w = t$, so $x = -s + t$ and $z = -2t$:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \qquad \operatorname{Ker} B = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

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Two free columns, giving us two spanning vectors. As neither is a multiple of the other, we need both and we get a plane.

Independence of rows and columns

Observe: RREF(A) had as many non-zero rows as independent columns. No coincidence!

PROPOSITION 8.4

For an $m \times n$-matrix A, the number of linearly independent rows is the same as the number of linearly independent columns.

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For an $m \times n$-matrix A, the number of linearly independent rows is the same as the number of linearly independent columns.

PROOF.

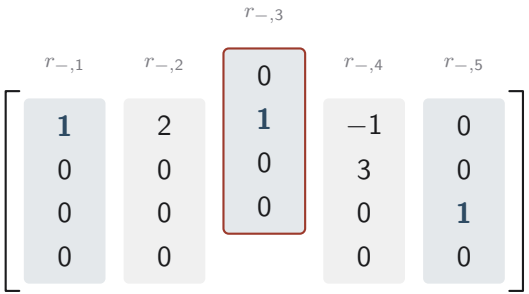
Row operations preserve the number of independent rows, and for columns even *which* ones are, as Ker A is unchanged. So assume $A = \text{RREF}(A)$, with r pivots.
The r non-zero rows are independent, as each pivot is alone in its column. The r pivot columns are $\mathbf{e}_1, \ldots, \mathbf{e}_r$, so they are independent too, and every other column is zero below row r, hence a combination of them. Both numbers are r. □

Pivots to the front

$$\begin{matrix} r_{-,1} & r_{-,2} & r_{-,3} & r_{-,4} & r_{-,5} \end{matrix}$$
$$\begin{bmatrix} \mathbf{1} & 2 & 0 & -1 & 0 \\ 0 & 0 & \mathbf{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here is RREF(A) of a 4×5 matrix, with pivots in columns $1, 3, 5$.

Pivots to the front



Not a row operation! Swapping columns changes the matrix, and with it $\operatorname{Ker} A$ and $\operatorname{Im} A$. We do it *only* to see what is going on.

Let us move the pivot columns to the front. First, swap columns 2 and 3.

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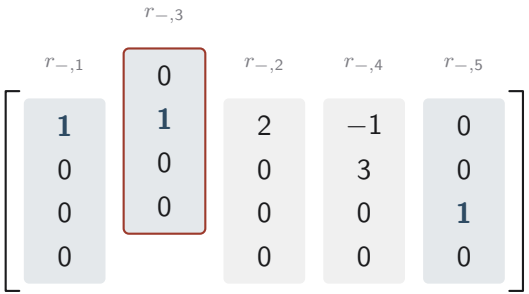

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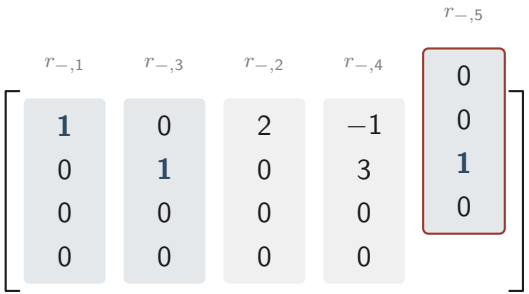

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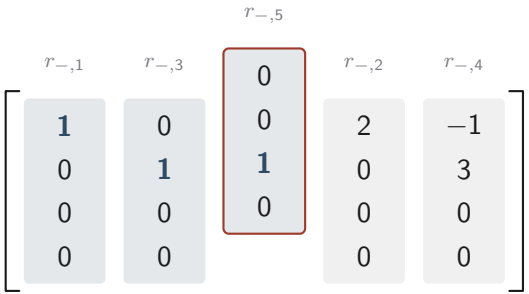
Then move column 5 to the third place.

Pivots to the front



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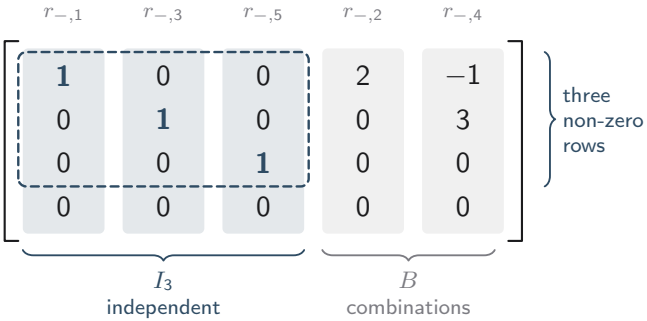
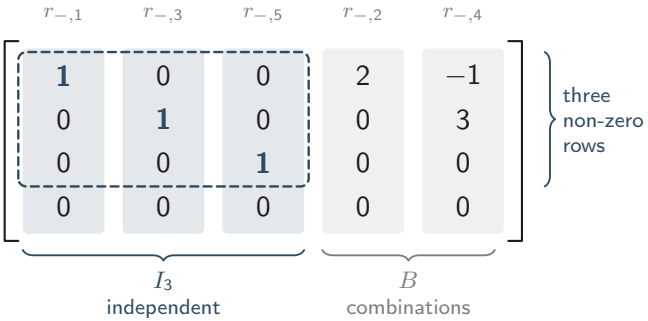


The pivot columns now form the identity I_3, with the two other columns B beside it and a zero row below.

Pivots to the front



In general, with r pivots,

$$\begin{bmatrix} I_r & B \\ 0 & 0 \end{bmatrix}$$

so there are r independent rows and columns.

Three independent rows and columns; the rest are combinations of those.

SECTION 4

Membership of the column space

The question, and its translation

Given an $m \times n$ matrix A and a vector $\mathbf{b} \in \mathbb{R}^m$, how do we decide whether $\mathbf{b} \in \text{Im}\, A$?

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KEY IDEA

Since $\operatorname{Im} A = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$,

$$\mathbf{b} \in \operatorname{Im} A \qquad \Longleftrightarrow \qquad A\mathbf{x} = \mathbf{b} \text{ has a solution.}$$

Note that we are **not** asked what $\mathbf{x}$ is, merely if it exists. Equivalently, if the system of linear equations defined by A is **consistent**.

A solution is called a *witness* or *representative*.

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A solution is called a *witness* or *representative*.

So we row-reduce the augmented matrix $[\,A \mid \mathbf{b}\,]$ and look at what the bottom rows say.

Row operations change the linear equations, and as such, we must change $\mathbf{b}$ in exactly the same way to ensure that we are asking equivalent questions.

The one thing to look for

THE IMPOSSIBLE ROW

If the reduction ever produces a row of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \mid c \end{bmatrix} \qquad \text{with} \quad c \neq 0,$$

the system is inconsistent as that row says $0 = c$. In that case, $\mathbf{b} \notin \operatorname{Im} A$.

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the system is inconsistent as that row says $0 = c$. In that case, $\mathbf{b} \notin \operatorname{Im} A$.

If no such row appears, the system is consistent and $\mathbf{b} \in \operatorname{Im} A$.

Note that the pivot must sit in the *last* column of $[\,A \mid \mathbf{b}\,]$ for this to happen, the one holding $\mathbf{b}$. A pivot in the augmented column is the definition of failure here.

A full example (1/2) ▸

Decide whether $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ lies in $\operatorname{Col} A$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}.$$

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Augment and clear the first column:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 3 \\ 1 & 2 & 1 & 2 & 2 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

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The last two rows are identical, so one of them will vanish.

A full example (2/2) ▸

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{array}\right]$$

A full example (2/2) ▸

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{array}\right]$$

The bottom row reads $0 = 0$, which is no obstruction at all. The system is consistent, so **Yes.** $\mathbf{b} \in \operatorname{Im} A$.

A full example (2/2) ▸

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{array}\right]$$

The bottom row reads $0 = 0$, which is no obstruction at all. The system is consistent, so **Yes.** $\mathbf{b} \in \operatorname{Im} A$.

We can read of a witness (representative) as well. Let $y = w = 0$. The rows give $z = 1$ and $x = 1$, and indeed

$$1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \mathbf{b}.$$

Why one vector was in the image and the other not ▸

Change the last entry of **b** from 2 to 3, and run exactly the same operations:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 3 \\ 1 & 2 & 1 & 2 & 3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{array}\right]$$

Why one vector was in the image and the other not ▸

Change the last entry of $\mathbf{b}$ from 2 to 3, and run exactly the same operations:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 3 \\ 1 & 2 & 1 & 2 & 3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} \end{array}\right]$$

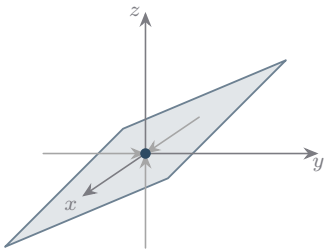
No. The bottom row now reads $0 = 1$ and the system is inconsistent. Hence $\mathbf{b} \notin \operatorname{Im} A$.

Seen from outside ▸

There $a_{-,2} = 2a_{-,1}$ and $a_{-,4} = a_{-,1} + a_{-,3}$, so

$$\operatorname{Im} A \;=\; \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\},$$

a plane in $\mathbb{R}^3$. Its equation is $x - y + z = 0$.



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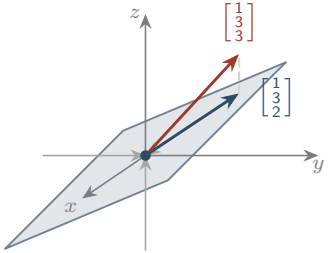
a plane in $\mathbb{R}^3$. Its equation is $x - y + z = 0$.

And sure enough

$$1 - 3 + 2 \;=\; 0, \qquad 1 - 3 + 3 \;=\; 1 \neq 0.$$

One **b** lies on the plane, the other just misses it.



Summary of Lecture 8

SAME THING	$A\mathbf{x} = \mathbf{b}$ is a system of equations. Row operations leave its solutions alone, so $\operatorname{Ker} A = \operatorname{Ker} \operatorname{RREF}(A)$, but $\operatorname{Im} A$ is *not* preserved.
REF	Zero rows at the bottom, each pivot strictly right of the one above. RREF adds taht pivots are 1 and they are alone in their columns.
THEOREM	For any matrix, $\operatorname{RREF}(A)$ is unique; $\operatorname{REF}(A)$ is not.
KERNEL	1. Reduce, 2. pivot columns are basic, rest free, 3. parametrise free variables, 4. spann parameters. One spanning vector per free col.
MEMBERSHIP	$\mathbf{b} \in \operatorname{Im} A$ exactly when $A\mathbf{x} = \mathbf{b}$ is consistent. Reduce $[\,A \mid \mathbf{b}\,]$ and look for a row $[\,0 \cdots 0 \mid c\,]$, $c \neq 0$.
IN SHORT	A pivot in the augmented column means no.

Before next time

AT HOME

1. Find RREF(A) and a spanning set of $\operatorname{Ker} A$ ▸ for $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \\ -1 & -3 & 2 \end{bmatrix}$.
2. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$. Decide whether $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^{\mathsf{T}}$ ▸ and $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^{\mathsf{T}}$ ▸ lie in $\operatorname{Col} A$.
3. An $m \times n$ matrix in reduced row echelon form has r pivots. How many vectors does our recipe produce for $\operatorname{Ker} A$, and what can you say when $r = n$?

END OF LECTURE 8

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Solution: question 1 ▸

SOLUTION

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \\ -1 & -3 & 2 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A).$$

Only one pivot, in column 1; columns 2 and 3 are free.

Solution: question 1 ▸

SOLUTION

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \\ -1 & -3 & 2 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A).$$

Only one pivot, in column 1; columns 2 and 3 are free.

Setting $y = s$, $z = t$ gives $x = -3s + 2t$, so $\text{Ker}\, A = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is aa plane.
Observe again, two free columns gave two spanning vectors.

Solution: question 2

$\begin{bmatrix}1 & 2 & 3\end{bmatrix}^{\mathsf{T}}$: YES ▸

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 3 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_2]{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Yes. Consistent, so the vector lies in Col A.

Solution: question 2

$\begin{bmatrix}1 & 2 & 3\end{bmatrix}^{\mathsf{T}}$: YES ▸

$$\left[\begin{array}{ccc|c}1 & 0 & 2 & 1\\ 0 & 1 & 1 & 2\\ 1 & 1 & 3 & 3\end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_2]{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|c}1 & 0 & 2 & 1\\ 0 & 1 & 1 & 2\\ 0 & 0 & 0 & 0\end{array}\right]$$

Yes. Consistent, so the vector lies in Col A.

No. for $\begin{bmatrix}1 & 2 & 4\end{bmatrix}^{\mathsf{T}}$ ▸. The same operations end with the row $[0\ 0\ 0|1]$, since $4 - 1 - 2 = 1$, and that reads $0 = 1$.

Solution: question 3 ▸

SOLUTION

There are n columns, and r of them carry a pivot. Thus $n - r$ are free and our little algorithm produces one spanning vector for each. Hence, $\operatorname{Ker} A$ is spanned by $n - r$ vectors.

Solution: question 3 ▸

SOLUTION

There are n columns, and r of them carry a pivot. Thus $n - r$ are free and our little algorithm produces one spanning vector for each. Hence, Ker A is spanned by $n - r$ vectors.

If $r = n$, no variable is free and the only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$; that is, Ker $A = \{\mathbf{0}\}$.

The number r is called the **rank** of A, and the relation between r, $n - r$ and the two subspaces is one of the theorems this module is heading towards.