LECTURE 7

Spanning sets and the image

Defining subspaces by finitely a handful of vectors, considering the column space

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29 September 2026 · Winter 2026

Where we got to

LAST LECTURE

- Span(S) is the set of all linear combinations of S. It is always a subspace, indeed the smallest one containing S.
- Every subspace U satisfies $U = \text{Span}(U)$, in particular, every subspace is a span.
- A linear combination of other vectors may be dropped without worry.
- Ker $A = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$ is a subspace of the source $\mathbb{R}^n$, and we described several of them as spans.

Today we give the phrase "S spans V" its official name, and meet the subspace hiding in the columns of a matrix.

Warm-up

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Let $A = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$.

1. Write $\operatorname{Ker} A$ as a span ▸.
2. Is $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathsf{T}} \in \operatorname{Ker} A$?

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2. Is $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathsf{T}} \in \operatorname{Ker} A$?

1 The single equation is $x - 2y + z = 0$. Setting $y = s$, $z = t$ gives $x = 2s - t$, so $\operatorname{Ker} A = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, a plane.

2 Yes: $1 - 2 + 1 = 0$. Recall, testing if $\mathbf{v} \in \operatorname{Ker} A$ is cheap.

SECTION 1

Spanning sets

Definition

DEFINITION 7.1 ▸ · Spanning set

Let V be a vector space and $S \subseteq V$. We say that S **spans** V, or that V is **generated by** S, if

$$\text{Span}(S) = V.$$

Such an S is called a **spanning set** (or generating set) of V.

Since $\text{Span}(S)$ is always a subspace, this says exactly that every vector of V can be written as a linear combination of elements of S — and that nothing outside V can.

$\text{Span}(S) \subseteq V$ is automatic. The content of the definition is the other inclusion.

A word on the words

Recall the lemma from last lecture: If $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$, then $\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{v}\}$. The set spans; the vector $\mathbf{w}$ contributes nothing.

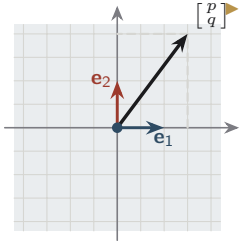
Saying that each vector *is* needed requires something more, which we do not yet have. That extra condition is **linear independence**, and a spanning set with it is a **basis**.

The standard spanning set of $\mathbb{R}^2$

With $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, every vector of $\mathbb{R}^2$ is reached in exactly one way:

$$\begin{bmatrix} p \\ q \end{bmatrix} = p\,\mathbf{e}_1 + q\,\mathbf{e}_2.$$

So $\{\mathbf{e}_1, \ldots, \mathbf{e}_2\}$ spans $\mathbb{R}^2$. The coefficients are simply the coordinates, which is why this set is called **standard**.



Note, it can be misleading to think that there exist "*special*" vectors in a vector space. There are not (unless $\mathbf{0}$). The reason $\mathbf{e}_1$ and $\mathbf{e}_2$ seem special is because we have already chosen them as a basis without realising it, the second we drew the space on the board/page.

Adding a third vector ▸

$$\text{Span}\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_1+\mathbf{e}_2\} = \mathbb{R}^2$$

The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_2\}$ also spans $\mathbb{R}^2$. Indeed, as the first two already do, and adding to it can not reduce the span.

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IT STILL DOES NOT SPAN $\mathbb{R}^3$

All three vectors live in $\mathbb{R}^2$ ▸, so every linear combination of them does too. No collection of vectors of $\mathbb{R}^2$ will ever span $\mathbb{R}^3$, no matter how many we take.

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Spanning is always spanning *of something*. The ambient space is part of the statement, and adding vectors never escapes it.

The standard spanning set of $\mathbb{R}^n$

Let $\mathbf{e}_i$ be the vector with a 1 in position i and zeros everywhere else. For any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \sum_{i=1}^{n} x_i \mathbf{e}_i,$$

so $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ spans $\mathbb{R}^n$.

Polynomials

EXAMPLE 7.2 ▸ · The obvious one

P_2 is spanned by $\{1, x, x^2\}$, since every element is $a + bx + cx^2$ by definition.

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P_2 is also spanned by $\{1,\ 1+x,\ 1+x+x^2\}$. Indeed

$$1 \ = \ 1, \qquad x \ = \ (1+x) - 1, \qquad x^2 \ = \ (1+x+x^2) - (1+x),$$

so the three standard polynomials are combinations of these three, and hence so is everything else.

> WE ONLY NEED TO CHECK A LINEAR MAP ON GENERATORS
>
> This is general and very useful way of showing that S is a spanning set: Show that you can get every element of another set S' already known to be a spanning set.

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> CAUTION
>
> $\mathbb{R}[x]$ admits no finite spanning set as linear combinations of polynomials can never exceed the largest degree among them.

$\mathrm{Mat}_{2\times 2}(\mathbb{R})$ is spanned by the four **matrix units**

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aE_{11} + bE_{12} + cE_{21} + dE_{22}$.

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since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aE_{11} + bE_{12} + cE_{21} + dE_{22}$.

In general $\mathrm{Mat}_{m\times n}(\mathbb{R})$ is spanned by the mn matrix units E_{ij}, one for each position.

A second spanning set of the same size ▸

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

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These four also span $\mathrm{Mat}_{2\times 2}(\mathbb{R})$, because each matrix unit can be recovered by subtracting neighbours:

$$E_{11} = B_1, \qquad E_{12} = B_2 - B_1, \qquad E_{21} = B_3 - B_2, \qquad E_{22} = B_4 - B_3.$$

SECTION 2

The Image and the column space

Column Spaces

A matrix is a list of columns, and each column is a vector of $\mathbb{R}^m$.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix}$$

Why columns not rows? Multiply A by the standard vectors $\mathbf{e}_1$, $\mathbf{e}_2, \ldots$. What do you observe?

DEFINITION 7.4 ▸ · Column space

The **column space** of an $m \times n$ matrix A is the span of its columns,

$$\operatorname{Col} A = \operatorname{Span}\{a_{-,1}, \ldots, a_{-,n}\} \subseteq \mathbb{R}^m.$$

Being a span, it is automatically a subspace. Nothing to check.

Three descriptions of the same set ▸

For an $m \times n$ matrix A:

BY COLUMNS $\quad \operatorname{Col} A = \operatorname{Span}\{a_{-,1}, \ldots, a_{-,n}\}$

BY ALL PRODUCTS $\quad \operatorname{Col} A = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$

BY SOLVABILITY $\quad \operatorname{Col} A = \{\mathbf{b} \in \mathbb{R}^m \mid \exists\, \mathbf{x} \in \mathbb{R}^n \text{ with } A\mathbf{x} = \mathbf{b}\}$

Note, $\exists$ is the mathematical symbol for *there exists*.

The image

We can regard our matrix as a map $A : \mathbb{R}^n \to \mathbb{R}^m$ by $\mathbf{x} \mapsto A\mathbf{x}$. This is a deep and important construction we will get back to in about two lectures, but for now, let us simply think of it as a map.

The **image** of $A : \mathbb{R}^n \to \mathbb{R}^m$ is the set

$$\text{Im}(A) \ := \ \{\mathbf{b} \in \mathbb{R}^m \mid \exists\, \mathbf{x} \in \mathbb{R}^n \text{ with} A\mathbf{x} = \mathbf{b}\} \subseteq \mathbb{R}^m$$

and thus,

$$\text{Col}\, A \ = \ \text{Im}\, A$$

.

In plane english, the image is the set of vectors "hit" if we map every vector $\mathbf{a}$ of $\mathbb{R}^n$ to $\mathbb{R}^m$ as $A\mathbf{a}$.

The kernel and image vs. the null space and the column space [Safely ignored]

> The image is the more general and basis-independent concept. It belongs to the linear map itself and can be defined without choosing a basis. That is why images and kernels continue to appear throughout abstract algebra.
>
> The column space is what the image looks like after we choose bases and write the map as a matrix. It is therefore an extremely useful computational tool, particularly when solving concrete problems. It is not a different mathematical phenomenon, but a coordinate description of the same one.
>
> So we should understand the image conceptually and use the column space computationally. TL;DR: Horses for courses.

Both are extremley important throughout maths and science, not just in this one course. Column space is preferred in fields such as Physics, Engineering, Data Science, Bioinformatics, Chemistry, Computer Scinece, and others. Image and Kernel (especially together) are are one of the most essential instruments in algebra and are fundamental for groups, rings, modules, sheaves, $\cdots$. Indeed, the whole field of homological algebra (where numerous fields medals were won) grows out of measuring how far the image of one map falls short of the kernel of the next.

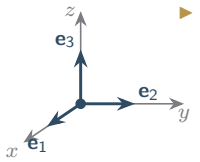
The first example: The identity

The columns of I_n are exactly $\mathbf{e}_1, \ldots, \mathbf{e}_n$, so

$$\operatorname{Col} I_n \;=\; \operatorname{Span}\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \;=\; \mathbb{R}^n.$$

Read through the second description this says that every $\mathbf{b}$ is $I_n\mathbf{b}$. But this is just the definition of the identity.

$$I_3 \;=\; \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

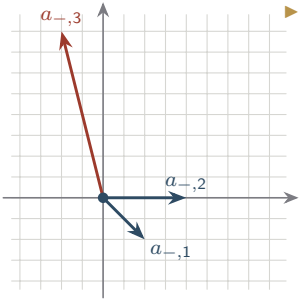


A wide matrix: Three columns in the plane

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix}$$

The three columns live in $\mathbb{R}^2$. The first two are not parallel, so they already span the plane, and

$$\text{Col}\, A = \mathbb{R}^2.$$



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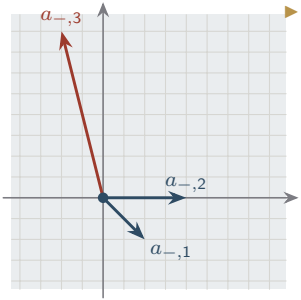
The third column is then *necessarily* redundant, and one may check that

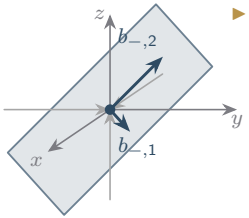
$$\begin{bmatrix} -1 \\ 4 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$



A tall matrix: Two columns in space

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Now take two columns, both in $\mathbb{R}^3$, but not parallel. So $\operatorname{Col} B$ is a plane through the origin. If they were parellel, we would get a line, noever all of $\mathbb{R}^3$.



SECTION 3

Kernel against image

Two subspaces, opposite behaviour

Every $m \times n$ matrix carries two subspaces, convenient in exactly opposite ways.

Ker $A \subseteq \mathbb{R}^n$

Given by an *equation*: $A\mathbf{x} = \mathbf{0}$.

Subspace of the source/domain

- Membership is easy, only a single multiplication.
- Spanning set is work, as we must solve the whole system.

Im $A \subseteq \mathbb{R}^m$

Given by a *list* of the columns.

Subspace of the target/range

- Spanning set is free (definition).
- Membership is work, as we must solve $A\mathbf{x} = \mathbf{b}$.

At their core, "spanning set for Ker" and "membership of Im" are the same question.

This is the proverbial tip of the duality-iceberg that exists in algebra, and this is just a very special case. Another special case you are familiar with (but likely never saw as a duality) is multplication and addition.

The question for next time

KEY IDEA

Given a matrix A:

1. Can we find a finite spanning set of $\operatorname{Ker} A$, with no vector to spare?
2. Given $\mathbf{b}$, can we decide whether $\mathbf{b} \in \operatorname{Col} A$?

The question for next time

KEY IDEA

Given a matrix A:

1. Can we find a finite spanning set of $\operatorname{Ker} A$, with no vector to spare?
2. Given $\mathbf{b}$, can we decide whether $\mathbf{b} \in \operatorname{Col} A$?

Both are questions about the system $A\mathbf{x} = \mathbf{b}$. The first asks for all the solutions to $\mathbf{b} = \mathbf{0}$. The second only asks whether a solution exists.

Both can be solved with something we are familiar with: Row operations.

Summary of Lecture 7

SPANS	S **spans** V, and V is **generated by** S, when $\mathrm{Span}(S) = V$.
LANGUAGE	We say the *set* generates V. Claiming that each vector is needed requires linear independence, which is still to come.
STANDARD	$\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ spans $\mathbb{R}^n$; $\{1, x, \ldots, x^n\}$ spans P_n; the mn matrix units span $\mathrm{Mat}_{m\times n}(\mathbb{R})$.
NOT UNIQUE	$\{1, 1+x, 1+x+x^2\}$ also spans P_2, and $B_1, \ldots, B_4$ also span $\mathrm{Mat}_{2\times 2}(\mathbb{R})$: a smallest spanning set is not unique, only its size.
COLUMN SPACE	$\mathrm{Col}\, A = \mathrm{Span}\{a_{-,1}, \ldots, a_{-,n}\} = \{A\mathbf{x}\} = \{\mathbf{b} \mid A\mathbf{x} = \mathbf{b} \text{ solvable}\}$, a subspace of $\mathbb{R}^m$. Also called the image $\mathrm{Im}\, A$, or the range.
OPPOSITES	Ker: Membership is easy but spanning set is hard. Col: Spanning set is easy but membership is hard.

Excersies to practice on

AT HOME

1. Does $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}\right\}$ span $\mathbb{R}^2$? ▸ Does $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$? ▸ Justify both.
2. Show that $\{1+x,\ x+x^2,\ 1+x^2\}$ spans P_2 ▸.

   Recover 1, x and x^2 from them.
3. Let $A = \begin{bmatrix}1 & 2 & 1\\0 & 1 & 1\\1 & 1 & 0\end{bmatrix}$. Write down a spanning set of $\operatorname{Col} A$ without any computation, then decide whether $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^{\mathsf{T}}$ lies in $\operatorname{Col} A$ ▸.

END OF LECTURE 7

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Solution: question 1

THE FIRST SET: YES ▸

$\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ are not parallel, and Det $\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = -2 \neq 0$, so already those two span $\mathbb{R}^2$; the third changes nothing.

Solution: question 1

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THE SECOND SET: NO ▸

$\begin{bmatrix}2\\4\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix}$, so the second vector is redundant and the span is only the line Span$\{\begin{bmatrix}1\\2\end{bmatrix}\}$, i.e. $y = 2x$. For instance $\begin{bmatrix}1\\0\end{bmatrix}$ is not in it.

Solution: question 2 ▸

SOLUTION

Write $p_1 = 1 + x$, $p_2 = x + x^2$, $p_3 = 1 + x^2$. Then

$$p_1 - p_2 + p_3 \;=\; (1 + x) - (x + x^2) + (1 + x^2) \;=\; 2,$$

so $1 = \frac{1}{2}(p_1 - p_2 + p_3)$. Subtracting, $x = p_1 - 1$ and $x^2 = p_3 - 1$.

Solution: question 2 ▸

SOLUTION

Write $p_1 = 1 + x$, $p_2 = x + x^2$, $p_3 = 1 + x^2$. Then

$$p_1 - p_2 + p_3 \;=\; (1 + x) - (x + x^2) + (1 + x^2) \;=\; 2,$$

so $1 = \frac{1}{2}(p_1 - p_2 + p_3)$. Subtracting, $x = p_1 - 1$ and $x^2 = p_3 - 1$.

Since 1, x and x^2 are all linear combinations of p_1, p_2, p_3, so is every element of P_2.

The same manoeuvre as with the matrices B_i: recover the standard spanning set, and the rest follows.

Solution: question 3

A SPANNING SET, FOR FREE

$\text{Col}\, A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. I.e, just the columns.

Solution: question 3

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$\text{Col}\, A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. I.e, just the columns.

MEMBERSHIP, WITH WORK ▸

Here $\text{Det}\, A = 0$, so we cannot conclude at once. But the third column is the second minus the first, so $\text{Col}\, A$ is the plane $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$. Now $a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ forces $b = 1$ from the middle row and then $a = -1$ from the top, while the bottom row reads $-1 + 1 = 0 \neq 1$. So **No.**