LECTURE 6

Span and kernel

Two ways of building subspaces: outwards from vectors, or inwards from equations

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
22 September 2026 · Winter 2026

Recall

LAST LECTURE

- A **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is $\lambda_1\mathbf{v}_1 + \cdots + \lambda_k\mathbf{v}_k$.
- Deciding whether a given $\mathbf{w}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ becomes a system of linear equations.
- It is decided by the invertibility of the column matrix $A = \big[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\big]$.

This can be used to generate subsspaces as will show today.

Warm-up

WARM-UP

Is $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

Find the coefficients, and write the question as a matrix equation $A\mathbf{x} = \mathbf{w}$.

Warm-up

WARM-UP

Is $\begin{bmatrix}7\\4\end{bmatrix}$ a linear combination of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$?

Find the coefficients, and write the question as a matrix equation $A\mathbf{x} = \mathbf{w}$.

SOLUTION

We need $a+3b=7$ and $2a+b=4$. The second gives $b=4-2a$, so $a+12-6a=7$, giving $a=1$, $b=2$: indeed $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]+2\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]=\left[\begin{smallmatrix}7\\4\end{smallmatrix}\right]$.

In matrix form: Is $\left[\begin{smallmatrix}1&3\\2&1\end{smallmatrix}\right]\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]=\left[\begin{smallmatrix}7\\4\end{smallmatrix}\right]$ solvable? Yes, and uniquely, since $\mathrm{Det}=-5\neq 0$.

SECTION 1

Span

Definition

DEFINITION 6.1 ▸ · Span

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a finite set of vectors in V. Its **span** is the set

$$\text{Span}(S) \;=\; \{\lambda_1\mathbf{v}_1 + \cdots + \lambda_k\mathbf{v}_k \mid \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}$$

of *all* linear combinations, where λ_i run through all combinations of real numbers.

- Each $\mathbf{v}_i$ lies trivially in $\text{Span}(S)$ with take $\lambda_i = 1$ and $\lambda_j = 0$ for all $j \neq i$.
- $\mathbf{0} \in \text{Span}(S)$ by setting every coefficient zero.
- By convention, $\text{Span}(\emptyset)\,\text{Span}(\mathbf{0}) = \{\mathbf{0}\}$.

A span is always a subspace

THEOREM 6.2

For any $S \subseteq V$, the subset $\mathrm{Span}(S) \subseteq V$ *is a subspace.*

PROOF.

(U1) $\mathbf{0} \in \mathrm{Span}(S)$ as argued above, (U2) Adding two combinations and collecting terms,

$$\Big(\sum_i \lambda_i \mathbf{v}_i\Big) + \Big(\sum_i \mu_i \mathbf{v}_i\Big) \;=\; \sum_i (\lambda_i + \mu_i)\mathbf{v}_i,$$

is again a combination of the same vectors. (U3) Scaling by μ replaces each λ_i by $\mu\lambda_i$. $\square$

Nothing here used $\mathbb{R}^n$. The proof is the same in P_2, in $\mathrm{Mat}_{m\times n}(\mathbb{R})$, in $\mathcal{F}(\mathbb{R},\mathbb{R})$...

Span as the smallest subspace: The intersection of all

THE SMALLEST SUBSPACE CONTAINING S

Span(S) is not merely *a* subspace containing S. It is *the smallest* one. Any subspace containing S must, by (U2) and (U3), contain all of Span(S).

Span as the smallest subspace: The intersection of all

THE SMALLEST SUBSPACE CONTAINING S

Span(S) is not merely *a* subspace containing S. It is *the smallest* one. Any subspace containing S must, by (U2) and (U3), contain all of Span(S).

INTERSECTION OF SUBSPACES

For any subset S the intersection of all subspaces containing S, being $V_S := \bigcap_{S \subseteq V_i} V_i$ is a subspace since the (arbitrary) intersection of subspaces remains one. We have

$$\text{Span}(S) \;=\; V_S$$

making Span(S) the smallest subspace containing S.

Thus *every subspace is a span* and so, describing all subspaces and describing all spans are is the same problem.

But the generating set is not unique

DIFFERENT SETS, SAME SPAN

$$\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix}\right\} = \mathbb{R}^2.$$

But the generating set is not unique

DIFFERENT SETS, SAME SPAN

$$\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix}\right\} = \mathbb{R}^2.$$

Even more simply, $\text{Span}\{\begin{bmatrix}1\\2\end{bmatrix}\} = \text{Span}\{\begin{bmatrix}2\\4\end{bmatrix}\}$ since they are each others multiples.

TAKING SPANS LOOSES INFORMATION

We conclude that a subspace does not remember its generator set. This is mostly fine, but it raises an imporant question: Are some sets better than others to span a vecotr space?

$U = \text{Span}(U)$ and mostly infinite, but we can often find a finite S with $U = \text{Span}(S)$. Hence some are better. But is there a *best*? The answer is **No.** , but there is a *best class*. We come back to it when discussing **basis**.

Why a redundant vector changes nothing

LEMMA

If $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$ *then* $\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

PROOF.

Any combination using $\mathbf{w}$ can be rewritten without it as

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \;=\; a\mathbf{u} + b\mathbf{v} + c(\alpha\mathbf{u} + \beta\mathbf{v}) \;=\; (a + c\alpha)\mathbf{u} + (b + c\beta)\mathbf{v}.$$

So the left span sits inside the right, and the reverse is clear. □

Why a redundant vector changes nothing

LEMMA

If $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$ *then* $\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

PROOF.

Any combination using $\mathbf{w}$ can be rewritten without it as

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \;=\; a\mathbf{u} + b\mathbf{v} + c(\alpha\mathbf{u} + \beta\mathbf{v}) \;=\; (a + c\alpha)\mathbf{u} + (b + c\beta)\mathbf{v}.$$

So the left span sits inside the right, and the reverse is clear. □

We call such $\mathbf{w}$ **linearly dependent** on $\mathbf{v}, \mathbf{u}$, and is useless addition as a generator. We will learn how to spot them. If any vector $\mathbf{v} \in S$ is linearly dependent on other vectors of S we call S linearly dependent. If no such vector exsts, we call S **linearly independent**. — Don't worry, we come back to it later and make it more formal, and clear.

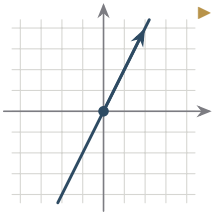
SECTION 2

Spans, in pictures

One vector

$$\text{Span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \left\{\begin{bmatrix}t\\2t\end{bmatrix} \,\middle|\, t \in \mathbb{R}\right\}$$

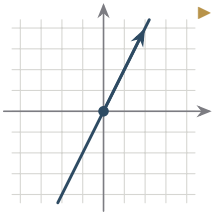
The multiples of a single non-zero vector form the line through the origin in its direction.
Here the line is $y = 2x$.



Instead of non-zero, it would be better to say: **The span of a single linearly independent vector is a line**.
It might sound odd to think of $\mathbf{0}$ linearly dependent, (as the question: "But linearly dependent to what" is so apparant), but it is mathematically correct: $\mathbf{0}$ is linearly dependent.

One vector

$$\text{Span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} = \left\{\begin{bmatrix}t\\2t\end{bmatrix} \,\middle|\, t \in \mathbb{R}\right\}$$



The multiples of a single non-zero vector form the line through the origin in its direction. Here the line is $y = 2x$.

The span of a single non-zero vector is always a line. For $\mathbf{0}$, the span is $\{\mathbf{0}\}$: a single point, not a line.

Instead of non-zero, it would be better to say: **The span of a single linearly independent vector is a line**. It might sound odd to think of $\mathbf{0}$ linearly dependent, (as the question: "But linearly dependent to what" is so apparant), but it is mathematically correct: $\mathbf{0}$ is linearly dependent.

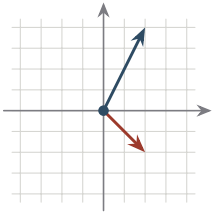
Two vectors in $\mathbb{R}^2$

What is Span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$?

Given a target $\begin{bmatrix} p \\ q \end{bmatrix}$, we must solve

$$a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

,



Two vectors not on the same line fill the plane.

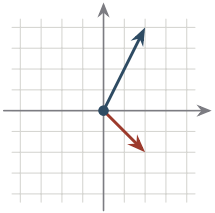
Another (better) way of saying this is **two linearly independent vectors span a plane.**

Two vectors in $\mathbb{R}^2$

What is Span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$? ,

Given a target $\begin{bmatrix} p \\ q \end{bmatrix}$, we must solve

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$



Two vectors not on the same line fill the plane.

Another (better) way of saying this is **two linearly independent vectors span a plane.**

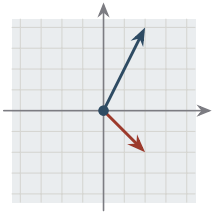
Two vectors in $\mathbb{R}^2$

What is Span$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$? ,

Given a target $\begin{bmatrix} p \\ q \end{bmatrix}$, we must solve

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

The determinant is $-1 - 2 = -3 \neq 0$, so the matrix is invertible and a solution exists for *every* target, making the span all of $\mathbb{R}^2$.



Two vectors not on the same line fill the plane.

Another (better) way of saying this is **two linearly independent vectors span a plane.**

More vectors need not mean a bigger span ▸

It is clear that two-dimensional vectors, no matter how many, can not span anything bigger than $\mathbb{R}^2$. Indeed

$$\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\} = \mathbb{R}^2,$$

because the first two already span $\mathbb{R}^2$, and the third is redundant: $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

More vectors need not mean a bigger span ▸

It is clear that two-dimensional vectors, no matter how many, can not span anything bigger than $\mathbb{R}^2$. Indeed

$$\text{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\} = \mathbb{R}^2,$$

because the first two already span $\mathbb{R}^2$, and the third is redundant: $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

In $\mathbb{R}^3$, on the other hand, three can be exactly enough:

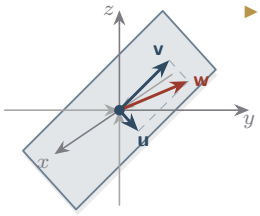
$$\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\} = \mathbb{R}^3, \quad \text{since} \quad \text{Det}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} = 1 \neq 0.$$

KEY IDEA

Counting the vectors is not enough. What matters is whether any of them was already reachable from the others. (Question of linear (in-)dependence.)

Three vectors in $\mathbb{R}^3$ that do not span it

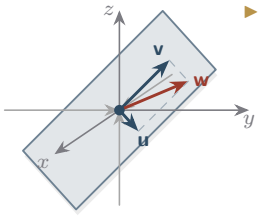
$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$



Another way of saying, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is not linearly independent.

Three vectors in $\mathbb{R}^3$ that do not span it

$$\mathbf{u} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}1\\2\\1\end{bmatrix}$$



No two of these lie on a common line.
But $\mathbf{w} = \mathbf{u} + \mathbf{v}$, so by the lemma

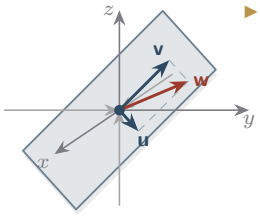
$$\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{v}\},$$

which is a plane, not all of $\mathbb{R}^3$.

Another way of saying, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is not linearly independent.

Three vectors in $\mathbb{R}^3$ that do not span it

$$\mathbf{u} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}1\\2\\1\end{bmatrix}$$



No two of these lie on a common line.
But $\mathbf{w} = \mathbf{u} + \mathbf{v}$, so by the lemma

$$\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{v}\},$$

which is a plane, not all of $\mathbb{R}^3$.

Its equation is $x - y + z = 0$, as all three vectors satisfy.

Another way of saying, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is not linearly independent.

Reading a span off a parametrisation

Are the following vector spaces?

$$H = \left\{ \begin{bmatrix} 2s \\ 1-s \end{bmatrix} \middle| \, s \in \mathbb{R} \right\} \qquad H = \left\{ \begin{bmatrix} 2s \\ -s \end{bmatrix} \middle| \, s \in \mathbb{R} \right\}$$

Reading a span off a parametrisation

Are the following vector spaces?

$$H = \left\{ \begin{bmatrix} 2s \\ 1-s \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\} \qquad H = \left\{ \begin{bmatrix} 2s \\ -s \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

No. $\mathbf{0} \in H$ needs $2s = 0$ and $1 - s = 0$ at once, which cannot happen. This is an *affine* line. ▸

Reading a span off a parametrisation

Are the following vector spaces?

$$H = \left\{ \begin{bmatrix} 2s \\ 1-s \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

No. $\mathbf{0} \in H$ needs $2s = 0$ and $1 - s = 0$ at once, which cannot happen. This is an *affine* line. ▸

$$H = \left\{ \begin{bmatrix} 2s \\ -s \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

Yes. Every coordinate is a multiple of s, so factor it out: ▸

$$\begin{bmatrix} 2s \\ -s \end{bmatrix} = s \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad H = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}.$$

Reading a span off a parametrisation

Are the following vector spaces?

$$H = \left\{ \begin{bmatrix} 2s \\ 1-s \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

No. $\mathbf{0} \in H$ needs $2s = 0$ and $1 - s = 0$ at once, which cannot happen. This is an *affine* line. ▸

$$H = \left\{ \begin{bmatrix} 2s \\ -s \end{bmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

Yes. Every coordinate is a multiple of s, so factor it out: ▸

$$\begin{bmatrix} 2s \\ -s \end{bmatrix} = s \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad H = \mathrm{Span}\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}.$$

Factor the parameters out: if fixed vectors remain, the set is their span, hence a subspace. A leftover constant means affine.

Two parameters ▸

Is this a vector space?

$$H = \left\{ \begin{bmatrix} s+t \\ s-t \\ 2s \end{bmatrix} \middle| \, s,t \in \mathbb{R} \right\}$$

Two parameters ▸

Is this a vector space?

$$H = \left\{ \begin{bmatrix} s+t \\ s-t \\ 2s \end{bmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

Separate the parameters, collecting the s terms and the t terms. Here,

$$\begin{bmatrix} s+t \\ s-t \\ 2s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Two parameters ▸

Is this a vector space?

$$H = \left\{ \begin{bmatrix} s+t \\ s-t \\ 2s \end{bmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

Separate the parameters, collecting the s terms and the t terms. Here,

$$\begin{bmatrix} s+t \\ s-t \\ 2s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Yes. So $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$, a plane through the origin.

One parameter gives a line, two independent ones a plane. We will make "independent" precise soon enough.

SECTION 3

Kernel

A matrix as a map

MATRICES ARE MAPS BETWEEN VECTORSPACES

An $m \times n$ matrix A turns a vector of $\mathbb{R}^n$ into a vector of $\mathbb{R}^m$, making it a map

$$A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad \mathbf{x} \longmapsto A\mathbf{x}.$$

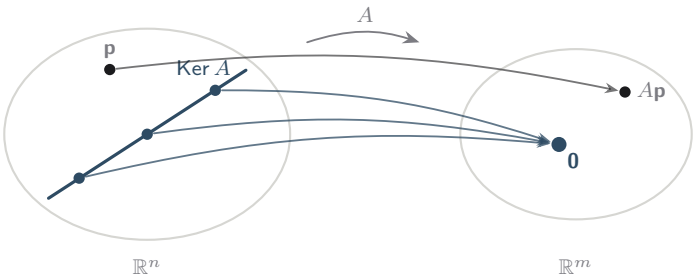
DEFINITION 6.3 ▸ · Kernel

The **kernel** of A is the set of vectors it sends to zero,

$$\operatorname{Ker} A \;=\; \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \;=\; \mathbf{0}\}.$$

Some books call it the null space and write $\operatorname{Nul} A$. Please don't.

The kernel, illustrated



A whole line of $\mathbb{R}^n$ is crushed onto the single point $\mathbf{0}$. The kernel keeps exactly what the image of A forgets.

A bit aside but this is as true as it gets in vector spaces: For $A : V \to V$, $V = \mathrm{Ker}(A) \oplus \mathrm{Im}(A)$.

Membership is easy to test

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix}$, our matrix from last lecture.

EXAMPLE 6.4 ▸

Is $\mathbf{x} = \begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix}$ in Ker A? Multiply it out and see:

Membership is easy to test

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix}$, our matrix from last lecture.

EXAMPLE 6.4 ▸

Is $\mathbf{x} = \begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix}$ in Ker A? Multiply it out and see:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8-6-2 \\ -8+0+8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \text{so yes.}$$

Membership is easy to test

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix}$, our matrix from last lecture.

EXAMPLE 6.4 ▸

Is $\mathbf{x} = \begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix}$ in Ker A? Multiply it out and see:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8-6-2 \\ -8+0+8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \text{so yes.}$$

And $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is not, since $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \neq \mathbf{0}$.

The kernel is a subspace

THEOREM 6.5

For any $m \times n$ matrix A, $\operatorname{Ker} A \subseteq \mathbb{R}^n$ is a subspace.

PROOF.

All three conditions follow from the fact that multiplication by A respects addition and scaling. First $A\mathbf{0} = \mathbf{0}$, which is (U1). If $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$ then

$$A(\mathbf{u}+\mathbf{w}) \;=\; A\mathbf{u} + A\mathbf{w} \;=\; \mathbf{0}+\mathbf{0} \;=\; \mathbf{0}, \qquad A(\lambda\mathbf{u}) \;=\; \lambda A\mathbf{u} \;=\; \lambda\mathbf{0} \;=\; \mathbf{0},$$

which are (U2) and (U3). □

Note that $\operatorname{Ker} A$ lives in $\mathbb{R}^n$, the *domain* (also called source — basically, the left hand side).

Testing an entry is easy, fully describing it requires work

Checking whether one given vector lies in $\operatorname{Ker} A$ is a single matrix multiplication. Describing the *whole* kernel is another matter.

> KEY IDEA
>
> Finding $\operatorname{Ker} A$ means solving $A\mathbf{x} = \mathbf{0}$ for *all* solutions at once, and then writing the answer as a span.

Two small kernels

For $A = \begin{bmatrix} 1 & -1 \end{bmatrix}$ ▸ we have $A\mathbf{x} = x - y$, so $\operatorname{Ker} A = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x = y\}$.

For $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ▸ the system reads $x + 2y = 0$ and $2x + 4y = 0$.

Two small kernels

For $A = \begin{bmatrix} 1 & -1 \end{bmatrix}$ ▸ we have $A\mathbf{x} = x - y$, so $\operatorname{Ker} A = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x = y\}$.

Setting $y = t$ forces $x = t$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \operatorname{Ker} A = \operatorname{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}.$$

For $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ▸ the system reads $x + 2y = 0$ and $2x + 4y = 0$.

Two small kernels

For $A = \begin{bmatrix} 1 & -1 \end{bmatrix}$ ▸ we have $A\mathbf{x} = x - y$, so $\operatorname{Ker} A = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x = y\}$.

Setting $y = t$ forces $x = t$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \operatorname{Ker} A = \operatorname{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}.$$

For $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ▸ the system reads $x + 2y = 0$ and $2x + 4y = 0$.

The second is twice the first, so says nothing new; $y = t$ gives $x = -2t$:

$$\operatorname{Ker} A = \operatorname{Span}\Big\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\Big\}.$$

Two small kernels

For $A = \begin{bmatrix} 1 & -1 \end{bmatrix}$ ▸ we have $A\mathbf{x} = x - y$, so $\operatorname{Ker} A = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x = y\}$.

Setting $y = t$ forces $x = t$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \operatorname{Ker} A = \operatorname{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}.$$

For $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ ▸ the system reads $x + 2y = 0$ and $2x + 4y = 0$.

The second is twice the first, so says nothing new; $y = t$ gives $x = -2t$:

$$\operatorname{Ker} A = \operatorname{Span}\Big\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\Big\}.$$

ASIDE

In the second case $\operatorname{Det} A = 0$. A square matrix has a kernel bigger than $\{\mathbf{0}\}$ exactly when its determinant vanishes. The collapse from lecture 1, seen from the other side.

Back to our running matrix ▸

Solve $A\mathbf{x} = \mathbf{0}$ for $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} a & b & c \end{bmatrix}^{\mathsf{T}}$ is equivalent to

$$a + 2b - c = 0, \qquad -a + 4c = 0.$$

Back to our running matrix ▸

Solve $A\mathbf{x} = \mathbf{0}$ for $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} a & b & c \end{bmatrix}^{\mathsf{T}}$ is equivalent to

$$a + 2b - c = 0, \qquad -a + 4c = 0.$$

The second equation gives $a = 4c$. Substituting into the first gives

$$4c + 2b - c = 0 \implies 2b = -3c \implies b = -\tfrac{3}{2}c.$$

Back to our running matrix ▸

Solve $A\mathbf{x} = \mathbf{0}$ for $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} a & b & c \end{bmatrix}^{\mathsf{T}}$ is equivalent to

$$a + 2b - c = 0, \qquad -a + 4c = 0.$$

The second equation gives $a = 4c$. Substituting into the first gives

$$4c + 2b - c = 0 \implies 2b = -3c \implies b = -\tfrac{3}{2}c.$$

So, taking $c = 2t$ to clear the fraction, we get

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = t \begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix}, \qquad \operatorname{Ker} A = \operatorname{Span}\left\{ \begin{bmatrix} 8 \\ -3 \\ 2 \end{bmatrix} \right\}.$$

The very vector we tested three slides ago. This is where it came from.

A two-parameter kernel ▸

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$. Every row is a multiple of the first, so the system collapses to the single equation $x + y + z = 0$.

A two-parameter kernel ▸

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$. Every row is a multiple of the first, so the system collapses to the single equation $x + y + z = 0$.

Two of the three unknowns are now free. Setting $y = s$ and $z = t$ forces $x = -s - t$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

A two-parameter kernel ▸

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$. Every row is a multiple of the first, so the system collapses to the single equation $x + y + z = 0$.

Two of the three unknowns are now free. Setting $y = s$ and $z = t$ forces $x = -s - t$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence $\operatorname{Ker} A = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, a plane through the origin in $\mathbb{R}^3$.

Two constructions, one pattern

Both of today's constructions produce subspaces, and they approach the task from opposite directions.

SPAN

Builds a subspace *outwards*, from a list of vectors, meaning

$$S \subseteq \text{Span}(S).$$

KERNEL

Cuts a subspace *out* of $\mathbb{R}^n$, meaning

$$\text{Ker}(A) \subseteq \mathbb{R}^n.$$

And every kernel we computed today ended up written as a span. That is not a coincidence either.

Summary of Lecture 6

SPAN	Span(S) is the set of all linear combinations of S. It is always a subspace. Indeed it is the smallest one containing S.
CONSEQUENCE	Every subspace is a span, so it is one problem, not two.
NOT UNIQUE	Different sets can have the same span and we may drop some.
PARAMETRISED	Factor the parameters out and if fixed vectors remain, the set is their span.
KERNEL	Ker $A = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$ is a subspace of the domain.
DESCRIBING	Solve $A\mathbf{x} = \mathbf{0}$, parametrise and read off as a span.

Practice

AT HOME

1. Describe Span$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ as simply as you can ▸, and find an equation for it ▸.
2. Find Ker A ▸ for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$, and write your answer as a span.
3. Let $H = \left\{ \begin{bmatrix} s+2t \\ s-t \\ 3t \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$. Write H as a span, and then find a matrix A with Ker $A = H$ ▸.

END OF LECTURE 6

ilia.pirashvili@universityofgalway.ac.ie

Solution to question 1 ▶

SOLUTION

The second vector is twice the first, so by the lemma it may be dropped. Thus

$$\text{Span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}2\\-2\\4\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\},$$

is a plane through the origin.

Solution to question 1 ▸

SOLUTION

The second vector is twice the first, so by the lemma it may be dropped. Thus

$$\text{Span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}2\\-2\\4\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\},$$

is a plane through the origin.

A general element is $s\begin{bmatrix}1\\-1\\2\end{bmatrix}+t\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}s\\t-s\\2s+t\end{bmatrix}$. Writing (x,y,z) for it gives $s=x$, $t=x+y$, so $z=3x+y$, i.e. $3x+y-z=0$ ▸.

Solution to question 2 ▸

SOLUTION

The second row is twice the first, so the system says only $x + 2y + 3z = 0$. Setting $y = s$ and $z = t$ gives $x = -2s - 3t$, hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \qquad \operatorname{Ker} A = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

SOLUTION

The second row is twice the first, so the system says only $x + 2y + 3z = 0$. Setting $y = s$ and $z = t$ gives $x = -2s - 3t$, hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \qquad \operatorname{Ker} A = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A plane through the origin, as we have two spanning vectors.

Solution to question 3 ▸

SOLUTION

Factoring out the parameters,

$$\begin{bmatrix} s+2t \\ s-t \\ 3t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \qquad H = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\}.$$

Solution to question 3 ▸

SOLUTION

Factoring out the parameters,

$$\begin{bmatrix} s+2t \\ s-t \\ 3t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \qquad H = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\}.$$

H is a plane, so one equation should cut it out. Looking for $\alpha x + \beta y + \gamma z = 0$ satisfied by both spanning vectors: $\alpha + \beta = 0$ and $2\alpha - \beta + 3\gamma = 0$. Taking $\alpha = 1$ gives $\beta = -1$ and $\gamma = -1$, so $A = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$ has $\text{Ker}\, A = H$ ▸.