LECTURE 4

# Subspaces in practice

A gallery of tests, and the first hint of what comes next

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# Recall

## LAST LECTURE

- Functions form vector spaces, with everything defined *pointwise*:
$$(f+g)(x) \;=\; f(x)+g(x).$$
- $\mathbb{R}[x] \subset \mathcal{C}^\infty(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R}) \subset \mathcal{C}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R},\mathbb{R})$, each sitting inside the next.
- $U \subseteq V$ is a **subspace** if $\mathbf{0} \in U$ and $U$ is closed under $+$ and scaling.
- A subspace is a vector space in its own right.

# Warm-up

**WARM-UP**

Let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; xy = 0 \right\}$. Is $H$ a subspace?

Can you picture is geometrically?

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Can you picture is geometrically?

| | | |
|---|---|---|
| (U1) | $0 \cdot 0 = 0$, so $\mathbf{0} \in H$. | **Holds.** |
| (U2) | $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ lie in $H$, their sum $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ does not. | **Fails**, so $H$ is not a subspace. |
| (U3) | $(\lambda x)(\lambda y) = \lambda^2 xy = 0$. | **Holds.** |

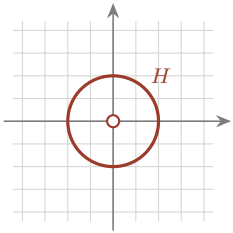
Two out of three is not enough — and remember this failure, we meet it again at the end of the lecture.

SECTION 1

# A gallery of tests

# The unit circle

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1 \right\}$$




Being centred on the origin is not the same as containing it.

# The unit circle

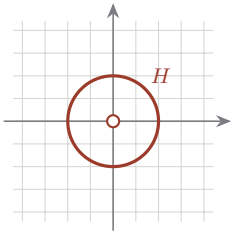
$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1 \right\}$$

**No.** The origin does not lie on the circle, so (U1) fails at once.

(U2) fails as well: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ both lie on the circle, but their sum $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has $1^2 + 1^2 = 2 \neq 1$.

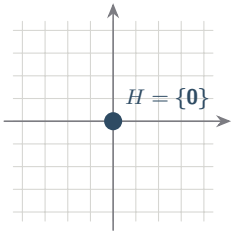
Indeed, (U3) fails as well for every point.

Being centred on the origin is not the same as containing it.



## Sometimes its hard to see what we described

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 + 1 = 1 \right\}$$




Simplify the condition *before* testing it. A quadratic description need not describe a curved set.

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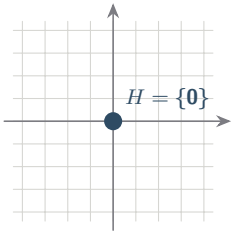
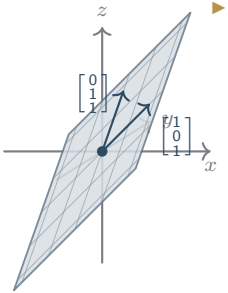



**Yes.** The condition says $x^2 + y^2 = 0$, and a sum of two squares vanishes only when both do. So $H = \{\mathbf{0}\}$, one of the two trivial subspaces.

Simplify the condition *before* testing it. A quadratic description need not describe a curved set.

# A plane through the origin

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x + y - z = 0 \right\}$$




Every point is a combination of the two arrows drawn.

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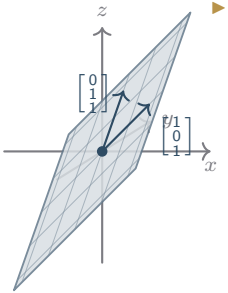



**Yes.** $\mathbf{0} \in H$. (U1): Let $x_1, y_1, z_1$ and $x_2, y_2, z_2$ satisfy $x_1 + y_1 - z_1 = 0$ and $x_2 + y_2 - z_2 = 0$. Then

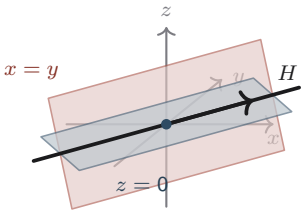
$$(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 - z_1) + (x_2 + y_2 - z_2) = 0 + 0 = 0.$$

Likewise when scaling.

Every point is a combination of the two arrows drawn.

## Two conditions at once ▸

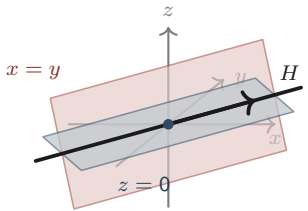
$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x - y = 0,\ z = 0 \right\}$$



## Two conditions at once ▸

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x - y = 0,\ z = 0 \right\}$$

**Yes.** Each condition is linear with right-hand side 0, so each defines a plane through the origin. A point of $H$ must satisfy both, so $H$ is the *intersection* of the two planes: The line through $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{\mathsf{T}}$.



## Intersections, in general ▸

*THEOREM 4.1*

*If $U$ and $W$ are subspaces of $V$, then $U \cap W$ is a subspace of $V$.*

PROOF.

(U1) $\mathbf{0}$ lies in both. (U2) If $\mathbf{u}, \mathbf{v} \in U \cap W$ then $\mathbf{u} + \mathbf{v} \in U$ because $U$ is a subspace, and $\mathbf{u} + \mathbf{v} \in W$ for the same reason. An identical argument shows (U3). □

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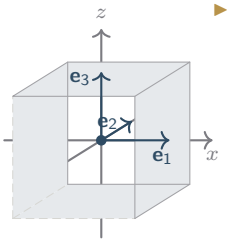
KEY IDEA

This is why a *system* of homogeneous linear equations always defines a subspace. Each equation yields alone a subspace and several of them just define their intersections.

Note, you can not say the intersection of vector spaces is a vector space. Not, because the "vector space" part would break, but because the "intersection" part breaks: You can only intersect sets if they lie in a common bigger space.

# A condition that never fails

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 \geq 0 \right\}$$



## A condition that never fails

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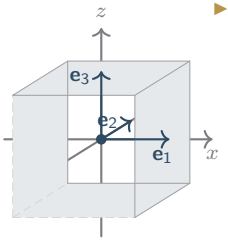
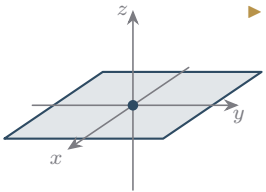
**Yes.** Every vector of $\mathbb{R}^3$ satisfies this, so $H = \mathbb{R}^3$, the other trivial subspace.

Same as with $x^2 + y^2 + 1 = 1$, conditions can somtimes make things more complicated than they are.



# The same trick, one dimension down

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 \geq 0,\ z = 0 \right\}$$



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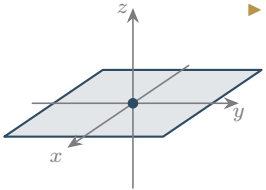
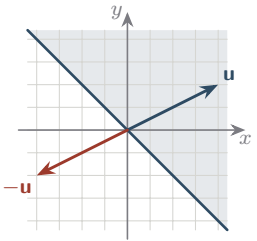



**Yes.** The first condition is satisfied by everything, so it says nothing at all. What is left is the plane $z = 0$, which is linear and homogeneous, hence a subspace.

As a vector space in its own right, it is an exact copy of $\mathbb{R}^2$.

# A half-plane

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; x + y \geq 0,\ z = 0 \right\}$$




Drawn inside the plane $z = 0$. A genuine inequality always breaks at $\lambda < 0$.

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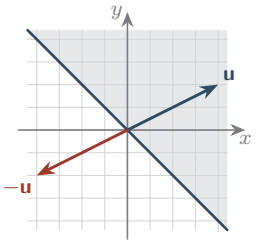



**No.** This one is closed under addition, and it does contain $\mathbf{0}$, but that is not enough. We lack inverses: Take $\mathbf{u} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\mathsf{T} \in H$, then $-\mathbf{u} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^\mathsf{T} \notin H$, so (U3) fails.

Drawn inside the plane $z = 0$. A genuine inequality always breaks at $\lambda < 0$.

## What the examples were really about

KEY IDEA

Every subspace we have drawn was *flat*. In $\mathbb{R}^3$ we only saw $\{\mathbf{0}\}$, lines through the origin, planes through the origin and $\mathbb{R}^3$ itself. We never had a valid example begin a curve, a polygon, a cone or anything like that. That is no coincidence: this is why the subject is called ***linear** algebra*.

- $x + y = 1$: **No.** Affine, so it misses $\mathbf{0}$.
- $x + y \geq 0$: **No.** A genuine inequality, so scaling by $\lambda < 0$ escapes.
- $x^2 + y^2 = 1$: **No.** Genuinely curved, and misses $\mathbf{0}$.
- $x^2 + y^2 + z^2 \geq 0$ or $x^2 + y^2 + 1 = 1$: **Yes.** But only "by coincidence", as they are actually just linear.

SECTION 2

# Towards linear combinations (spans)

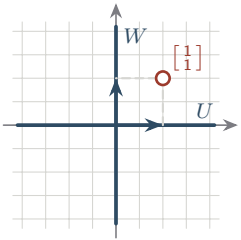
# Unions of subspaces

Intersections of subspaces were subspaces, but what about unions? The answer is **No.**

The $x$-axis $U$ and the $y$-axis $W$ are both subspaces of $\mathbb{R}^2$, and both contain $\mathbf{0}$. But their union, being our warmup is not. Indeed

$$\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

lies on neither axis, so $U \cup W$ fails (U2).



## Spanning subspaces from subsets

What if we have a subset $S$ that is not a subspace. Does it mean we can not use linear algebra at all? The answer is **No.** We can always **span** a subspace from our subset. This is the *smallest subspace containing* $S$. But we need a few steps before we come to that.

DEFINITION 4.2 ▸ · Linear combination

Let $V$ be a vector space. An element $\mathbf{v} \in V$ is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ if there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k.$$

KEY IDEA

By (U2) and (U3), if $\mathbf{v}_1, \ldots, \mathbf{v}_k \in U \subseteq V$, then $\mathbf{v} \in U$ **must** hold if $U$ is to be a subspace.

# Example: Linear combination

EXAMPLE 4.3 ▸ · The standard basis

Let $\mathbf{v_1} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{v_1} = \begin{bmatrix}0\\1\end{bmatrix}$. Then $\begin{bmatrix}1\\2\end{bmatrix} = 1 \cdot \mathbf{v_1} + 2 \cdot \mathbf{v_2}$ is a linear combination.

EXAMPLE 4.4 ▸ · Two vectors in the plane

Let $\mathbf{w_1} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{w_1} = \begin{bmatrix}2\\-1\end{bmatrix}$. Then $\begin{bmatrix}1\\2\end{bmatrix} = \frac{5}{3} \cdot \mathbf{w_1} - \frac{1}{3} \cdot \mathbf{w_2}$ is a linear combination.

Any vector in $\mathbb{R}^2$ can be expressed as a linear combination of $\{\mathbf{v_1}, \mathbf{v_2}\}$ **and** of $\{\mathbf{w_1}, \mathbf{w_2}\}$.

## Set of all linear combinations

EXAMPLE 4.5 ▸

In $\mathbb{R}^3$, the plane $x + y - z = 0$ from earlier is exactly the set of linear combinations of $\begin{bmatrix}1 & 0 & 1\end{bmatrix}^\mathsf{T}$ and $\begin{bmatrix}0 & 1 & 1\end{bmatrix}^\mathsf{T}$:

$$\lambda \begin{bmatrix}1\\0\\1\end{bmatrix} + \mu \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}\lambda\\ \mu\\ \lambda+\mu\end{bmatrix}, \qquad \text{and indeed} \qquad \underbrace{\lambda}_{x} + \underbrace{\mu}_{y} - \underbrace{(\lambda+\mu)}_{z} = 0.$$

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EXAMPLE 4.6

In $P_2$, every element is a linear combination of 1, $x$ and $x^2$. ▸ In $\mathrm{Mat}_{2\times 2}(\mathbb{R})$, every matrix is a linear combination of the four matrices ▸ $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$.

## Three questions for next time

- What is the relation between the set of *all* linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and subspaces?

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- How many are needed in a given situation?

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- What is the relation between the set of *all* linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and subspaces?
- Can every subspace be produced this way, from finitely many vectors?
- How many are needed in a given situation?

These are the questions of *span*, *spanning sets*, *linear independence* and, eventually, *dimension*.

# Summary of Lecture 4

| | |
|---|---|
| SUBSPACE IF: | $\mathbf{0} \in U$, closed under $+$ and scaling. Single faile means **No.** |
| SHAPE | Subspaces of $\mathbb{R}^3$ are $\{\mathbf{0}\}$, lines and planes through the origin, and $\mathbb{R}^3$. |
| BUT FIRST | Simplify before testing: $x^2 + y^2 + 1 = 1$ is $\{\mathbf{0}\}$, and $x^2 + y^2 \geq 0$ is the whole space. |
| OPERATIONS | $U \cap W$ is always a subspace; $U \cup W$ is never a subspace unless $U \subseteq W$ or $W \subseteq U$. |
| NEXT | $\mathbf{v} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_k \mathbf{v}_k$ is a **linear combination**. For $U$ to be a subspace, if $\mathbf{v_i} \in U$, then $\mathbf{v} \in U$ must hold. |

# Examples to consolidate understanding

AT HOME

1. Decide, with justification, which of these are subspaces of $\mathbb{R}^3$: (a) $\{(x, y, z) \mid x + 2y = 0 \text{ and} z = x\}$ ▸; (b) $\{(x, y, z) \mid x + y + z = 3\}$ ▸; (c) $\{(x, y, z) \mid x = y \text{ or} y = z\}$.
2. Write $\begin{bmatrix}4 & 1 & 3\end{bmatrix}^\mathsf{T}$ as a linear combination ▸ of $\begin{bmatrix}1 & 1 & 0\end{bmatrix}^\mathsf{T}$, $\begin{bmatrix}0 & 1 & 1\end{bmatrix}^\mathsf{T}$ and $\begin{bmatrix}1 & 0 & 1\end{bmatrix}^\mathsf{T}$, or show that it cannot be done.
3. Let $U$ and $W$ be subspaces of $V$. Show that $U + W \coloneqq \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U,\ \mathbf{w} \in W\}$ is a subspace of $V$. What is $U + W$ when $U$ and $W$ are the two axes of $\mathbb{R}^2$?

# END OF LECTURE 4

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# Solution: Question 1

THREE SUBSETS OF $\mathbb{R}^3$

**(a) Yes.** ▸ Two linear equations with right-hand side 0. Each defines a plane through the origin, and $H$ is their intersection, which is a subspace by the theorem.

# Solution: Question 1

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**(b) No.** ▸ $0 + 0 + 0 = 0 \neq 3$, so $\mathbf{0} \notin H$ and (U1) fails. Affine again.

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**(b) No.** ▸ $0 + 0 + 0 = 0 \neq 3$, so $\mathbf{0} \notin H$ and (U1) fails. Affine again.

**(c) No.** A union of two planes. $(1, 1, 0)$ satisfies $x = y$ and $(0, 1, 1)$ satisfies $y = z$, so both lie in $H$; but their sum $(1, 2, 1)$ has $x \neq y$ and $y \neq z$. (U2) fails.

## Solution: Question 2 ▸

SOLUTION

We need $a, b, c$ with

$$a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}a+c\\a+b\\b+c\end{bmatrix} = \begin{bmatrix}4\\1\\3\end{bmatrix}.$$

Subtracting the third equation from the second gives $a - c = -2$, and with $a + c = 4$ this forces $a = 1$ and $c = 3$; then $b = 0$.

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So $\begin{bmatrix}4 & 1 & 3\end{bmatrix}^{\mathsf{T}} = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^{\mathsf{T}} + 3\begin{bmatrix}1 & 0 & 1\end{bmatrix}^{\mathsf{T}}$: the middle vector was not needed at all.

That last observation is question 3 of the next lecture.

# Solution: Question 3

SOLUTION

(U1) $\mathbf{0} = \mathbf{0} + \mathbf{0}$ with one $\mathbf{0}$ from each, so $\mathbf{0} \in U + W$.

(U2) Given $\mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{u}_2 + \mathbf{w}_2$, regroup:

$$(\mathbf{u}_1 + \mathbf{w}_1) + (\mathbf{u}_2 + \mathbf{w}_2) = \underbrace{(\mathbf{u}_1 + \mathbf{u}_2)}_{\in U} + \underbrace{(\mathbf{w}_1 + \mathbf{w}_2)}_{\in W},$$

using (A2) and (A5). (U3) $\lambda(\mathbf{u} + \mathbf{w}) = \lambda\mathbf{u} + \lambda\mathbf{w}$ by (S1).

# Solution: Question 3

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For the two axes, every $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix}$, so $U + W = \mathbb{R}^2$.

The union failed; the *sum* repairs it, and gives the smallest subspace containing both.