LECTURE 3

More vector spaces, and the subspace test

Subspaces and examples

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
15 September 2026 · Winter 2026

Recall

LAST LECTURE

- The inverse A^{-1} of A exists exactly when $\operatorname{Det} A \neq 0$, is unique and we can use Gauss–Jordan finds it by reducing $[\,A \mid I\,]$ to $[\,I \mid A^{-1}\,]$.
- $\operatorname{Det}(AB) = \operatorname{Det}(A)\operatorname{Det}(B)$, $(AB)^{-1} = B^{-1}A^{-1}$ and $\operatorname{Det}(A^{-1}) = 1/\operatorname{Det} A$.
- A **real valued vector space** is a set V with an addition and a scaling by real numbers satisfying nine axioms: (A1)–(A5) make $(V, +)$ an abelian group, (S0)–(S4) govern the scaling.
- Examples so far: $\mathbb{R}^n$, $\operatorname{Mat}_{m\times n}(\mathbb{R})$, P_1, P_2, $\mathbb{R}[x]$, and $\mathbb{C}$ viewed over $\mathbb{R}$.

Today we will see many more examples, and uncover what many examples share.

Warm-up

WARM-UP

1. Is $\mathbb{R}^2$ with its usual addition and scaling given by $\lambda \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ y \end{bmatrix}$ a vector space?
2. Show that a vector space has only one zero vector.

Warm-up

WARM-UP

1. Is $\mathbb{R}^2$ with its usual addition and scaling given by $\lambda \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ y \end{bmatrix}$ a vector space?
2. Show that a vector space has only one zero vector.

1 (S2) fails: $(\lambda + \mu) \odot \mathbf{v} = \begin{bmatrix} (\lambda+\mu)x \\ y \end{bmatrix}$, but $\lambda \odot \mathbf{v} + \mu \odot \mathbf{v} = \begin{bmatrix} (\lambda+\mu)x \\ 2y \end{bmatrix}$.

2 If $\mathbf{0}$ and $\mathbf{0}'$ are both neutral then $\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}'$.

SECTION 1

Vector spaces, revisited

The nine axioms, for reference

For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\lambda, \mu \in \mathbb{R}$, we have

(A1)	$\mathbf{u} + \mathbf{v} \in V$	Closure under addition
(A2)	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associativity
(A3)	There exist $\mathbf{0} \in V$ with $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$	Neutral element
(A4)	For every $\mathbf{v}$ there exists $-\mathbf{v}$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	Inverses
(A5)	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutativity
(S0)	$\lambda \mathbf{u} \in V$	Closure under scaling
(S1)	$\lambda(\mathbf{u} + \mathbf{v}) = \lambda \mathbf{u} + \lambda \mathbf{v}$	Distributivity in V
(S2)	$(\lambda + \mu)\mathbf{v} = \lambda \mathbf{v} + \mu \mathbf{v}$	Distributivity in $\mathbb{R}$
(S3)	$(\lambda \mu)\mathbf{v} = \lambda(\mu \mathbf{v})$	Compatibility
(S4)	$1\,\mathbf{v} = \mathbf{v}$	Unit acts trivially

The model example: $\mathbb{R}^n$

$$\mathbb{R}^n = \left\{ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \middle| \, v_1, \ldots, v_n \in \mathbb{R} \right\}, \qquad \mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}, \qquad \lambda \mathbf{v} = \begin{bmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{bmatrix}.$$

THEOREM 3.1

$\mathbb{R}^n$ *with these operations is a real vector space.*

We can check every single axiom in the same way, by verifying them coordinatewise and since every one of these axioms clearly holds for the real numbers, it must thus hold over $\mathbb{R}^n$.

Let us do one properly to see exactly what is ment by the above.

Let us check one axiom by hand

LEMMA · Distributivity in V

For all $\lambda \in \mathbb{R}$ *and* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *we have* $\lambda(\mathbf{u}+\mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$. *(S1)*

PROOF.

We have to show

$$\lambda \left(\begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix} \right) = \lambda \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix} + \lambda \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix}.$$

Let us check one axiom by hand

LEMMA · Distributivity in V

For all $\lambda \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we have $\lambda(\mathbf{u}+\mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$. *(S1)*

PROOF.

We have to show

$$\lambda \left(\begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix} \right) = \lambda \begin{bmatrix} u_1 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{bmatrix} + \lambda \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{bmatrix}.$$

Addition, scaling and equality are coordinatewise, so it is enough to check the boxed row, for each i. □

The proof at a single coordinate

PROOF OF (S1).

At coordinate i, the left-hand side reads

$$\big(\lambda(\mathbf{u}+\mathbf{v})\big)_i \;=\; \lambda\,(u_i + v_i),$$

and the right-hand side reads

$$\big(\lambda\mathbf{u} + \lambda\mathbf{v}\big)_i \;=\; (\lambda\mathbf{u})_i + (\lambda\mathbf{v})_i \;=\; \lambda u_i + \lambda v_i.$$

These agree because distributivity holds in $\mathbb{R}$ $\big(\text{e.g.} 7(2+3) = 7\cdot 2 + 7\cdot 3\big)$, and two vectors agreeing at every coordinate are equal. □

The other eight axioms go the same way.

Example: Set of all functions

DEFINITION 3.2

Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of *all* functions $f : \mathbb{R} \to \mathbb{R}$, with

$$(f + g)(x) \coloneqq f(x) + g(x), \qquad (\lambda f)(x) \coloneqq \lambda\, f(x) \qquad \text{for all } x \in \mathbb{R}.$$

EVERYTHING HAPPENS POINTWISE

Note that functions are added by adding their values, at each point. That is the only way we can add "a circle and a line", by algebraically adding them at every point.

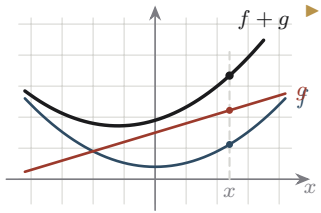
Two functions are equal exactly when they agree at every single x. Hence, that too is pointwise and this is how we can check each axiom.

Pointwise, in a picture

At the marked x, the height of $f + g$ is the height of f plus the height of g. This is pointwise addition.

Scaling works likewise: λf stretches every height by λ, leaving the input untouched.

So the vector $\mathbf{0}$ here is the constant function 0, the one whose height is 0 everywhere — meaning, the x-axis.



Checking the axioms for $\mathcal{F}(\mathbb{R}, \mathbb{R})$

The zero vector is the constant function 0, and the negative of f is $-f$, because at every x

$$(f+0)(x) \;=\; f(x)+0 \;=\; f(x), \qquad \text{and} \qquad \big(f+(-f)\big)(x) \;=\; f(x)-f(x) \;=\; 0.$$

Checking the axioms for $\mathcal{F}(\mathbb{R}, \mathbb{R})$

The zero vector is the constant function 0, and the negative of f is $-f$, because at every x

$$(f+0)(x) \;=\; f(x)+0 \;=\; f(x), \qquad \text{and} \qquad \big(f+(-f)\big)(x) \;=\; f(x)-f(x) \;=\; 0.$$

Associativity holds because it holds at every $x \in \mathbb{R}$:

$$\big((f+g)+h\big)(x) \;=\; \big(f(x)+g(x)\big)+h(x) \;=\; f(x)+\big(g(x)+h(x)\big) \;=\; \big(f+(g+h)\big)(x).$$

Checking the axioms for $\mathcal{F}(\mathbb{R}, \mathbb{R})$

The zero vector is the constant function 0, and the negative of f is $-f$, because at every x

$$(f+0)(x) \;=\; f(x)+0 \;=\; f(x), \qquad \text{and} \qquad \big(f+(-f)\big)(x) \;=\; f(x)-f(x) \;=\; 0.$$

Associativity holds because it holds at every $x \in \mathbb{R}$:

$$\big((f+g)+h\big)(x) \;=\; \big(f(x)+g(x)\big)+h(x) \;=\; f(x)+\big(g(x)+h(x)\big) \;=\; \big(f+(g+h)\big)(x).$$

Much as with $\mathbb{R}^n$, every axiom reduces to the same axiom in $\mathbb{R}$, applied at each point. So $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space.

Four more, quickly

There are countless similar examples of functions of a particular type.

- $\mathcal{C}(\mathbb{R})$: Continuous functions.

  Sums and multiples of continuous functions are continuous.

- $\mathcal{C}^1(\mathbb{R})$, $\mathcal{C}^\infty(\mathbb{R})$: Fuctions differentiable (at least) once and infinitely differentiable functions. The key formula here is $(f+g)' = f' + g'$.
- $\mathbb{R}[x, x^{-1}]$: The **Laurent polynomials**.
- $\mathbb{R}[x]$: We already met (all) polynomials.

Notice that each sits inside the one before it. Hold that thought.

How far does this reach?

FAR AFIELD

The supply of interesting vector spaces is vast, because linear algebra sits at the foundation of practically every quantitative science, sometimes transparently and sometimes not. To name only a few of the stranger ones: The space of distributions, formal combinations of knots in $\mathbb{R}^3$, sequences of DNA words over the field $\mathbb{F}_4$, the weight space of a neural network, payoff spaces of game theory.

None of these look like arrows in plane and are yet the same thing, deep down.

SECTION 2

Counterexamples

Polynomials of a fixed degree ▸

Let $D_1 = \{a + bx \mid b \neq 0\}$, the polynomials of degree *exactly* 1. This is not a vector space.

Polynomials of a fixed degree ▸

Let $D_1 = \{a + bx \mid b \neq 0\}$, the polynomials of degree *exactly* 1. This is not a vector space.

TWO SEPARATE FAILURES

$$(1 + x) + (1 - x) = 2 \notin D_1 \qquad \text{and} \qquad 0 \notin D_1.$$

The sum of two degree-one polynomials need not have degree one, and the zero polynomial does not have degree one either.

Degree *at most* 1 is fine; degree *exactly* 1 is not.

The degree of the zero polynomial is a topic of its own. Let us simplify our lives and declare it to be $-\infty$.

The non-negative reals

Let $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$, with the usual addition and scaling. Adding two non-negative numbers is no problem, but scaling proves problematic here, as

$$2 \in \mathbb{R}_{\geq 0}, \qquad (-1) \cdot 2 \; = \; -2 \notin \mathbb{R}_{\geq 0}.$$

So (A4) and (S0) both fail.

The non-negative reals

Let $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$, with the usual addition and scaling. Adding two non-negative numbers is no problem, but scaling proves problematic here, as

$$2 \in \mathbb{R}_{\geq 0}, \qquad (-1) \cdot 2 \;=\; -2 \notin \mathbb{R}_{\geq 0}.$$

So (A4) and (S0) both fail.

> FAILING USEFULLY
>
> A set closed under addition and under scaling by *non-negative* numbers only is called a **convex cone**. Cones are not vector spaces, but they are far from useless. Linear programming, useful in optimisation, scheduling and economics, is the study of what happens when the vector space axioms are weakened in exactly this way.

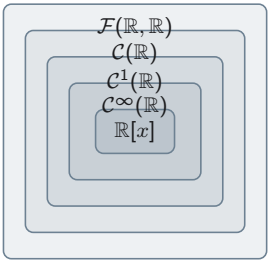
SECTION 3

Subspaces

A pattern in the "boring" examples

$$\mathbb{R}[x] \subset \mathcal{C}^\infty(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R}) \subset \mathcal{C}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R}, \mathbb{R})$$

- Each is a *subset* of the next, with the same operations.
- So (A2), (A5) and (S1)–(S4), being *equations*, hold automatically. What is true of all functions is true of the continuous ones.
- A subset can therefore fail *only* by leading outside itself (closure).



Definition

DEFINITION 3.3 ▸ · Subspace

Let V be a vector space. A subset $U \subseteq V$ is a **subspace** of V if

(U1)	$\mathbf{0} \in U$	Non-empty; equivalently, contains the zero vector
(U2)	$\mathbf{u} + \mathbf{w} \in U$ for all $\mathbf{u}, \mathbf{w} \in U$	Closed under addition
(U3)	$\lambda \mathbf{u} \in U$ for all $\lambda \in \mathbb{R},\ \mathbf{u} \in U$	Closed under scaling

KEY IDEA

Note, we only need to chekc three, not nine, conditions. This can make life much easier.

Why three conditions are enough

THEOREM 3.4

A subspace $U \subseteq V$ is itself a vector space, under the operations inherited from V.

Why three conditions are enough

THEOREM 3.4

A subspace $U \subseteq V$ is itself a vector space, under the operations inherited from V.

PROOF.

(U1) is just (A3).

$\mathbf{0}$ satisfies $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for every $\mathbf{v} \in V$ and every $\mathbf{u} \in U$ also lies inside V, clearly $\mathbf{0} + \mathbf{u} = \mathbf{u}$ for every $\mathbf{u} \in U$.

□

Why three conditions are enough

THEOREM 3.4

A subspace $U \subseteq V$ is itself a vector space, under the operations inherited from V.

PROOF.

(U1) is just (A3).

(U2) and (U3) are just (A1) and (A4), meaning closure under addition and scaling.

(A4) is actually not needed in our axiom lists, as it follows from (S0) for $\lambda = -1$.

□

Why three conditions are enough

THEOREM 3.4

A subspace $U \subseteq V$ is itself a vector space, under the operations inherited from V.

PROOF.

(U1) is just (A3).

(U2) and (U3) are just (A1) and (A4), meaning closure under addition and scaling.

Axioms (A2), (A5), (S1) – (S4) hold automatically subsubsets.

Note that if an equation holds in a big set, it will always hold in a smaller one. If $2 < 5$ as real numbers, then 2 is also smaller as intigers.

□

The definition, applied to what we have seen

Subspaces.

- $P_2 \subseteq \mathbb{R}[x]$: Adding and scaling polynomials of degree ≤ 2 again yield polynomials of degree ≤ 2, and $0 \in P_2$.
- $\mathcal{C}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$. Indeed, the whole chain from before.

Not subspaces.

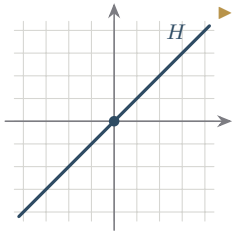
- $D_1 \subseteq \mathbb{R}[x]$: Fails (U1) and (U2).
- $\mathbb{R}_{\geq 0} \subseteq \mathbb{R}$: Fails (U3).

TWO TRIVIAL SUBSPACES

Every subspace V has at least two subspaces: $\{\mathbf{0}\}$ and V itself. Note for $V = \{\mathbf{0}\}$ only, these two coincide. Everything else is nested inbetween.

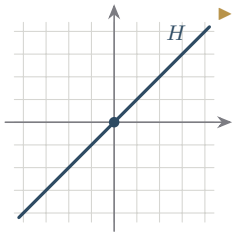
Examples: A line through the origin

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x - y = 0 \right\}$$



Examples: A line through the origin

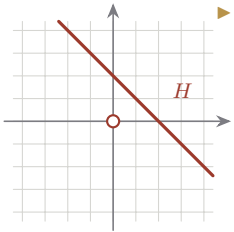
$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x - y = 0 \right\}$$



Yes. (U1): $0 - 0 = 0$, so $\mathbf{0} \in H$. (U2): If $x_1 = y_1$ and $x_2 = y_2$ then $(x_1 + x_2) - (y_1 + y_2) = 0$. (U3): $\lambda x - \lambda y = \lambda(x - y) = 0$. All three hold, so H is a subspace and thus, a vector space itself.

Examples: A line that misses the origin

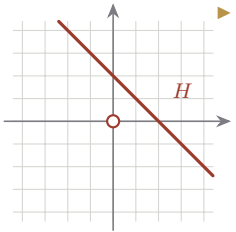
$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x + y = 1 \right\}$$



Despite everything failing, this is still a near misses, and still useful. They are called **affine spaces**.

Examples: A line that misses the origin

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x + y = 1 \right\}$$



No. Already (U1) fails: $0 + 0 = 0 \neq 1$, so $\mathbf{0} \notin H$, and we may stop there.

For the record, (U2) fails too: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ lie in H, but their sum $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not, since $1 + 1 = 2$. Indeed, even (U3) fails, since $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in H$ but $2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin H$.

Despite everything failing, this is still a near misses, and still useful. They are called **affine spaces**.

Summary of Lecture 3

FUNCTIONS	$\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space, with $(f+g)(x) = f(x) + g(x)$ and the zero vector being the constant function "0 everywhere". So were $\mathcal{C}(\mathbb{R})$, $\mathcal{C}^1(\mathbb{R})$, $\mathcal{C}^\infty(\mathbb{R})$, $\mathbb{R}[x]$, $\mathbb{R}[x, x^{-1}]$.
METHOD	Verify the axioms *coordinatewise/pointwise* in $\mathbb{R}^n$ and function spaces. Each reduces to the same axiom in $\mathbb{R}$.
SUBSPACE	$U \subseteq V$ is a subspace if $\mathbf{0} \in U$, and its closed under $+$ and saling.
FAILURES	Degree *exactly* 1 fails (U1) and (U2); $\mathbb{R}_{\geq 0}$ fails (U3) and affine lines fail everything.
WHY	A subspace is a vector space in its own right.
TRIVIAL	We always have the subspaces $\{\mathbf{0}\}$ and V for any vector space V.

Examples to consolidate understanding

AT HOME

1. Show that $W = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f(0) = 0\}$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. Is $\{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f(0) = 1\}$ one?
2. Which of these are subspaces of $\mathbb{R}^3$? (a) $\{(x, y, z) \mid x = 2y\}$; (b) $\{(x, y, z) \mid x + y + z = 0$ and$x - z = 0\}$; (c) $\{(x, y, z) \mid x^2 = y^2\}$.
3. Show that the symmetric matrices $\{A \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid A^{\mathsf{T}} = A\}$ form a subspace of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$. Is $\{A \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid a_{1,1} = 1\}$ one?

END OF LECTURE 3

ilia.pirashvili@universityofgalway.ac.ie

Solution: question 1

$f(0) = 0$: YES

(U1) The constant function 0 satisfies $0(0) = 0$. (U2) If $f(0) = g(0) = 0$ then $(f + g)(0) = f(0) + g(0) = 0$. (U3) $(\lambda f)(0) = \lambda f(0) = 0$.

All three conditions hold, so W is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$, and by the theorem it is a vector space.

Solution: question 1

$f(0) = 0$: YES

(U1) The constant function 0 satisfies $0(0) = 0$. (U2) If $f(0) = g(0) = 0$ then $(f + g)(0) = f(0) + g(0) = 0$. (U3) $(\lambda f)(0) = \lambda f(0) = 0$.

All three conditions hold, so W is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$, and by the theorem it is a vector space.

$f(0) = 1$: NO

The constant function 0 has $0(0) = 0 \neq 1$, so (U1) fails. Note also $(f + g)(0) = 2$: the same affine near miss as the line $x + y = 1$ ▸, dressed as functions.

Solution: question 2

THREE SUBSETS OF $\mathbb{R}^3$

(a) Yes. ▸ $x = 2y$ is linear with right-hand side 0: $\mathbf{0}$ qualifies; if $x_1 = 2y_1$ and $x_2 = 2y_2$ then $x_1 + x_2 = 2(y_1 + y_2)$; and $\lambda x = 2\lambda y$.

Solution: question 2

THREE SUBSETS OF $\mathbb{R}^3$

(a) Yes. ▸ $x = 2y$ is linear with right-hand side 0: $\mathbf{0}$ qualifies; if $x_1 = 2y_1$ and $x_2 = 2y_2$ then $x_1 + x_2 = 2(y_1 + y_2)$; and $\lambda x = 2\lambda y$.

(b) Yes. Both conditions are linear with right-hand side 0, and each survives addition and scaling as in (a). The set is a line through the origin.

Solution: question 2

THREE SUBSETS OF $\mathbb{R}^3$

(a) Yes. ▸ $x = 2y$ is linear with right-hand side 0: $\mathbf{0}$ qualifies; if $x_1 = 2y_1$ and $x_2 = 2y_2$ then $x_1 + x_2 = 2(y_1 + y_2)$; and $\lambda x = 2\lambda y$.

(b) Yes. Both conditions are linear with right-hand side 0, and each survives addition and scaling as in (a). The set is a line through the origin.

(c) No. (U1) and (U3) hold, since $(\lambda x)^2 = (\lambda y)^2$. But $(1, 1, 0) + (1, -1, 0) = (2, 0, 0)$, and $4 \neq 0$: (U2) fails.

Solution: question 3

SYMMETRIC MATRICES: YES

The zero matrix is symmetric. If $A^{\mathsf{T}} = A$ and $B^{\mathsf{T}} = B$ then $(A+B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}} = A + B$, and $(\lambda A)^{\mathsf{T}} = \lambda A^{\mathsf{T}} = \lambda A$. So all three conditions hold.

Solution: question 3

SYMMETRIC MATRICES: YES

The zero matrix is symmetric. If $A^{\mathsf{T}} = A$ and $B^{\mathsf{T}} = B$ then $(A+B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}} = A + B$, and $(\lambda A)^{\mathsf{T}} = \lambda A^{\mathsf{T}} = \lambda A$. So all three conditions hold.

$a_{1,1} = 1$: NO

The zero matrix has $a_{1,1} = 0$, so (U1) fails. Adding two such matrices gives $a_{1,1} = 2$, so (U2) fails as well.

Every failure so far has been one of the same three: the origin is missing, addition escapes, or scaling escapes.