MA313 · LINEAR ALGEBRA I

LECTURE 2

Inverses and vector spaces

Dr Ilia Pirashvili
Department of Mathematics, University of Galway
9 September 2026 · Winter 2026

Recap

MATRICES AND VECTORS

- A matrix is a rectangle of numbers, to define AB the inner dimensions must agree, and $AB \neq BA$ in general.
- $(AB)_{i,j} = a_{i,-} \cdot b_{-,j}$, and $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$.
- $\mathbb{R}^n$, meaning vectors, is the set of $n \times 1$ matrices, and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathsf{T}} \mathbf{v}$.
- Det is defined for square matrices only and given by minors. For 3×3 we can use Sarrus.
- $|\operatorname{Det} A|$ is the area factor, $\operatorname{Det}(AB) = \operatorname{Det} A \operatorname{Det} B$ and $\operatorname{Det}(A) \neq 0$ iff A is invertible (more on this now).

Warm-up

WARM-UP

1. Compute $\operatorname{Det}\left[\begin{smallmatrix} 4 & 2 \\ 3 & 1 \end{smallmatrix}\right]$ ▸ and $\operatorname{Det}\left[\begin{smallmatrix} 2 & 6 \\ 1 & 3 \end{smallmatrix}\right]$ ▸. Which of the two do you expect to be invertible?
2. Let A, B be 3×3 with $\operatorname{Det} A = 3$ and $\operatorname{Det} B = -2$. Find $\operatorname{Det}(AB)$ and $\operatorname{Det}(A^{\mathsf{T}}B)$.

Warm-up

WARM-UP

1. Compute $\operatorname{Det}\left[\begin{smallmatrix} 4 & 2 \\ 3 & 1 \end{smallmatrix}\right]$ ▸ and $\operatorname{Det}\left[\begin{smallmatrix} 2 & 6 \\ 1 & 3 \end{smallmatrix}\right]$ ▸. Which of the two do you expect to be invertible?
2. Let A, B be 3×3 with $\operatorname{Det} A = 3$ and $\operatorname{Det} B = -2$. Find $\operatorname{Det}(AB)$ and $\operatorname{Det}(A^{\mathsf{T}} B)$.

1 $4 - 6 = -2$ and $6 - 6 = 0$. The first should be invertible; the second collapses the plane onto a line.

2 $\operatorname{Det}(AB) = 3 \cdot (-2) = -6$, and since $\operatorname{Det}(A^{\mathsf{T}}) = \operatorname{Det} A$, also $\operatorname{Det}(A^{\mathsf{T}} B) = -6$.

SECTION 1

Inverses

What an inverse is, and why we cannot divide

DEFINITION 2.1 ▸ · Inverse

A square matrix A is called **invertible** if there is a matrix A^{-1} with $AA^{-1} = A^{-1}A = I$.

What an inverse is, and why we cannot divide

DEFINITION 2.1 ▸ · Inverse

A square matrix A is called **invertible** if there is a matrix A^{-1} with $AA^{-1} = A^{-1}A = I$.

NO FRACTIONS

Never write $\frac{B}{A}$. Multiplication is non-commutative, so $A^{-1}B$ and BA^{-1} are different matrices, and the notation does not distinguish them.

Some people write $B/_A$ for BA^{-1} and ${}_A\backslash B$ for $A^{-1}B$. Please do not:), at least in this module.

There is only one inverse

PROPOSITION 2.2

If A is invertible, its inverse is unique. We may therefore speak of ***the*** *inverse A^{-1}.*

PROOF.

Suppose B and C both undo A, so $BA = I$ and $AC = I$. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

□

The 2×2 formula

THEOREM 2.3

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *with* Det $A = ad - bc \neq 0$. *Then*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The 2×2 formula

THEOREM 2.3

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *with* $\operatorname{Det} A = ad - bc \neq 0$. *Then*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

EXAMPLE 2.4 ▸

$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ has $\operatorname{Det} A = 1$, so $A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$, and indeed

$$AA^{-1} = \begin{bmatrix} 6-5 & -3+3 \\ 10-10 & -5+6 \end{bmatrix} = I.$$

Swap the diagonal, negate the off-diagonal, then divide by the determinant.

What the inverse does to area

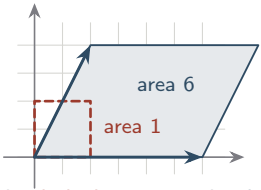
If A multiplies area by $\operatorname{Det} A$, then whatever undoes A must divide it back.

THEOREM 2.5

$$\operatorname{Det}(A^{-1}) = \frac{1}{\operatorname{Det} A}.$$

PROOF.

$$AA^{-1} = I$$



A sends the dashed square to the shaded one.
A^{-1} sends it back

In the image, A has columns $\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, so $\operatorname{Det} A = 6$ and $\operatorname{Det}(A^{-1}) = \frac{1}{6}$.

What the inverse does to area

If A multiplies area by $\operatorname{Det} A$, then whatever undoes A must divide it back.

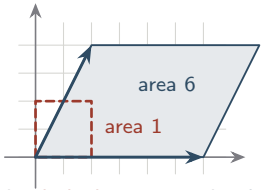
THEOREM 2.5

$$\operatorname{Det}(A^{-1}) = \frac{1}{\operatorname{Det} A}.$$

PROOF.

$$\operatorname{Det}(AA^{-1}) = \operatorname{Det} I$$

If two matrices are equal, everything is equal, including their determinants.



A sends the dashed square to the shaded one.
A^{-1} sends it back

In the image, A has columns $\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, so $\operatorname{Det} A = 6$ and $\operatorname{Det}(A^{-1}) = \frac{1}{6}$.

What the inverse does to area

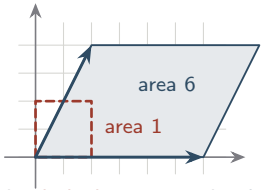
If A multiplies area by $\operatorname{Det} A$, then whatever undoes A must divide it back.

THEOREM 2.5

$$\operatorname{Det}(A^{-1}) = \frac{1}{\operatorname{Det} A}.$$

PROOF.

$$\operatorname{Det}(AA^{-1}) = 1 \qquad \square$$



A sends the dashed square to the shaded one.
A^{-1} sends it back

In the image, A has columns $\left[\begin{smallmatrix}3\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$, so $\operatorname{Det} A = 6$ and $\operatorname{Det}(A^{-1}) = \frac{1}{6}$.

The three elementary row operations ▸

Consider the following building blocks of matrix manipulations:

1. **Swap** two rows: $R_i \leftrightarrow R_j$.
2. **Scale** a row by a non-zero λ: $R_i \leftarrow \lambda R_i$, $\lambda \neq 0$.

   Technically, we need λ to be invertible, but $\lambda \in \mathbb{R}$, is invertile iff its non-zero.
3. **Add** a multiple of one row to another: $R_i \leftarrow R_i + \lambda R_j$ with $i \neq j$.

> WHY ARE THESE SO SPECIAL?
>
> Each operation is reversible, which is exactly why it destroys no information.
> Expressed more algebraically, each one is multiplication by an invertible matrix.

Scaling by $\lambda = 0$ is not allowed: it cannot be undone.

The Gauss-Jordan algorithm

Write A and I side-by-side and apply the elementary row operations on **both** A and I **simultaniously** until the left block is I. Expressed

$$[\,A \mid I\,] \xrightarrow{\text{row operations}} \left[\,I \mid A^{-1}\,\right].$$

If the left block cannot be brought to I, then A is not invertible. This happens exactly when $\operatorname{Det} A = 0$.

This algorithm answers existnance and what the inverse is at once.

Gauss–Jordan: worked example (1/2) ▸

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Gauss–Jordan: worked example (1/2) ▸

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Clear the first column, then the second:

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

Gauss–Jordan: worked example (2/2) ▸

$$\xrightarrow{R_3 \leftarrow -\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Gauss–Jordan: worked example (2/2) ▸

$$\xrightarrow{R_3 \leftarrow -\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \qquad A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

It is good practice to verify $AA^{-1} = I$, since it can catch arithmetic mistakes.

Two remarks on the method

- We never needed a row swap here, but its used in more complicated examples. One is needed as soon as a pivot position holds a zero.
- But when can we *definitely* say the left block cannot be brought to I (as opposed to perhaps us struggling)? When we reduce a row entirely to zeros, it is definite proof of non-invertibility.

EXAMPLE 2.6 ▸

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has Det $= 0$. The operation $R_2 \leftarrow R_2 - 2R_1$ kills the second row,

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

and the process stops.

The rules of invertibility

THEOREM 2.7 · Inversion criterion

A square matrix A is invertible if and only if $\det A \neq 0$.

THEOREM 2.8 · Products and transposes

If A and B are invertible, then so are AB and A^{T}, and

$$(AB)^{-1} = B^{-1}A^{-1}, \qquad (A^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}.$$

The rules of invertibility

THEOREM 2.7 · Inversion criterion

A square matrix A is invertible if and only if $\det A \neq 0$.

THEOREM 2.8 · Products and transposes

If A and B are invertible, then so are AB and A^{T}, and

$$(AB)^{-1} = B^{-1}A^{-1}, \qquad (A^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}.$$

PROOF.

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$

Same for transpose (try it yourself at home!) □

"Socks and shoes" law: To undo "socks, then shoes" you take off the shoes first then socks.

A far aside before we change subject

INVERTIBLE MATRICES FORM A GROUP (NOT RELEVANT FOR THE MODULE)

The set of invertible $n \times n$ matrices over $\mathbb{R}$ is denoted by $\mathrm{GL}_n(\mathbb{R})$ The last slide says exactly that $\mathrm{GL}_n(\mathbb{R})$ is closed under multiplication and under taking inverses, that I is the neutral element, and multiplication is associative. These axioms together form the definition of a **group**.

While we will not pursue groups as such and may be ignored, we will run into a group in a few slides again, as part of one of our main definitions: Vector spaces.

Indeed, in maths, physics and numerous other ares, it is not really possible to escape groups if one pays attention, as they really are just symmetries and thus, visibly or invisibly, underline everything.

SECTION 2

From $\mathbb{R}^n$ to vector spaces

What do we actually use about vectors?

Working in $\mathbb{R}^n$ for a fixed n, we have used only the following:

- We can add two vectors, and the result is again a vector.

What do we actually use about vectors?

Working in $\mathbb{R}^n$ for a fixed n, we have used only the following:

- We can add two vectors, and the result is again a vector.
- Addition does not care about order or bracketing.

  $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ and $\mathbf{v} + (\mathbf{w} + \mathbf{u}) = (\mathbf{v} + \mathbf{w}) + \mathbf{u}$.

What do we actually use about vectors?

Working in $\mathbb{R}^n$ for a fixed n, we have used only the following:

- We can add two vectors, and the result is again a vector.
- Addition does not care about order or bracketing.

  $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ and $\mathbf{v} + (\mathbf{w} + \mathbf{u}) = (\mathbf{v} + \mathbf{w}) + \mathbf{u}$.
- There is a zero vector, and every vector has a negative.

What do we actually use about vectors?

Working in $\mathbb{R}^n$ for a fixed n, we have used only the following:

- We can add two vectors, and the result is again a vector.
- Addition does not care about order or bracketing.

  $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$ and $\mathbf{v}+(\mathbf{w}+\mathbf{u})=(\mathbf{v}+\mathbf{w})+\mathbf{u}$.
- There is a zero vector, and every vector has a negative.
- We can scale a vector by a real number.

  E.g., terms like $3.14\mathbf{v}$ is again a vector.

What do we actually use about vectors?

Working in $\mathbb{R}^n$ for a fixed n, we have used only the following:

- We can add two vectors, and the result is again a vector.
- Addition does not care about order or bracketing.

  $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$ and $\mathbf{v}+(\mathbf{w}+\mathbf{u})=(\mathbf{v}+\mathbf{w})+\mathbf{u}$.
- There is a zero vector, and every vector has a negative.
- We can scale a vector by a real number.

  E.g., terms like $3.14\mathbf{v}$ is again a vector.
- Scaling distributes over addition, both ways.

  $\lambda(\mathbf{v}+\mathbf{w})=\lambda\mathbf{v}+\lambda\mathbf{w}$.

And what do we not use?

- There is no way to multiply two vectors of $\mathbb{R}^n$ and get a vector of $\mathbb{R}^n$, not in a way that works for every n.
- In particular, there is no division of vectors either.
- We never mix vectors of different lengths.

WHAT WE DELIBERATELY LEAVE OUT

The dot product eats two vectors and returns a *number*, so it is not a multiplication on $\mathbb{R}^n$. The vector product does return a vector, but only in $\mathbb{R}^3$, and it is neither associative nor commutative. Both are genuine extra structure, useful later, but not part of what we are about to call a vector space.

The definitions first half

DEFINITION 2.9 · Abelian group

A set V with an operation $+$ is an **abelian group** if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

(A1)	$\mathbf{u} + \mathbf{v} \in V$	closure
(A2)	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	associativity
(A3)	There is $\mathbf{0} \in V$ with $\mathbf{v} + \mathbf{0} = \mathbf{v}$	neutral element
(A4)	Each $\mathbf{v}$ has $-\mathbf{v}$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	inverses
(A5)	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	commutativity

Without (A5), we still have group, just like $\mathrm{GL}_n(\mathbb{R})$. The last condition makes it into a commutative group, which out of respect to the Norwegian Mathematician Niels Abel is called *abelian*.

The full definition

Having defined the abelian group part, all we need now are the scalars.

DEFINITION 2.10 · Vector space over $\mathbb{R}$

An ($\mathbb{R}$ valued) **vector space** is an abelian group $(V, +)$ together with a scaling $\mathbb{R} \times V \to V$ satisfying

(S0)	$\lambda \mathbf{u} \in V$	closure under scaling
(S1)	$\lambda(\mathbf{u} + \mathbf{v}) = \lambda \mathbf{u} + \lambda \mathbf{v}$	distributivity in V
(S2)	$(\lambda + \mu)\mathbf{v} = \lambda \mathbf{v} + \mu \mathbf{v}$	distributivity in $\mathbb{R}$
(S3)	$(\lambda \mu)\mathbf{v} = \lambda(\mu \mathbf{v})$	compatibility
(S4)	$1\,\mathbf{v} = \mathbf{v}$	the unit acts trivially

for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in V$.

Nine axioms altogether. Anything satisfying them *is* a vector space over $\mathbb{R}$. Note, we never mentioned arrows.

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$0\mathbf{v}$

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$0\mathbf{v} = (0+0)\mathbf{v}$$

Clearly, $0 = 0 + 0$.

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$0\mathbf{v} = \boxed{0\mathbf{v} + 0\mathbf{v}}$

(S2): $(0+0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}$.

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

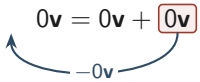
On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.



□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$0\mathbf{v} - 0\mathbf{v} = 0\mathbf{v}$$

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$\boxed{\mathbf{0}} = 0\mathbf{v}$

(A4): $0\mathbf{v} + (-0\mathbf{v}) = \mathbf{0}$.

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$\boxed{0\mathbf{v} = \mathbf{0}}$

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\mathbf{v} = \mathbf{0} \qquad \lambda\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$\lambda\mathbf{0}$

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

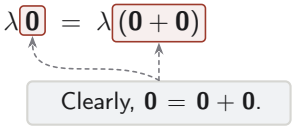
On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.



□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\lambda\mathbf{0} = \boxed{\lambda\mathbf{0} + \lambda\mathbf{0}}$$

(S1): $\lambda(\mathbf{0}+\mathbf{0}) = \lambda\mathbf{0}+\lambda\mathbf{0}$.

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\lambda\mathbf{0} = \lambda\mathbf{0} + \boxed{\lambda\mathbf{0}}$$

$$-\lambda\mathbf{0}$$

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\lambda\mathbf{0} - \lambda\mathbf{0} = \lambda\mathbf{0}$$

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\mathbf{0} = \boxed{\lambda \mathbf{0}}$$

(A4): $\lambda \mathbf{0} + (-\lambda \mathbf{0}) = \mathbf{0}$.

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\mathbf{v} = \mathbf{0} \qquad \lambda\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\boxed{\lambda\mathbf{0} = \mathbf{0}}$$

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\mathbf{v} + (-1)\mathbf{v}$$

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\mathbf{v} + (-1)\mathbf{v} = \boxed{1}\mathbf{v} + (-1)\mathbf{v}$$

(S4): $1\mathbf{v} = \mathbf{v}$.

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\mathbf{v} + (-1)\mathbf{v} = \boxed{(1 + (-1))}\,\mathbf{v}$$

(S2), applied in reverse.

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\mathbf{v} + (-1)\mathbf{v} = \boxed{0}\,\mathbf{v}$$

Clearly, $1 + (-1) = 0$.

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\mathbf{v} + (-1)\mathbf{v} = \boxed{\mathbf{0}}$$

By (i), shown above.

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

On zeroes and ones

PROPOSITION 2.11

In any vector space V, and for every $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$, we have

$$0\,\mathbf{v} = \mathbf{0} \qquad \lambda\,\mathbf{0} = \mathbf{0} \qquad (-1)\mathbf{v} = -\mathbf{v}$$

PROOF.

$$\boxed{(-1)\mathbf{v} = -\mathbf{v}}$$

□

Note, all this might sound trivial, but we did not assume anything from basic arithmetic, we derived everything from our axioms!

Our main example: $\mathbb{R}^n$

Consider $V = \mathbb{R}^n$ together with our entrywise operations:

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}, \qquad \lambda \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{bmatrix}.$$

- Clearly, (A2), (A5), (S1)-(S4) hold, since they hold in $\mathbb{R}$ and thus, hold in each coordinate.
- Setting $\mathbf{0} = (0, \ldots, 0)^\mathsf{T}$ fulfils (A3) and (A4) holds if we set $-\mathbf{v} = (-v_1, \ldots, -v_n)^\mathsf{T}$. (A1) and (S0) are clear.

Thus, everything we show for vector spaces in general, *must* hold for our main example as well.

Not just thin matricies: $m \times n$-shaped matrices for fixed m, n are vector spaces.

Consider $\mathrm{Mat}_{m\times n}(\mathbb{R})$ with pointwise (entrywise, coordinatewise) addition ▸ and scaling ▸.

$$\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix} + \begin{bmatrix}3 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}4 & 2\\ 1 & 0\end{bmatrix}, \qquad 3\begin{bmatrix}1 & -2\\ 0 & 4\end{bmatrix} = \begin{bmatrix}3 & -6\\ 0 & 12\end{bmatrix}$$

Not just thin matricies: $m \times n$-shaped matrices for fixed m, n are vector spaces.

Consider $\mathrm{Mat}_{m\times n}(\mathbb{R})$ with pointwise (entrywise, coordinatewise) addition ▸ and scaling ▸.

$$\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix} + \begin{bmatrix}3 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}4 & 2\\ 1 & 0\end{bmatrix}, \qquad 3\begin{bmatrix}1 & -2\\ 0 & 4\end{bmatrix} = \begin{bmatrix}3 & -6\\ 0 & 12\end{bmatrix}$$

The zero vector is the zero matrix, and the negative of A is $(-1)A$.

> The vector space structure on $\mathrm{Mat}_{m\times n}(\mathbb{R})$ only uses *addition and scaling*, not multiplication.

> CAUTION
>
> Note, matrices actually play a different role for vectorspaces for us, and there, their full structure will shine, including multiplication.

Example: linear polynomials

There are examples that look nothing like $\mathbb{R}^n$. That is exactly why its worthwhile to explore these topics in the abstract. Results proved once in generality holds for every special case! For example, define

$$P_1 := \{a + bx \mid a, b \in \mathbb{R}\}$$

P_1 is **closed** under addition and scaling, meaning to say

$$(2 + 3x) + (1 - 5x) = 3 - 2x, \qquad 4(2 + 3x) = 8 + 12x.$$

Example: linear polynomials

There are examples that look nothing like $\mathbb{R}^n$. That is exactly why its worthwhile to explore these topics in the abstract. Results proved once in generality holds for every special case! For example, define

$$P_1 := \{a + bx \mid a, b \in \mathbb{R}\}$$

P_1 is **closed** under addition and scaling, meaning to say

$$(2 + 3x) + (1 - 5x) = 3 - 2x, \qquad 4\,(2 + 3x) = 8 + 12x.$$

- The polynomial 0 plays the zero vector and the negation of $a + bx$ is $-a - bx$.
- All nine axioms hold, coefficient by coefficient.

Clearly, (a, b) recover $a + b$ in full. Hence P_1 is actually just $\mathbb{R}^2$ written differently. Is this a coincidence?

An example of abstraction

Not important for our module.

FAR ASIDE: THE RUBIK'S CUBE

Proving things in the abstract can have very powerful applications. Have you ever wondered how there can be ways to solve the Rubik's from any position?

It is actually just a bit of group theory. Note that every rotation (which should be seen as our element!) has an inverse (rotating back). We can "add" rotations by performing them one after the other. The "zero rotation" is just doing nothing. And its clearly associative.

Hence, we have a (finite) group! and thus, can use abstract results that always hold.

Example: Quadratic polynomials

$$P_2 = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}$$

$$(1 + 2x + x^2) + (3 - 2x - x^2) = 4, \qquad -2(1 + 2x + x^2) = -2 - 4x - 2x^2.$$

Example: Quadratic polynomials

$$P_2 = \{a + bx + cx^2 : a, b, c \in \mathbb{R}\}$$

$$(1 + 2x + x^2) + (3 - 2x - x^2) = 4, \qquad -2(1 + 2x + x^2) = -2 - 4x - 2x^2.$$

READ THE DEFINITION CAREFULLY

P_2 is the set of polynomials of degree *at most* 2. The polynomials of degree *exactly* 2 do not form a vector space as the sum of $x^2 + x + 1$ and $-x^2$ is $x + 1$, which has left the set.

Meaning, it is not closed under addition (axiom (A1), fails).

Example: All polynomials

$$\mathbb{R}[x] \;=\; \{a_0 + a_1 x + \cdots + a_k x^k : k \in \mathbb{N},\; a_i \in \mathbb{R}\}$$
$$(1 + x^3) + (2x - x^3) \;=\; 1 + 2x$$

Do not be alarmed that the degree is unbounded: so are the natural numbers. As long as each individual polynomial is finite, all is well.

Example: All polynomials

$$\mathbb{R}[x] \;=\; \{a_0 + a_1 x + \cdots + a_k x^k : k \in \mathbb{N},\ a_i \in \mathbb{R}\}$$
$$(1 + x^3) + (2x - x^3) \;=\; 1 + 2x$$

Do not be alarmed that the degree is unbounded: so are the natural numbers. As long as each individual polynomial is finite, all is well.

> ASIDE
>
> Here, no finite list of coordinates will do: $1, x, x^2, x^3, \ldots$ are all needed. This is our first **infinite-dimensional** example, and almost everything we prove later will quietly assume finite dimension. There are, however, many very interesting examples of infinite-dimensional spaces, particularly were analysis meets linear algebra, e.g., Fourier series, quantum mechanics.

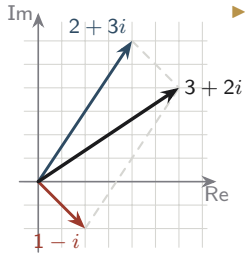
Example: the complex numbers over $\mathbb{R}$

We have already met a vector space without calling it one: the complex numbers $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$, with the usual addition and with scaling by *real* numbers only,

$$(2 + 3i) + (1 - i) \;=\; 3 + 2i$$

$$\lambda(a + bi) \;=\; \lambda a + \lambda b\, i.$$

This is exactly the parallelogram rule, with (a, b) as coordinates.



$\mathbb{C}$: what we deliberately forget

As a real vector space, $\mathbb{C}$ is two-dimensional: every element is $a \cdot 1 + b \cdot i$, so $\{1, i\}$ plays the role of the coordinate axes and $\mathbb{C}$ is a copy of $\mathbb{R}^2$.

$\mathbb{C}$: what we deliberately forget

As a real vector space, $\mathbb{C}$ is two-dimensional: every element is $a \cdot 1 + b \cdot i$, so $\{1, i\}$ plays the role of the coordinate axes and $\mathbb{C}$ is a copy of $\mathbb{R}^2$.

THE MULTIPLICATION IS NOT PART OF THE STRUCTURE

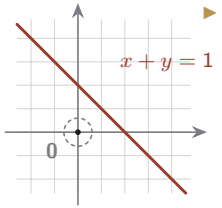
Of course $(2 + 3i)(1 - i) = 5 + i$, but that product is *extra*: the nine axioms never mention it. Calling $\mathbb{C}$ a real vector space is a decision to remember only $+$ and scaling by reals.

Forgetting structure is not a loss. It is what lets one theorem serve $\mathbb{R}^n$, matrices, polynomials and $\mathbb{C}$ at the same time.

Three near misses

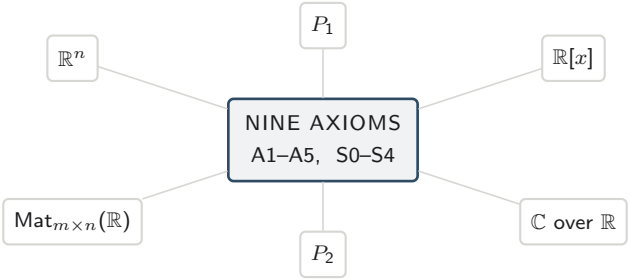
Failing even one axiom is fatal.

- $U = \{(x, y) : x + y = 1\}$. It has no zero vector, and $(1, 0) + (0, 1) = (1, 1) \notin U$.
- The first quadrant $\{(x, y) : x, y \geq 0\}$. Closed under addition, but $-1 \cdot (1, 1)$ escapes it.
- $\mathbb{R}^2$ with scaling redefined as $\lambda \odot (x, y) = (\lambda x, 0)$. Everything else is fine, but $1 \odot (x, y) = (x, 0) \neq (x, y)$: axiom (S4) fails.



A line that misses the origin is not a vector space.

One definition, many faces



KEY IDEA

Not abstraction for its own sake: Each theorem proved once instead of six times. Indeed, there are countless examples of vectorspaces.

Summary of Lecture 2

INVERSE	$AA^{-1} = A^{-1}A = I$; it is unique, so we say *the* inverse. There is no division: write $A^{-1}B$ or BA^{-1}.
2×2	$A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
ALGORITHM	Gauss–Jordan: $[A \mid I] \to \left[I \mid A^{-1}\right]$ by row operations; it fails exactly when $\det A = 0$.
RULES	A invertible $\iff \det A \neq 0$; $(AB)^{-1} = B^{-1}A^{-1}$; $(A^{\mathsf{T}})^{-1} = (A^{-1})^{\mathsf{T}}$; $\det(A^{-1}) = 1/\det A$.
VECTOR SPACE	Nine axioms: (A1)–(A5) make $(V, +)$ an abelian group, (S0)–(S4) govern scaling. They give $0\mathbf{v} = \mathbf{0}$ and $(-1)\mathbf{v} = -\mathbf{v}$.
EXAMPLES	$\mathbb{R}^n$, $\mathrm{Mat}_{m\times n}(\mathbb{R})$, P_1, P_2, $\mathbb{R}[x]$, and $\mathbb{C}$ over $\mathbb{R}$.

Before next time

AT HOME

1. Invert $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ by Gauss–Jordan ▸, or explain why no inverse exists. Check your answer.
2. Is $W = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ a vector space under the operations of $\mathbb{R}^3$? What about $U = \{(x, y, z) : x + y + z = 1\}$? Justify both answers.
3. Which of these are vector spaces over $\mathbb{R}$? (a) the 2×2 matrices with Det $= 0$; (b) the polynomials p with $p(1) = 0$; (c) $\mathbb{R}^2$ with the usual addition but $\lambda \odot (x, y) = (\lambda x, \lambda y + 1)$.

END OF LECTURE 2

ilia.pirashvili@universityofgalway.ac.ie

Solution: question 1 ▸

Det $A = 7 \neq 0$, so an inverse exists. Reducing $[A \mid I]$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 7 & -1 & 2 & 1 \end{array}\right]$$

Solution: question 1 ▸

Det $A = 7 \neq 0$, so an inverse exists. Reducing $[A \mid I]$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 7 & -1 & 2 & 1 \end{array}\right]$$

Then $R_3 \leftarrow \frac{1}{7} R_3$, $R_2 \leftarrow R_2 - 3R_3$, $R_1 \leftarrow R_1 - 2R_2$ give

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -2 & 6 \\ 3 & 1 & -3 \\ -1 & 2 & 1 \end{bmatrix}, \qquad \text{and indeed } AA^{-1} = I.$$

Solution: question 2

W: YES ▸

$\mathbf{0} = (0,0,0)$ satisfies $0+0+0=0$, so W is not empty. If $\mathbf{u}, \mathbf{v} \in W$ then the coordinates of $\mathbf{u}+\mathbf{v}$ sum to $0+0=0$, and those of $\lambda\mathbf{u}$ to $\lambda \cdot 0 = 0$: both stay in W. The remaining axioms are inherited from $\mathbb{R}^3$, since the operations are the same ones.

Solution: question 2

W: YES ▸

$\mathbf{0} = (0,0,0)$ satisfies $0+0+0=0$, so W is not empty. If $\mathbf{u}, \mathbf{v} \in W$ then the coordinates of $\mathbf{u}+\mathbf{v}$ sum to $0+0=0$, and those of $\lambda\mathbf{u}$ to $\lambda \cdot 0 = 0$: both stay in W. The remaining axioms are inherited from $\mathbb{R}^3$, since the operations are the same ones.

U: NO ▸

$(0,0,0) \notin U$, so (A3) fails at once. Closure fails too: $(1,0,0)$ and $(0,1,0)$ lie in U, but their sum $(1,1,0)$ has coordinate sum 2.

W passing every test by inheritance is not an accident. Next lecture gives it a name.

Solution: question 3

WHICH ARE VECTOR SPACES?

(a) No. $\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$ both have determinant 0, but their sum is I: closure fails.

Solution: question 3

WHICH ARE VECTOR SPACES?

(a) No. $\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$ both have determinant 0, but their sum is I: closure fails.

(b) Yes. If $p(1) = q(1) = 0$ then $(p+q)(1) = 0$ and $(\lambda p)(1) = 0$; the zero polynomial qualifies, and the remaining axioms are inherited from $\mathbb{R}[x]$.

Solution: question 3

WHICH ARE VECTOR SPACES?

(a) No. $\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$ both have determinant 0, but their sum is I: closure fails.

(b) Yes. If $p(1) = q(1) = 0$ then $(p+q)(1) = 0$ and $(\lambda p)(1) = 0$; the zero polynomial qualifies, and the remaining axioms are inherited from $\mathbb{R}[x]$.

(c) No. $1 \odot (x, y) = (x, y+1) \neq (x, y)$, so axiom (S4) fails — the one axiom that looks too obvious to be worth stating.