LECTURE 1

Matrices, vectors and determinants

Basics of linear algebra

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Outline

1. Matrices, concretely
2. Vectors: Just thin matrices
3. Determinants

SECTION 1

Matrices, concretely

Matrices are rectangles of numbers

DEFINITION 1.1 · Matrix

An $m \times n$ **matrix** over $\mathbb{R}$ is a rectangular array (list) of real numbers with m rows and n columns. The set of all of them is written $\mathrm{Mat}_{m\times n}(\mathbb{R})$.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{2\times 3} \qquad \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{2\times 2} \qquad \underbrace{\begin{bmatrix} 7 \\ 0 \\ 2 \end{bmatrix}}_{3\times 1} \qquad \underbrace{\begin{bmatrix} 3 & \pi \end{bmatrix}}_{1\times 2}$$

Rows first (y-coordinate), then columns (x-coordinate). A 2×3 matrix is short and wide, not tall and thin.

Naming the entries

We write $a_{i,j}$ for the entry in **row** i, **column** j:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix} \qquad \text{for example} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

row 2

column 3

The entry $a_{2,3} = 6$ is exactly where the two boxes cross.

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row 2

column 3

The entry $a_{2,3} = 6$ is exactly where the two boxes cross. A dash means *"let this index run"*:

$$a_{2,-} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \quad \text{(the 2nd row)}, \qquad a_{-,3} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{(the 3rd column)}.$$

Two easy operations

EXAMPLE 1.2 · Addition ▸ and scalar multiplication ▸ both work entry by entry

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}, \qquad 3 \cdot \begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 0 & 12 \end{bmatrix}.$$

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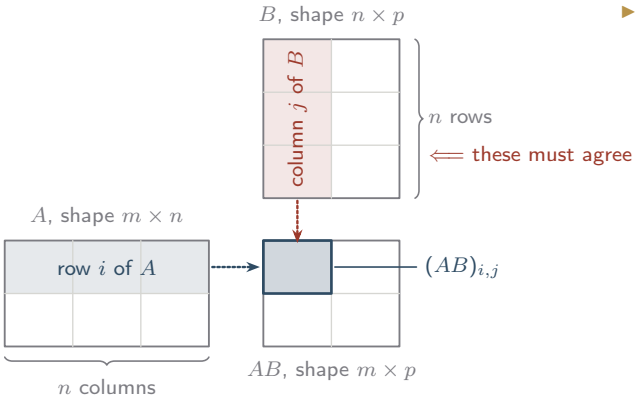
Of course, we can also combine the operations:

$$2 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} - 3 \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 4 \\ -3 & -5 \end{bmatrix}.$$

CAUTION

For addition, both matrices must have the same shape: $\begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not defined ▸.

Where each entry of a product comes from



$$(AB)_{i,j} \;=\; a_{i,-} \cdot b_{-,j}$$

When is a product defined? ▸

Pairing a row of A with a column of B needs the two to have the same length, so unlike addition, multiplication is not always defined.

- $(2 \times 3) \times (3 \times 2)$: allowed, and the product is 2×2.
- $(2 \times 3) \times (2 \times 3)$: not allowed.
- $(1 \times n) \times (n \times 1)$: allowed; the product is a 1×1 matrix, i.e. a number. Remember this one.

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KEY IDEA

AB exists *only* if the columns of A match the rows of B. The two inner numbers then cancel:

$$\underset{m\times n}{A} \times \underset{n\times p}{B} = \underset{m\times p}{AB}.$$

Multiplication: a worked example ▸

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

Multiplication: a worked example ▸

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & \square \\ \square & \square \end{bmatrix}$$

For example, entry $(1,1)$ is built from **row 1** and **column 1**:

$$1 \cdot 3 \;+\; 0 \cdot 2 \;+\; 2 \cdot 1 \;=\; 5.$$

Shapes: $(2 \times 3) \times (3 \times 2)$, so the answer is 2×2 — neither the shape of A nor the shape of B.

Multiplication: the other three slots ▸

They are filled in exactly the same way:

$$(1,1):\ 1\cdot 3+0\cdot 2+2\cdot 1\ =\ 5 \qquad (1,2):\ 1\cdot 1+0\cdot 1+2\cdot 0\ =\ 1$$

$$(2,1):\ (-1)\cdot 3+3\cdot 2+1\cdot 1\ =\ 4 \qquad (2,2):\ (-1)\cdot 1+3\cdot 1+1\cdot 0\ =\ 2$$

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and so

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}.$$

The rule: in words and in symbols

IN WORDS

For the (i, j)-th entry of AB: walk along row i of A and down column j of B, multiply the paired numbers, and add up the results.

FORMALLY EXPRESSED:

$$(AB)_{i,j} = \sum_{k=1}^{n} a_{i,k}\, b_{k,j}$$

Order matters

EXAMPLE 1.3 ▸

Take $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \qquad BA = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}.$$

NON-COMMUTATIVITY

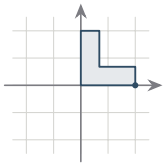
$AB \neq BA$ in general. Indeed AB can be defined while BA is not, and even when both are defined they may have different shapes.

Almost everything that feels strange about matrices comes from this one fact.

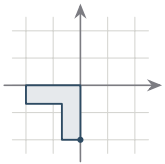
Why order matters: a matrix is an instruction

Let R rotate the plane by 90° and S reflect it in the horizontal axis. The dot follows the corner $\begin{bmatrix}1\\0\end{bmatrix}$. Consider the shape F below. What do these operations do to it?

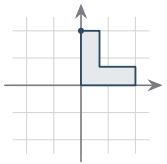
$$R = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \quad S = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \qquad SR = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}, \quad RS = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$



The shape F



SR: Rotate, then reflect



RS: Reflect, then rotate

AB means *do B, then do A*. Composing two instructions in the other order is a different instruction — that is all non-commutativity is.

The zero and the identity

Much as for numbers we have a "0" and a "1", but one must be a little careful. There is one of each *per shape*.

DEFINITION 1.4 ▸ · Zero and identity matrices

The identity mateix $I = I_n = \mathrm{Id}$ is *square*, with ones on the diagonal and zeros elsewhere. The zero matrix is a matrix that has only zeros, irrespective of shape.

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad I_1 = \begin{bmatrix} 1 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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They behave as one would expect, so long as the shape allows it,

$$0 \times M = M \times 0 = 0, \qquad I \times M = M \times I = M.$$

These are among the very few matrices that commute with everything.

Transpose: reflection in the diagonal

DEFINITION 1.5 ▸ · Transpose

A^{T} is the matrix with $(A^{\mathsf{T}})_{i,j} = a_{j,i}$. Rows and columns switch, so an $m \times n$ matrix becomes $n \times m$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \Longrightarrow \quad A^{\mathsf{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

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Transposing reverses products: $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$. On the example from before,

$$\left(\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \right)^{\mathsf{T}} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}.$$

SECTION 2

Vectors: Just thin matrices

Vectors need no new definition

KEY IDEA

A **column vector** in $\mathbb{R}^n$ is an $n \times 1$ matrix, and a **row vector** is a $1 \times n$ matrix.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3 \ _{(3\times 1)}, \qquad \mathbf{u}^{\mathsf{T}} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \ _{(1\times 3)}.$$

Adding and scaling vectors are just the matrix operations. They are done one coordinate at a time. Everything from the last few slides applies unchanged.

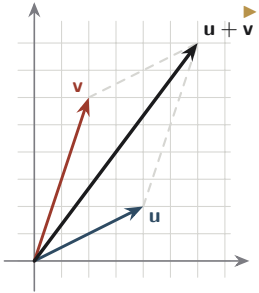
Unless we say otherwise, *vector* will always mean *column vector*. It matters as soon as we multiply.

The geometry behind addition

Adding vectors means laying them nose to tail. The sum is the diagonal of the parallelogram they span.

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Order is irrelevant here: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ traces the two other sides of the same parallelogram.

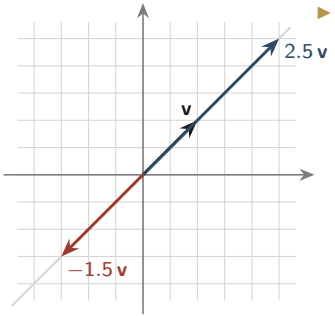


The geometry behind scaling

Multiplying by λ stretches by $|\lambda|$ and flips the direction when $\lambda < 0$. All the multiples of one non-zero vector,

$$\{\lambda \mathbf{v} \mid \lambda \in \mathbb{R}\},$$

form a line through the origin. Two vectors are **parallel** exactly when each is a multiple of the other.



That line will be our first example of a subspace, next lecture.

The dot product is a matrix product

The dot product looks like something new, but if you simpy consider it as a shorthand for the *closest sensible operation*, it starts to make sense: Two shapes $(1 \times n)$ and $(1 \times n)$ do not multiply, so we **transpose** one of them to get a number.

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the product $\mathbf{u}^\mathsf{T}\mathbf{v}$ has shape $(1 \times n)(n \times 1) = 1 \times 1$, i.e. it is a number:

$$\mathbf{u}^\mathsf{T}\mathbf{v} \;=\; \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \;=\; \sum_{k=1}^{n} u_k v_k \;=\; u_1 v_1 + \cdots + u_n v_n \;=\; \mathbf{u} \cdot \mathbf{v}.$$

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EXAMPLE 1.6 ▸

For $\mathbf{u} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\mathsf{T}$ and $\mathbf{v} = \begin{bmatrix} 4 & -1 & 0 \end{bmatrix}^\mathsf{T}$ we have $\mathbf{u}^\mathsf{T}\mathbf{v} = 4 - 2 + 0 = 2$.

Two consequences worth keeping

A MATRIX PRODUCT IS A TABLE OF DOT PRODUCTS

$$(AB)_{i,j} \;=\; a_{i,-} \cdot b_{-,j}$$

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$\mathbf{u}^{\mathsf{T}}\mathbf{v}$ AND $\mathbf{u}\mathbf{v}^{\mathsf{T}}$ ARE NOT THE SAME THING

Swapping the transpose swaps the shapes: $(n \times 1)(1 \times n)$ is an $n \times n$ matrix, not a number.

$$\begin{bmatrix}1\\2\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}3\\4\end{bmatrix} \;=\; 11, \quad \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix}^{\mathsf{T}} \;=\; \begin{bmatrix}3&4\\6&8\end{bmatrix}.$$

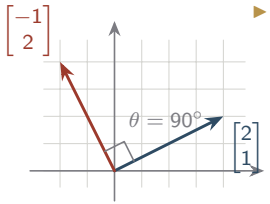
The latter always has determinant zero. Can you see why?

What the dot product measures

The length of a vector is $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$, and for the angle θ between two non-zero vectors

$$\mathbf{u} \cdot \mathbf{v} \;=\; \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta.$$

So the dot product is zero exactly when the vectors are **perpendicular**. We need not compute anything else.



EXAMPLE 1.7

$\begin{bmatrix}2\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\2\end{bmatrix} = -2 + 2 = 0$, so these two are perpendicular.

A word on the vector product

ASIDE

Some of you have met the *vector product* $\mathbf{u} \times \mathbf{v}$, which intakes two vectors and returns a vector. This is *not* a general matrix operation, but

- Neither is it a general vector operation. It only exists in $\mathbb{R}^3$.
- It is neither commutative nor associative: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- We will not use it in any way, so can be safely ignored.

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FAR ASIDE

There are numerous other such "accidental" structures in mathematics, and do not invalidate the general priciple. For example, we have a multiplication on $\mathbb{R}^2$ defined that makes it into $\mathbb{C}$ while preserving everything. No such operation exists for $\mathbb{R}^3$!. We have something similar (but a bit weaker — no longer commutative) for $\mathbb{R}^4$, being the **quaternions** $\mathbb{H}$, introduced by the Irish mathematician *Hamilton*, and then another even weaker structure for $\mathbb{R}^8$, and nothing else sensible for any other powers of $\mathbb{R}$!. There are countless such results in mathematics.

SECTION 3

Determinants

What a determinant is for

We can attach a single number $\operatorname{Det} A \in \mathbb{R}$ to a*square* matrix A that carries a significant amount of information of A. For this, we have

- $\operatorname{Det} A \neq 0$ exactly when the action of A can be undone, i.e. is invertible.

  More precisely: $\operatorname{Det} A$ is invertible in R if and only if A is invertible over R.
- $|\operatorname{Det} A|$ is the factor by which A scales area (in 2D) or volume (in 3D+).
- Its sign records whether A preserves or reverses orientation.

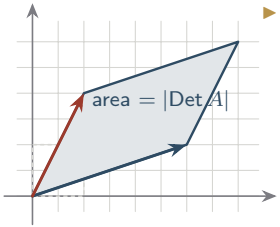
> CAUTION
>
> Non-square matrices have no determinant at all. It is not zero, it is undefined.
>
> But we can still do "magic" with them, as we will see at the end of the module!

The 2×2 case

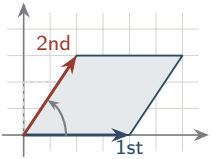
$$\text{Det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\text{Det} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = 6 - 5 = 1$$

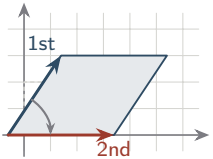
The two columns $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ span a parallelogram, and $|ad - bc|$ is its area. The unit square has area 1, so this is the area factor promised on the last slide.



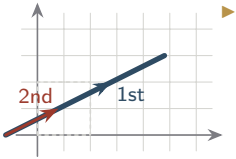
The sign, and the collapse



Det $= +3$
1st to 2nd: counterclockwise



Det $= -3$
1st to 2nd: clockwise



Det $= 0$
The columns are parallel

KEY IDEA

If you reverse the order of two columns/rows, you keep the are, but sign flips. Det $A = 0$ means the image of the square is squashed onto a line. In 2D it means the columns are parallel.

If you collapse 3 vecotrs in 3D to one plane, the volumn becomes zero and the determinant is thus also zero. Note, the area need not be zero in 3D, so long as the volumn vanishes. In other words, determinant asks "did you really need all the vectors to describe your object?".

Determinants by minors ▸

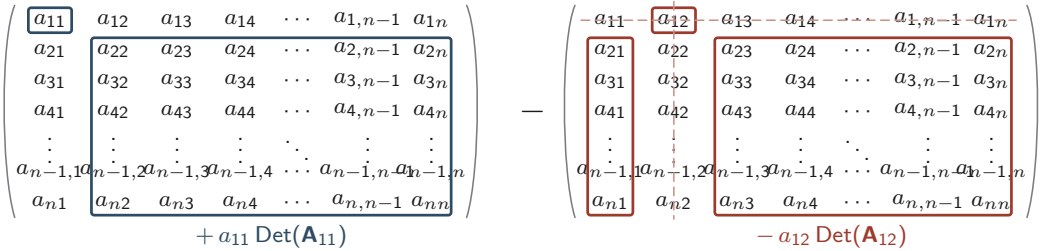
For an $n \times n$ matrix $\mathbf{A} = (a_{i,j})$, write $\mathbf{A}_{1j}$ for the matrix obtained by deleting row 1 and column j.

$$\mathrm{Det}(\mathbf{A}) \;=\; a_{1,1}\,\mathrm{Det}(\mathbf{A}_{11}) - a_{1,2}\,\mathrm{Det}(\mathbf{A}_{12}) + \cdots + (-1)^{1+n} a_{1,n}\,\mathrm{Det}(\mathbf{A}_{1n}).$$

KEY IDEA

The determinant of a large matrix is defined from determinants of smaller matrices, with signs alternating along the row. The 2×2 formula is where the recursion stops.

Determinants by minors: what gets deleted



Each term deletes the first row and one column; the boxed part is what is left, and its determinant is computed the same way again.

Minors in practice ▸

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Minors in practice ▸

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Expanding along the first row, with signs $+, -, +$:

$$\operatorname{Det} A = 1 \cdot \begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix}$$

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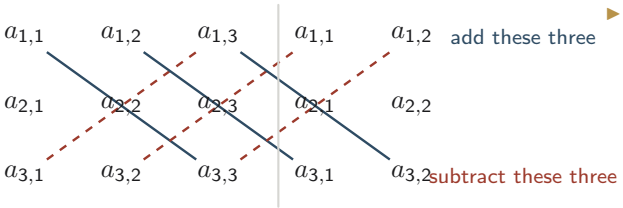
$$\operatorname{Det} A = 1 \cdot \begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix}$$

$$= 1 \cdot (0 - 24) - 2 \cdot (0 - 20) + 3 \cdot (0 - 5) = -24 + 40 - 15 = 1.$$

Three 2×2 determinants for one 3×3. For a 4×4 it is four 3×3s, so twelve 2×2s. The cost grows fast.

The 3×3 case: Sarrus' scheme



Copy the first two columns to the right, then run down the three solid diagonals (adding) and up the three dashed ones (subtracting).

This trick works for 3×3 only. It is false for 4×4 ▸, where it would give six terms instead of the correct twenty-four.

Sarrus in practice ▸

The same matrix as two slides ago, so the two methods must agree:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Sarrus in practice ▸

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Down the solid diagonals:

$$1 \cdot 1 \cdot 0 \; + \; 2 \cdot 4 \cdot 5 \; + \; 3 \cdot 0 \cdot 6 \; = \; 0 + 40 + 0 \; = \; 40.$$

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Up the dashed ones:

$$3 \cdot 1 \cdot 5 \; + \; 1 \cdot 4 \cdot 6 \; + \; 2 \cdot 0 \cdot 0 \; = \; 15 + 24 + 0 \; = \; 39.$$

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Hence $\operatorname{Det} A = 40 - 39 = 1$, as before.

Two properties we will use constantly

THEOREM 1.8 · Transpose invariance

$\mathrm{Det}(A^{\mathsf{T}}) = \mathrm{Det}(A)$.

$$\mathrm{Det}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = 1 = \mathrm{Det}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Two properties we will use constantly

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THEOREM 1.9 · Multiplicativity

$\mathrm{Det}(AB) = \mathrm{Det}(A)\,\mathrm{Det}(B)$ *for square* A, B *of the same size.*

$$A = \begin{bmatrix}3 & 1\\ 5 & 2\end{bmatrix}, \quad B = \begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix}, \quad AB = \begin{bmatrix}7 & 1\\ 13 & 2\end{bmatrix}, \qquad 1 \cdot 1 = \mathrm{Det}(AB) = 14 - 13.$$

Consequences, immediately

- Rows and columns play symmetric roles: Anything true of rows is true of columns, by transposing.
- $AB \neq BA$, yet $\mathrm{Det}(AB) = \mathrm{Det}(BA)$ holds always, as the determinant is a projection to numbers, and they commute.
- Areas multiply when maps are composed.

  Algebraically, this is summed up by the concept of multiplicativity.

THE QUESTION THIS LEAVES OPEN

A collapse cannot be undone, so $\mathrm{Det}\, A \neq 0$ is *necessary* for A to have an inverse. Is it *sufficient*? And if it is, how do we find the inverse?

Summary of Lecture 1

SHAPE	$m \times n$ is rows $\times$ columns. $A + B$ needs equal shapes; AB needs the inner ones to agree, and $(m \times n)(n \times p) = (m \times p)$.
PRODUCT	$(AB)_{i,j} = a_{i,-} \cdot b_{-,j}$, a table of dot products. $AB \neq BA$: a matrix is an instruction, and order matters.
TRANSPOSE	$(A^{\mathsf{T}})_{i,j} = a_{j,i}$, and $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$.
VECTORS	$\mathbb{R}^n$ is the $n \times 1$ matrices; $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathsf{T}}\mathbf{v}$; zero means perpendicular.
DETERMINANT	Square only. 2×2: $ad - bc$; 3×3: Sarrus or minors. $\lvert\mathrm{Det}\, A\rvert$ is the area factor, its sign the orientation, and $\mathrm{Det}\, A = 0$ means collapse.
RULES	$\mathrm{Det}(A^{\mathsf{T}}) = \mathrm{Det}\, A$ and $\mathrm{Det}(AB) = \mathrm{Det}\, A\, \mathrm{Det}\, B$.

Try a few of these before next time

AT HOME

1. For $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 5 & 1 \end{bmatrix}$: compute AB ▸ and BA ▸, note their shapes, and check $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$.
2. For $C = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$: compute $\operatorname{Det} C$ twice, by Sarrus ▸ and by minors ▸ along row 1. What does the answer suggest about inverting C?
3. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Show they are perpendicular, find the area of the parallelogram they span ▸, and explain why that area is $\|\mathbf{u}\|\|\mathbf{v}\|$.

END OF LECTURE 1

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Solution: question 1

SOLUTION

AB is 2×2 ▸ and BA is 3×3 ▸: the same two matrices, two different shapes.

$$AB = \begin{bmatrix} 2 & 5 \\ 23 & 9 \end{bmatrix}, \qquad BA = \begin{bmatrix} 8 & 1 & 8 \\ -3 & -1 & -4 \\ 13 & -4 & 4 \end{bmatrix}$$

Solution: question 1

SOLUTION

AB is 2×2 ▸ and BA is 3×3 ▸: the same two matrices, two different shapes.

$$AB = \begin{bmatrix} 2 & 5 \\ 23 & 9 \end{bmatrix}, \qquad BA = \begin{bmatrix} 8 & 1 & 8 \\ -3 & -1 & -4 \\ 13 & -4 & 4 \end{bmatrix}$$

And $B^{\mathsf{T}}A^{\mathsf{T}}$ has shape $(2 \times 3)(3 \times 2)$, so 2×2:

$$B^{\mathsf{T}}A^{\mathsf{T}} = \begin{bmatrix} 1 & 0 & 5 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 23 \\ 5 & 9 \end{bmatrix} = (AB)^{\mathsf{T}}.$$

Solution: question 2

BY SARRUS ▸

Down: $2{\cdot}3{\cdot}1+0{\cdot}2{\cdot}1+1{\cdot}1{\cdot}1 = 6+0+1 = 7$. Up: $1{\cdot}3{\cdot}1+2{\cdot}2{\cdot}1+0{\cdot}1{\cdot}1 = 3+4+0 = 7$.
So $\operatorname{Det} C = 7 - 7 = 0$.

Solution: question 2

BY SARRUS ▸

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So $\operatorname{Det} C = 7 - 7 = 0$.

BY MINORS ▸

$$\operatorname{Det} C \;=\; 2\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \;=\; 2(1) - 0 + 1(-2) \;=\; 0.$$

Solution: question 2

BY SARRUS ▸

Down: $2{\cdot}3{\cdot}1+0{\cdot}2{\cdot}1+1{\cdot}1{\cdot}1 = 6+0+1 = 7$. Up: $1{\cdot}3{\cdot}1+2{\cdot}2{\cdot}1+0{\cdot}1{\cdot}1 = 3+4+0 = 7$.
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BY MINORS ▸

$$\operatorname{Det} C \;=\; 2\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} \;=\; 2(1) - 0 + 1(-2) \;=\; 0.$$

So C squashes volume to nothing, and no inverse can exist. Strictly, we have only shown that Det $\neq 0$ is *necessary*; next lecture closes the gap.

Solution: question 3

SOLUTION

$\mathbf{u} \cdot \mathbf{v} = 2(-1) + 1(2) = 0$, so $\mathbf{u} \perp \mathbf{v}$, and the parallelogram they span has area

$$\left| \text{Det} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \right| = |4 + 1| = 5.$$

Both vectors have length $\sqrt{5}$, so $\|\mathbf{u}\|\|\mathbf{v}\| = 5$ as well: with perpendicular sides the parallelogram is a rectangle.

