Lecture 8: Probability Distributions

MA208 Quantitative Techniques for Business

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Poisson Distribution

The Poisson distribution is one of the most widely used probability distributions. It is usually used in scenarios where we are counting the occurrences of certain events in an interval of time or space. Here is an example of a scenario where a Poisson random variable might be used.

Example

Suppose that we are counting the number of customers who visit a certain store from 1pm to 2pm. Based on data from previous days, we know that on average $\lambda = 15$ customers visit the store. Of course, there will be more customers some days and fewer on others.

Here, we may model the random variable X showing the number of customers as a Poisson random variable with parameter $\lambda = 15$.

Poisson Distribution

The Poisson distribution is a discrete probability distribution that expresses the average number of events occuring in a fixed period of time or space if these events occur with a known average rate λ.

Poisson Probability Distribution

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

where λ is the average number of success per interval (also donated by μ), $e = 2.71828...$ and $x \in \{0, 1, 2, ...\}$.

Note: If we don't have a value for λ, we use $\lambda = np$.

Poisson Distribution

Properties of the Poisson distribution:

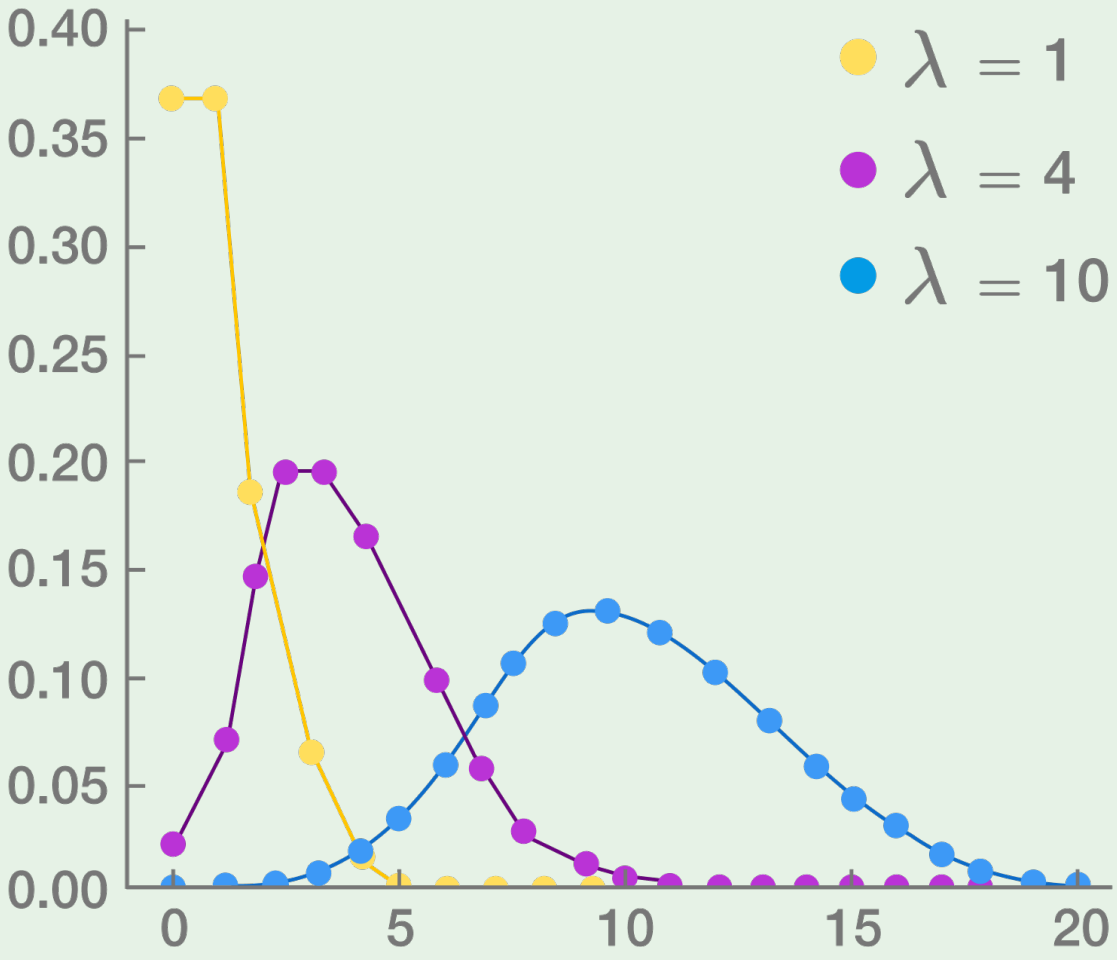
- The probability of an occurrence is the same for any two intervals of equal length.
- The occurrence or non-occurrence in any interval is independent of the occurrence or non-occurrence in any other interval.

The Poisson distribution is useful for modeling the number of times an event occurs in an interval of time or space, for example

- the number of patients arriving in an A&E between 11 and 12 pm,
- the number of occurrences of the DNA gene “ACGT” in a gene.

Poisson Distribution

Example



Example

Given that a bank receives on average six bad checks per day, what is the probability that it will receive four in a day?

Solution

We have $\lambda = 6$, $x = 4$, hence:

$$P(X = 4) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-6}6^{4}}{4!} \simeq 0.132$$

Example

A life insurance salesman sells on average three life insurance policies per week. Use the Poisson distribution formula to calculate the probability that in a given week he will sell

(a) some policies,

(b) two or more policies, but less than five policies.

Solution

We have $\lambda = 3$.

$$P(X > 0) = 1 - P(X = 0) = 1 - \frac{e^{-3}3^0}{0!}$$
$$= 1 - e^{-3} \simeq 1 - 4.9787 \times 10^{-3} \simeq 0.95021$$

$$
\begin{aligned}
P(2 \leqslant X \leqslant 5) &= P(X = 2) + P(X = 3) + P(X = 4) \\
&= \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!} + \frac{e^{-3}3^4}{4!} \\
&\simeq 0.61611
\end{aligned}
$$

Expected Value

If you have a collection of numbers $a_1, a_2, ..., a_n$, their average is a single number that describes the whole collection.

Now, consider a random variable X. We would like to define its average, or as it is called in probability, its **expected value** or **mean**.

The expected value is defined as the weighted average of the values in the range.

Expected Value

Suppose a random variable X can take

value	x_1	with probability	p_1,
"	x_2	"	p_2,
	...		
"	x_k	"	p_k.

Then the **expected value** of the random variable X is defined as

Expected Value

$$E(X) = \sum_{i=1}^{k} x_i p_i$$

Note: The expected value $E(X)$ is also called the **mean** of the probability distribution and often denoted by μ.

Consider the following example from Lecture 7:

Example

Two balls are drawn in random in a succession without replacement from an urn containing four red balls and six black balls. We found the possibilities of all possible outcomes.

1. What is the expected number of red balls?
2. If we perform the experiment 1000 times, how many red balls can we expect to get?

Below are the values we got in Lecture 7.

Possible outcomes	RR	RB	BR	BB
x_i	2	1	1	0
$P(x_i)$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{4}{15}$	$\frac{1}{3}$

Solution

$$\begin{aligned} E(X) &= \sum_{i=1}^{k} x_i p_i \\ &= 2 \cdot \frac{2}{15} + 1 \cdot \frac{4}{15} + 1 \cdot \frac{4}{15} + 0 \cdot \frac{1}{3} \\ &= \frac{4}{5} = 0.8 \end{aligned}$$

$$1000 \times E(X) = 1000 \times 0.8 = 800$$

If we perform the experiment 1000 times, we would expect to get 800 red balls.

The concept of expected value of a random variable is one of the most important concepts in probability theory. It was first devised in the 17th century to analyze gambling games and answer questions such as:

- How much do I gain - or lose - on average, if I repeatedly play a given gambling game?
- How much can I expect to gain - or lose - by performing a certain bet?

Example

If you play a game where you gain €2 with probability 1/2 and you lose €1 with probability 1/2, then the expected value of the game is 50 cent:

$$E(X) = \frac{1}{2} \cdot (+2€) + \frac{1}{2} \cdot (-1€) = 0.5€$$

Example

You are invited to play a game in which five fair coins are tossed in a fair way. Denote by X the number of heads that the five coins show together.

(a) What are the chances that (i) $X = 0$, (ii) $X = 1$, (iii) $X = 2$, (iv) $X = 3$, (v) $X = 4$, (v) $X = 5$?

(b) You win $€X + X^2$ if an even number of the coins show heads, and you lose $€X + X^2$ if an odd number of them show heads. What are your expected winnings in this game?

Solution

We have a binomial distribution:

$$\Rightarrow P(X = x) = \binom{n}{x} p^x \times (1-p)^{(n-x)} = \frac{1}{32}$$

$$\Rightarrow P(X = 0) = \binom{5}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$$\Rightarrow P(X = 1) = \binom{5}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4 = \frac{5}{32}$$

Similarly,

$$\begin{aligned} P(x=2) &= \tfrac{10}{32} \\ P(x=3) &= \tfrac{10}{32} \\ P(x=4) &= \tfrac{5}{32} \\ P(x=5) &= \tfrac{1}{32} \end{aligned}$$

(b)

$$\begin{aligned} \text{Expected winning} &= \sum P(X=x)\cdot \text{winnings} \\ &= \frac{1}{32}(0+0^2) - \frac{5}{32}(1+1^2) + \frac{10}{32}(2+2^2) \\ &\quad -\frac{10}{32}(3+3^2) + \frac{5}{32}(4+4^2) - \frac{1}{32}(5+5^2) \\ &= -\frac{10}{32} + \frac{60}{32} - \frac{120}{32} + \frac{100}{32} - \frac{30}{32} = 0! \end{aligned}$$

That is to say, we expect no winnings.

Variance

The **variance** is a measure of how spread out the distribution of a random variable is.

Variance

The variance of a random variable X, with mean $E(X)$, is

$$Var(X) = \sigma^2 = \sum_{i=1}^{k} (x_i - E(X))^2 \cdot P(x_i)$$

Example

Calculate the variance in our example from before (drawing balls from an urn containing four red balls and six black balls).

$$\begin{aligned} Var(X) &= \sigma^2 = \sum_{i=1}^{4} (x_i - 0.8)^2 \cdot P(x_i) \\ &= (2-0.8)^2 \cdot \frac{2}{15} + \cdots + (0-0.8)^2 \cdot \frac{1}{9} = \frac{6.4}{15} \approx 0.427 \end{aligned}$$

Standard Deviation

The **standard deviation**, usually shown as σ, is simply the square root of variance, is another measure of the spread of the distribution.

Variance

$$\sigma = \sqrt{Var(X)}$$

Example

In our example above the standard deviation is

$$\sigma = \sqrt{0.427} \approx 0.653.$$

Note: The standard deviation is important in finance, where the standard deviation on the *rate of return* (the profit on an investment over a period of time) on an investment is a measure of the *volatility* (the degree of variation of a trading price series) of the investment.

Why both?

Remark:

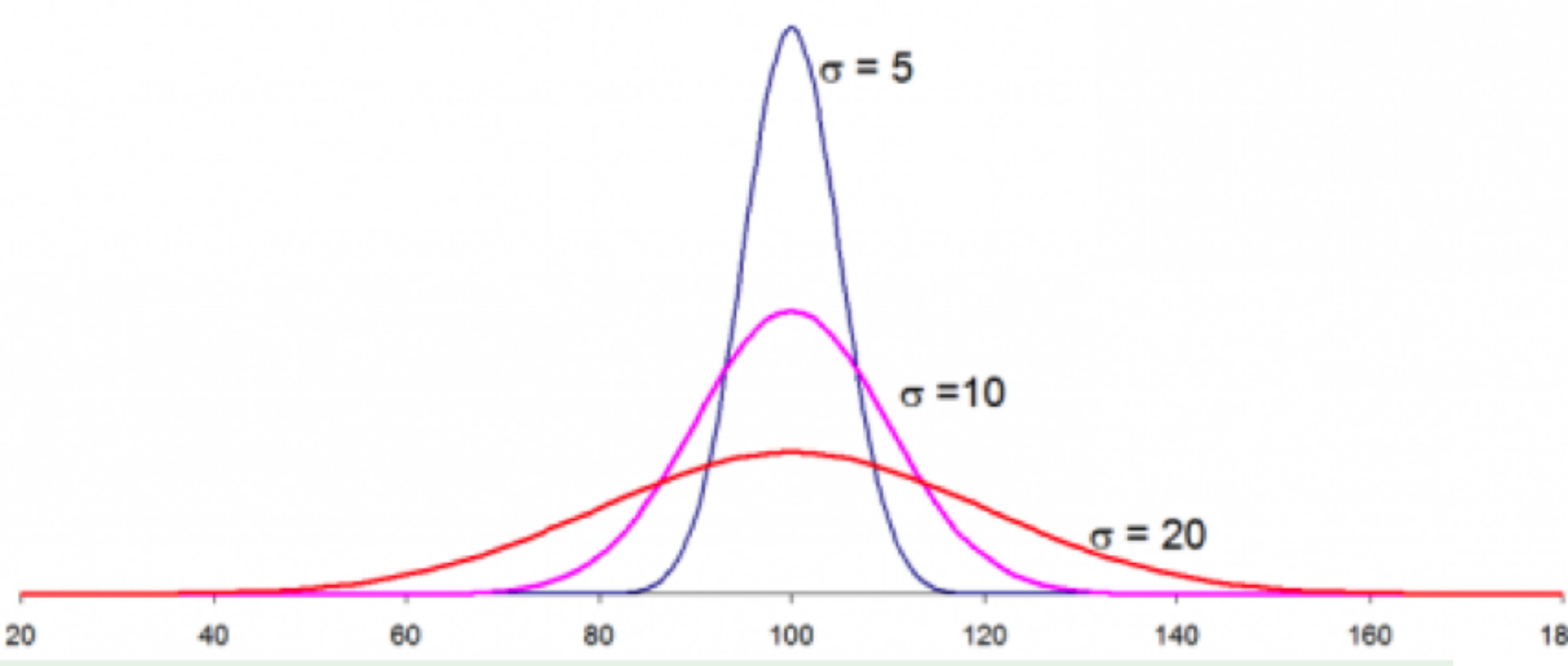
A common question we might ask would be *Why are we so interested in both*? Clearly, one defines the other, fully, hence, why care about both?
As a rule of thumb, we can think of it like this:

Variance:	Mathematically useful It has nice mathematical properties, such as $\mathrm{Var}(aX+b) = a^2 + \mathrm{Var}(X)$ $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ if $X \perp Y$.
Standard Deviation:	Intuitively useful Measures how far from the mean we expect to find most samples.

Maybe a reasonable "real world" comparison would be **Kelvin** and **Celsius**, when measuring temperature.

Standard Deviation

Examples



Examples

Find the standard deviation σ for the following distribution.

X	8	12	16	20	24
$P(X)$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{12}$

Solution

$$
\begin{aligned}
E(X) &= \mu = \sum_{i=1}^{5} x_i \cdot P(x_i) \\
&= 8\frac{1}{8} + 12\frac{1}{6} + 16\frac{3}{8} + 20\frac{1}{4} + 24\frac{1}{12} \\
&= 16 \\
Var(X) &= \sigma^2 = \sum_{i=1}^{5} \left[(x_i - \mu)^2 \cdot P(x)\right] \\
&= (8-16)^2 \cdot \frac{1}{8} + (12-16)^2 \cdot \frac{1}{6} + (16-16) \cdot \frac{1}{8} \\
&\quad + (20-16)^2 \cdot \frac{1}{4} + (24-16)^2 \cdot \frac{1}{12} = 20 \\
\sigma &= \sqrt{20} \approx 4.472
\end{aligned}
$$

Standard Deviation

Knowing the mean μ and standard deviation σ of the probability distribution of X, in conjunction with Chebyshev's Rule (Lecture 4), we can make statements about the likelihood that values of X will fall within the intervals $\mu + \sigma$, $\mu + 2\sigma$ and $\mu + 3\sigma$:

Chebyshev's Rule (actual result)

$$P(|X - \mu| \geqslant k\sigma) \leqslant \frac{1}{k^2}.$$

$$P(|X - \mu| < k\sigma) \geqslant 1 - \frac{1}{k^2}.$$

$$\begin{aligned} P(\mu - \sigma < x < \mu + \sigma) &\geqslant 0 \\ P(\mu - 2\sigma < x < \mu + 2\sigma) &\geqslant \frac{3}{4} \\ P(\mu - 3\sigma < x < \mu + 3\sigma) &\geqslant \frac{8}{9} \end{aligned}$$

Example

Suppose you invest a fixed sum of money in each of five Internet business ventures. Assume you know that 70% of such ventures are successful, the outcomes of the ventures are independent of one another, and the probability distribution for the number, x, of successful ventures out of five is

x	0	1	2	3	4	5
$P(x)$	0.002	0.029	0.132	0.309	0.360	0.168

1. Find $\mu = E(x)$. Interpret the result.
2. Find $\sigma = \sqrt{Var(x)}$.
3. Graph $P(x)$. Locate μ and the interval $(\mu - 2\sigma, \mu + 2\sigma)$. Use Chebyshev's Rule to approximate the probability that x falls in this interval. Compare this result with the actual probability.

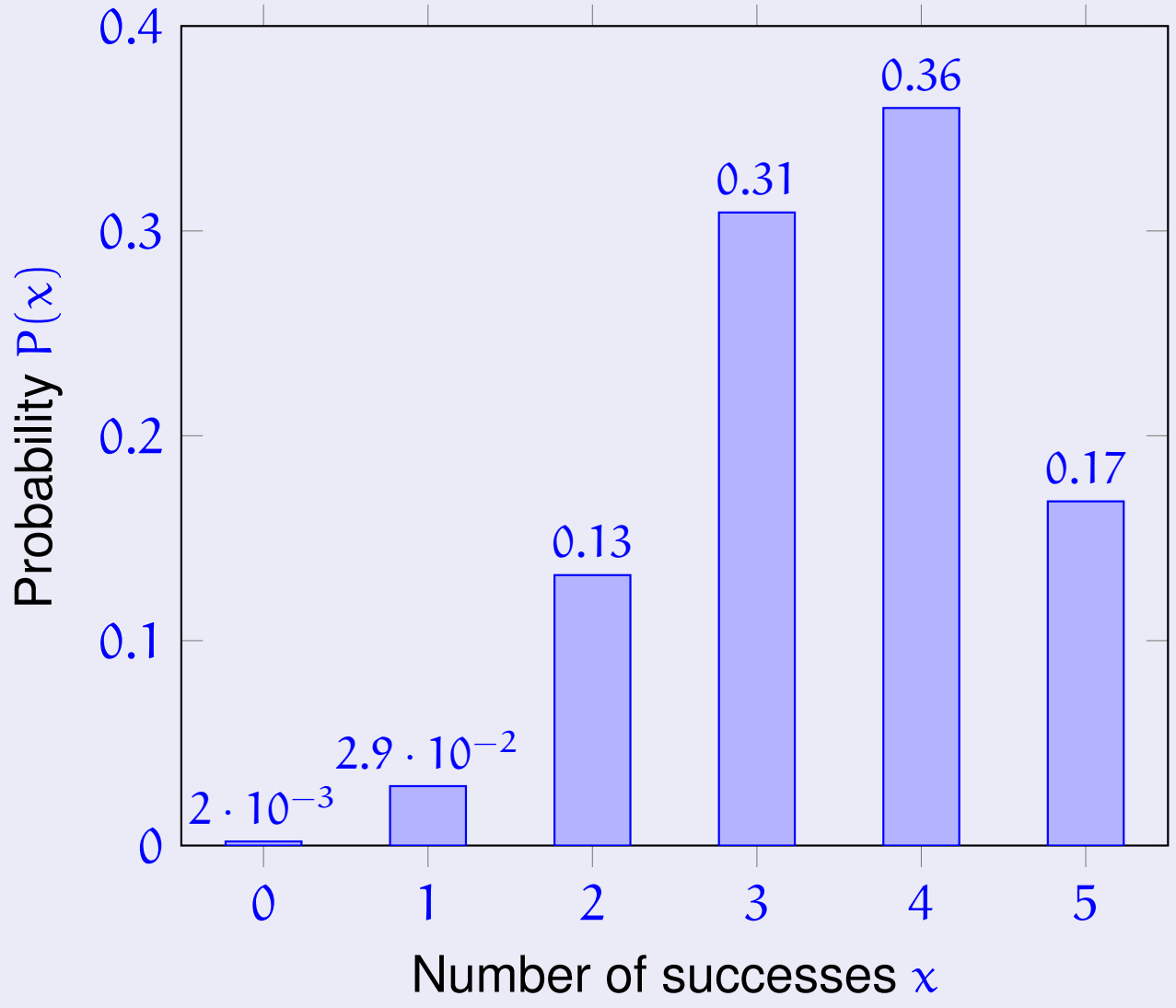
Solution

$\mu = E(X) = 0 \cdot 0.002 + 1 \cdot 0.029 + 2 \cdot 0.132 + 3 \cdot 0.309 + 4 \cdot 0.360 + 5 \cdot 0.168 = 3.5$

On Average, the number of successful ventures out of 5 is 3.5.

(2) $\sigma^2 = (0 - 3.5)^2 \cdot 0.002 + \cdots + (5 - 3.5)^2 \cdot 0.168 = 1.05$

(3)



Chebyshev: At least 75% of observed values are within $\mu \pm 2.6 = [1.46, 5.54]$.
In reality: The actual probability of successful ventures being in the interval $[1.46, 5.54] = [2, 5]$ is $0.13 + 0.31 + 0.36 + 0.17 = 0.969$.

Example (from Exam 2016/17)

A goup of students has the resources to sell any one of the following items at a market tomorrow: ice cream, hot drinks or soup. Their expected profit depends on the choice of item and the weather, and is summarized in the following table.

Weather	Ice cream	Hot drink	Soup
Sun	€ 100	€ 20	€ -30
Occasional showers	€ 30	€ 120	€ -10
Rain	€ -20	€ 70	€ 100

Assume probabilities of 30% for sun, 40% for occasional showers and 30% for rain.

(i) Which of the three items will give the maximum expected profit?

(ii) What would be the value, in €, for the students if they knew in advance how the weather is going to be?

Solution