Lecture 6: Probability

MA208 Quantitative Techniques for Business

Ilia Pirashvili, PhD

With special thanks to Dr. Kirsten Pfeiffer

School of Mathematical and Statistical Sciences
University of Galway

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Lectures 6 - Outline

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- Some terminology

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- Conditional probability
- Independent and dependent events

Probability is the measure of the likelihood that an event will occur (from Wikipedia).

Probability - Why?

Example

Suppose that the university's Parking Office claims that two-thirds (67%) of University of Galway students maintain a car. Then, suppose we take a random sample of 100 University of Galway students and determine that the proportion of students in the sample who maintain a car in the University of Galway is $\frac{69}{100} = 0.69$, that is 69%.

Now we need to ask the question: if the actual population proportion is 0.67, how likely is it that we would get a sample proportion of 0.69?

Any time we want to use a sample to draw a conclusion about some larger population, we need to answer the question "how likely is it...?" or "what is the probability...?". To answer such a question, we need to understand probability, probability rules, and probability models.

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$$S = \{\text{Head}, \text{Tail}\} = \{\text{H}, \text{T}\}.$$

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1. Suppose we are tossing a coin. The sample space is
$$S = \{\text{Head}, \text{Tail}\} = \{\text{H}, \text{T}\}.$$
2. Now we are tossing the coin twice. In this case
$$S = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}.$$

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3. If C is the event that a randomly selected student owns no more than five pairs of jeans:

$$E_C = \text{student owns no more than five} = \{0, 1, 2, 3, 4, 5\}.$$

Suppose an event E can happen in r ways, out of n possible equally likely ways. Then the probability of occurence of the event (success) is

Probability of an event

$$P(E) = \frac{r}{n}$$

The probability of non-occurence of the event (failure) is

Note: We have

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$$P(\overline{E}) = \frac{n-r}{n} = 1 - \frac{r}{n}$$

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2. What is the probability of getting a 5 rolling a fair die?

   Sample space $S = \{1, 2, 3, 4, 5, 6\}$, Event $E_5 = \{5\}$.
   As such, $n = 6$ and $r = 1$, giving us

   $$P(E_5) = \frac{1}{6}.$$

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   Sample space = $\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$, $E_{TT} = \{HTT, THT, TTH\}$. As such, we have $n = 8$, $r = 3$ and

$$P(E_{TT}) = \frac{3}{8}.$$

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   Alternatively: not $E_{A,B} = \overline{E_{A,B}} = \{CD\}$, $P(\overline{E_{A,B}}) = \frac{1}{6}$ and hence,
   $P(E_{A,B}) = 1 - P(\overline{E_{A,B}}) = 1 - \frac{1}{6} = \frac{5}{6}$.

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   $\overline{E_A} = \{BC, BD, CD\}$ with $P(\overline{E_A}) = \frac{3}{6} = \frac{1}{2}$.
   Or, using part (i) and that $P(E_A) + P(\overline{E_A}) = 1$.

Conditional Probability

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What is the probability of rain, given the forecast was for rain?

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If E_1 and E_2 are two events, the probability that E_2 occurs, given that E_1 has occured, is denoted by

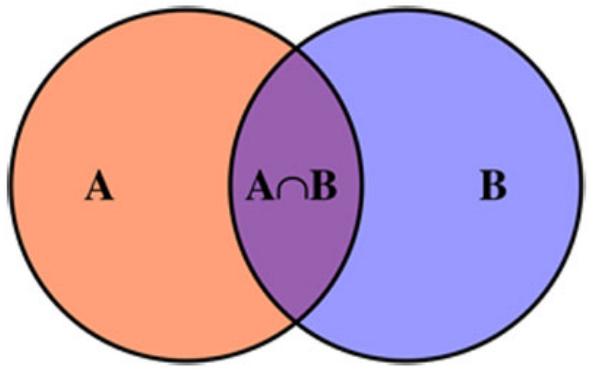
$$P(E_2|E_1).$$

$P(E_2|E_1)$ is called the ***conditional probability of*** E_2 ***given that*** E_1 ***has occured***.

Calculating Conditional Probability

Let E_1 and E_2 be two events defined in a sample space S such that $P(E_1) > 0$. The conditional probability of E_2, assuming that E_1 has already occured, is given by

$$P(E_2|E_1) = \frac{P(E_2 \text{ and } E_1)}{P(E_1)} = \frac{P(E_2 \cap E_1)}{P(E_1)}$$



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Let's get back to our example from above.

Example

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Let A = event "it rains" and B = event "forecast rain".
We are given the following information:

$$P(A \cap B) = 0.7 \tag{1}$$

$$P(B) = 0.8 \tag{2}$$

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So

$$P(A|B) = \frac{0.7}{0.8} = \frac{7}{8}.$$

Example

Let A denote the event "student wears glasses" and let B denote the event "student is French". In a class of 100 students, suppose 60 are French, and suppose that ten of the French students are wearing glasses. Find the probability that if I pick a French student, this student will wear glasses. That is, find $P(A|B)$.

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- 60 out of 100 students are French, so $P(B) = \frac{60}{100} = \frac{6}{10}$.
- So, the conditional probability is $P(A \parallel B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{10}}{\frac{6}{10}} = \frac{1}{6}$.

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   $P(E_{HH+} \mid E_{H+}) = \frac{P(E_{HH})}{P(E_{H+})} = \frac{\frac{1}{2}}{\frac{7}{8}} = \frac{4}{7}$.

Independent and Dependent Events

Definition

Two event E_1, E_2 are said to be independent events if

$$P(E_1) \cap P(E_2) = P(E_1) \times P(E_2).$$

We should understand this as "*the occurence or non-occurence of* E_1 *does not effect the probability of occurence of* E_2 (and vica-versa)".

Using the definition of conditional probability

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we immediately get the equivalent formulation:

Alternative definition

Two (well-defined) event E_1, E_2 are said to be independent events if

$$P(E_2|E_1) = P(E_2).$$

Example

A fair die is tossed twice. Find the probability of getting a 4 or 5 on the first toss and a 1, 2 or 3 on the second toss.

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Solution

Let $E_1 =$ "a 4 or 5 on the first toss" and $E_2 =$ "a 1, 2, or 3 on the second toss". Then

$$P(E_1) = \frac{2}{6} = \frac{1}{3}, \qquad P(E_2) = \frac{3}{6} = \frac{1}{2}.$$

Since the two tosses of the die are independent events,

$$P(E_1 \cap E_2) = P(E_1)\,P(E_2) = \frac{1}{3} \times \frac{1}{2} = \boxed{\frac{1}{6}}.$$

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Two balls are drawn successively without replacement from a box which contains 4 white balls and 3 red balls. Find the probability that

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(a) the first ball drawn is white and the second is red,

(b) both balls are red.

Solution (a): first white, second red

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Therefore, the required probability is

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Solution (b): both red

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Therefore, the required probability is

$$P(E_1 \cap E_2) = P(E_1)\,P(E_2 \mid E_1) = \frac{3}{7} \times \frac{1}{3} = \boxed{\frac{1}{7}}.$$

Examples

Suppose that the probability that a checked suitcase is lost on a single flight is

$$P_\ell = \frac{6}{10^3} = \frac{6}{1000}.$$

If a businesswoman takes 20 flights per year, what is the probability that at least one of her suitcases is lost within the next 20 years?

Assumptions:

- Different flights are independent;
- She checks a suitcase on every flight.

Solution

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Therefore, the probability that at least one suitcase is mishandled is

$$1 - \left(\frac{994}{1000}\right)^{400}.$$

Remark

The key point is that the question asks for **at least one suitcase lost** in n flights. One might wonder: why not just take

$$P(E)^n = \left(\frac{6}{1000}\right)^{400} ?$$

The reason is subtle: $P(E)^n$ would correspond to the probability that **every single flight has a suitcase lost**. But we are not asking for "all lost," we are asking for "at least one lost."

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Remember: Directly powering $P(E)$ does not capture the meaning of "at least one", it counts only the extreme case where every trial is a success (loss), which is far less likely.

By contrast, if we define E = "a suitcase is lost on a flight", then

$$1 - P(E) = \text{probability that a suitcase is NOT lost on a flight.}$$

If we raise $1 - P(E)$ to the nth power, $(1 - P(E))^n$, we get the probability that **no suitcases are lost in any of the n flights**. This is exactly the opposite of "at least one lost." Therefore, to answer the original question, we subtract:

$$P(\text{at least one lost}) = 1 - (1 - P(E))^n.$$

In short:

Raising $P(E)$ to n does not give "at least one success", it gives the probability that **every trial is a success**.

Raising $1 - P(E)$ to n gives "no successes," and subtracting from 1 gives the desired "at least one."